CHAPTER TWO

Electrostatic

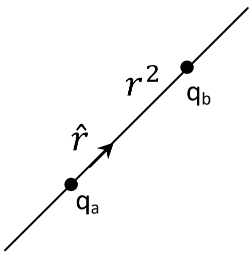
Electrostatic field:

In this chapter we shall restrict our attention to static electric fields in vacuum or free space. Such fields, for example, are found in the focusing and deflection systems of electrostatic cathode-ray tubes. For all practical purposes, our results will also be applicable to air and other gases.

Coulombs Law:

Coulomb stated that the force between two very small Objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F_{ab} = k\frac{q_a q_b}{r^2}\hat{r}$$



$q_a\ and\ q_b\ are\ two\ piont\ charges\ ,\qquad k = 9\times10^9\,\frac{N\cdot m^2}{c^2}$

$\epsilon = permitivity\ of\ free\ space = 8.8542\times10^{-12}\,\frac{F}{m}$

The coulombs force in nature is very enormous comparing with gravitational force. The force between two electrons:

$1-\ F_E = k\frac{q_1 q_2}{r^2}\hat{r} = 9\times10^9\,\frac{(1.6\times10^{-19})^2}{1}$

$2-\ F_G = G\frac{m_1 m_2}{r^2}\hat{r} = 6.6\times10^{-11}\,\frac{(9.1\times10^{-31})^2}{1}\quad ,\qquad G = 6.6\times10^{-11}\,\frac{N.m}{kg^2}$

$$\frac{F_E}{F_G} = \frac{23.04\times10^{-29}}{552.34\times10^{-73}} = 0.0417\times10^{44}\ \ N$$

$\therefore\ F_G \ll F_E$

Electric field intensity (E):

(E) is the force per unit charge exerted on a test charge.

$$F_{ab} = k\frac{q_a q_b}{r^2}\hat{r} = \frac{q_a q_b}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\therefore E_a = \frac{F_{ab}}{q_b} = \frac{q_a}{4\pi\varepsilon_0 r^2}\hat{r}$$

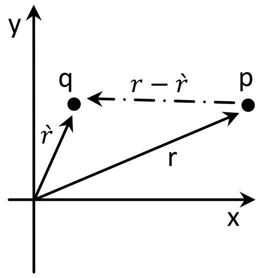
The electric field intensity is the same when ther q_a is present or not.

$$\left[E = \frac{q_a}{4\pi\varepsilon_0 r^2}\hat{r}\right]$$

$$E = \frac{q(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4\pi\varepsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}} \quad at\ the\ Origen\ point$$

$$E = \frac{q_a}{4\pi\varepsilon_0|r-\grave{r}|^2}(r-\grave{r}) \quad at\ any\ point$$

$$E = \frac{q((x-\grave{x})\hat{\imath}+(y-\grave{y})\hat{\jmath}+(z-\grave{z})\hat{k})}{4\pi\varepsilon_0[(x-\grave{x})^2+(y-\grave{y})^2+(z-\grave{z})^2]^{\frac{3}{2}}}$$



Field of n-point charge:

$$E = \frac{q_1}{4\pi\varepsilon_0|r-\grave{r}_1|^2}a_1 + \frac{q_2}{4\pi\varepsilon_0|r-\grave{r}_2|^2}a_2 + \frac{q_3}{4\pi\varepsilon_0|r-\grave{r}_3|^2}a_3 \dots\dots\dots + \frac{q_n}{4\pi\varepsilon_0|r-\grave{r}_n|^2}a_n$$

$$E = \sum_{m=1}^{n}\frac{q_m}{4\pi\varepsilon_0|r-\grave{r}_m|^2}a_m$$

Problem:

1- Two point charge $q_1 = 50\mu c$, $q_2 = 10\mu c$ are lactated at (-1,1,-3)m, and (3,1,0)m, respectively. Find the Force on q_1.
2- Find the electric field (E) at P(-4,6,-5)in the free space caused by a charge
   a- At the Origen.
   b- At (2,-1,3).
3- Point charge $q_1 = 2\mu c$ is locted at p$_1$(-3,7,-4) in free space while $q_2 = -5\mu c$ is locted at p$_2$(2,4,-1).
   At the point (12,12,15)find E, $|E|, a_E(unit\ vector)$

Charge Distribution:

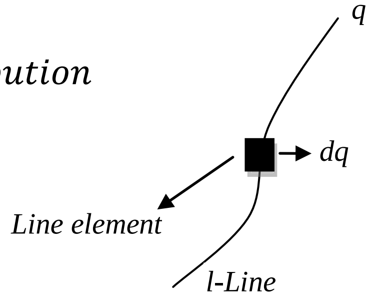
1- L in charge distribution:

$$\rho_l = \frac{q}{l} \quad \equiv \quad Line\ charge\ distribution$$

$$dq = \rho_l dl$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \int \frac{\rho_l dl}{4\pi\varepsilon_0 r^2}\hat{r}$$

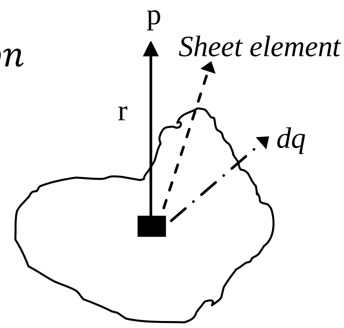


2- sheet charge distribution:

$$\rho_s = \frac{q}{s} \quad \equiv \quad Sheet\ charge\ distribution$$

$$dq = \rho_s ds$$

$$E = \int \frac{\rho_s ds}{4\pi\varepsilon_0 r^2}\hat{r}$$



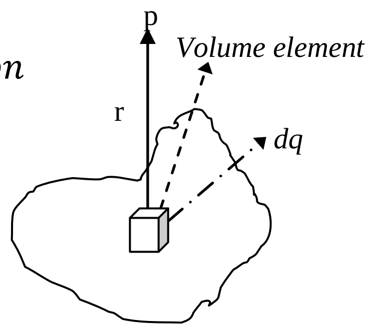
3- Volume charge distribution:

$$\rho_v = \frac{q}{v} \quad \equiv \quad Volume\ charge\ distribution$$

$$q = \rho_v v$$

$$dq = \rho_v dv$$

$$E = \int \frac{\rho_v dv}{4\pi\varepsilon_0 r^2}\hat{r}$$

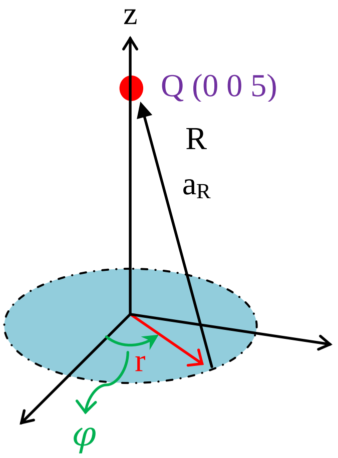


Standard charge distribution

1- Electric field intensity due to line charge, E =?. $\left(E = \frac{p_l}{2\pi\epsilon_0 r} a_r\right)$
2- Electric field intensity due to infinite plane sheet charge, E =?. $\left(E = \frac{p_s}{2\epsilon_0} a_n\right)$
3- Electric field intensity due to a point charge, E =?. $E = \frac{p}{4\pi\epsilon_0 r^2} a_r$

Example:

Find the force on a point charge of 50μc at(0,0,5) due to a charge of 500π μc, which uniformly distributed over the circular disk ($r \leq 5 \quad \varphi = 2\pi \quad z = 0$).



Sol/

$$F = Eq$$

$$E = \int \frac{\rho_s ds}{4\pi\varepsilon_0 R^2} a_R$$

$$F = qE = q\int \frac{\rho_s ds}{4\pi\varepsilon_0 R^2} a_R \quad , \qquad \rho_s = \frac{q}{s} = \frac{500\pi\ 10^{-6}}{\pi r^2} = \frac{500\pi\ 10^{-6}}{\pi\ 25} = 20\times10^{-6}$$

$$R = -r_{a_r} + z_{a_z} \quad , \qquad dl = dra_r + rd\varphi a_\varphi + dza_z \ , \qquad ds = (rdr\ d\varphi)a_z$$

$$F = q\int \frac{\rho_s rdrd\varphi}{4\pi\varepsilon_0 R^2} a_R = q\int \frac{\rho_s rdrd\varphi\ a_z}{4\pi\varepsilon_0(r^2+25)} \cdot \frac{(-r_{a_r} + 5_{a_z})}{\sqrt{r^2+25}}$$

$$= \frac{5q\rho_s}{4\pi\varepsilon_0}\int_0^5 \frac{rdr}{(r^2+25)^{\frac{3}{2}}}\int_0^{2\pi} d\varphi$$

$$= \frac{5.(50\times10^{-6})(20\times10^{-6})}{4\pi(\frac{10^{-9}}{36\pi})} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{r^2+25}}\Bigg|_0^5 \cdot 2\pi = 16.56N$$

Q/ Find E at (0,0,5)m due to Q1= 0.35μc at (0,4,0)m and Q2=0,55 μc at (3,0,0)m

The Electric Potential:

$$w = -\int_{p_1}^{p_2} F \cdot dl$$

The work required to move a point charge q from p_1 to p_2 at a constant speed in an electric field.

$$w = -\int_{p_1}^{p_2} Eq \cdot dl \quad if\ the\ path\ closed\ \ w = -\oint_{p_1}^{p_2} Eq \cdot dl = -q \oint_{p_1}^{p_2} E \cdot dl = 0$$

∴The same over the closed path = 0

$$\therefore \oint E\ dl = 0 \qquad E = conservative$$

The conservative field is 1-a function of position only 2- have a fixed value

Using stokes theorem $\oint Edl = \int_s (\nabla \times E) \cdot nda$

$$\therefore \int_s (\nabla \times E) \cdot nda = 0 \quad if\ \ E = -\nabla V \qquad p_2 < p_1$$

$$\nabla \times \nabla \mathrm{V} = 0$$

The (-)sing is required in order that or point toward a decrease in potential.

$$E = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$Edl = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right) = -dV$$

$$dV = -E \cdot dl$$

$$v_{12} = -\int_{r_1}^{r_2} dV = -\int_{r_1}^{r_2} E \cdot dl = -\int \frac{q}{4\pi\varepsilon_0 r^2} dr = -\frac{q}{4\pi\varepsilon_0}\int_{r_1}^{r_2} \frac{dr}{r^2}$$

$V_{1,2} -\ potential\ defirence$

$$V_{1,2} = V_2 - V_1$$

Is the work done in moving a charge q from one point to another

The potential is the work done per unit charge.

$$w = -\int Eq \cdot dl \qquad V_{1,2} = -\int_{r_1}^{r_2} E \cdot dl$$

If r_1 is at infinity

$$V_{1,2} = -\int_{\infty}^{r_2} dV = -\int_{\infty}^{r_2} E \cdot dl$$

$$V_{1,2} = V_1 - V_2 = \frac{q}{4\pi\varepsilon_0(\infty)^2} - \frac{q}{4\pi\varepsilon_0(r_2)^2} \quad , \qquad V_1 = 0 \, , \qquad V_2 = -\int_{\infty}^{r_2} E \cdot dl$$

$$in\ general \qquad V = -\int_{\infty}^{r} E \cdot dl$$

The potential:

Is the work required to bring a charge from infinity to a point considered in the electric field. Or potential is the work done per unit charge.

$$V = \frac{w}{q}$$

$$V = -\int_{\infty}^{r} E \cdot dl \qquad \therefore V = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} a_r \cdot dr a_r = -\frac{q}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{dr}{r^2}$$

$$V = -\frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r}\right]_{\infty}^{r}$$

$$= -\frac{q}{4\pi\varepsilon_0} \frac{1}{r} \quad potential\ due\ to\ a\ point\ charge\ at\ a\ distance\ (r)$$

Example:

Find the work done in carrying the charge from r_1 to r_2 along a radial path due to infinite line charge.

$$w = -\int Eq \cdot dl = -q \int \frac{\rho_l}{2\pi\varepsilon_0 r} a_r \cdot dr a_r = -\frac{q\rho_l}{2\pi\varepsilon_0} \int_{r_1}^{r_2} \frac{dr}{r} = -\frac{q\rho_l}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}$$

Q\ Find the work done in carrying the charge (q)from r_1 to r_2 along a radial path due to a point charge.

Example: Find the work done in moving a point charge q=5μc from the origin to $(2, \frac{\pi}{4}, \frac{\pi}{2})$ in the field $\left(E = 5e^{-\frac{r}{4}}a_r + \frac{10}{\text{r}\sin\theta}a_\varphi\right)$.

Sol\

$$w = -q\int_{r_1}^{r_2} Edl = -q\int_{r_1}^{r_2}\left(5e^{-\frac{r}{4}}a_r + \frac{10}{\text{r}\sin\theta}a_\varphi\right)\cdot(dra_r + rd\theta a_\theta + r\sin\theta\, d\varphi a_\varphi)$$

$$w = -5\times10^{-6}\left[\int_0^2 5e^{-\frac{r}{4}}dr + \int_0^{\frac{\pi}{2}}10\,d\varphi a_\varphi\right] = -5\times10^{-6}\left[20\left(e^{\frac{-2}{4}} - 1\right) + 5\pi\right]$$
$$= 117.9\times10^{-6}J$$

$$V_{12} = \frac{w}{q} = \frac{117.9\times10^{-6}J}{5\times10^{-6}} = 2.358\,volt.$$

<u>Electric Flux and Gausses Law:</u>

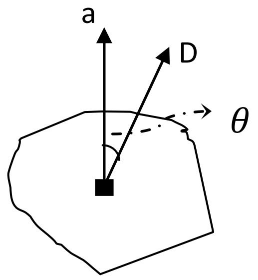
Electric flux (ψ):is the number of electrical lines.

1- Columbus of electrical charge gives rise.
2- Columbus of electric flux, flux(ψ)= φ_{charge}

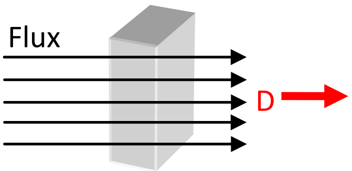
Electrical flux density (D): Is the number of lines per unit area

$$D = \frac{d\vec{\psi}}{d\vec{s}}$$

$$d\psi = \vec{D}.ds \implies d\psi = D\,s\cos\theta$$



D: Is the vector take its direction from the line of flux.

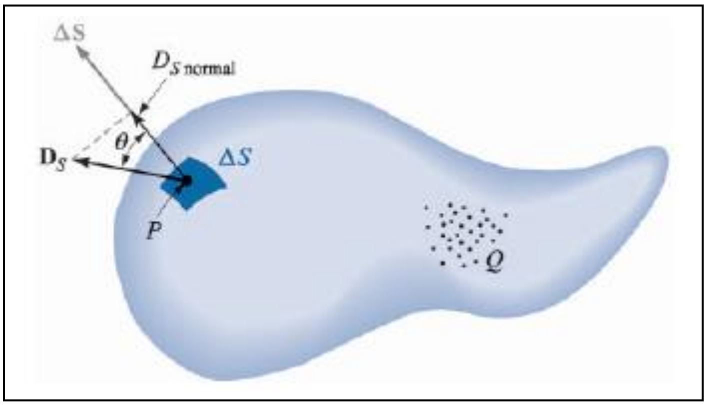


Gauss's Law:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$d\psi = D \cdot ds \qquad\qquad \oint d\psi = \oint D \cdot ds$$

$$\varphi_{enc} = \oint D \cdot ds$$



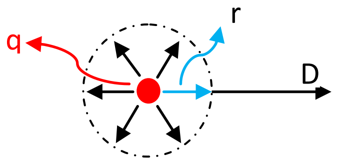
The electric flux density D_S
At P due to
Charge Q. The total flux passing through $\Delta S\ is\ \Delta D_S \cdot \Delta S$.

The ratio between flux density and electric field intensity

Relation between D and E :

$$D \cdot ds = \varphi_{enc} \quad \rightarrow \quad \int D \cdot ds\, a_r = \varphi_{enc} \quad \rightarrow \quad Dr^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = \varphi_{enc}$$

$$D4\pi r^2 = \varphi_{enc} \quad \rightarrow \quad D = \frac{\varphi_{enc}}{4\pi r^2}\, a_r$$



For appoint charge:

$$E = \frac{\varphi_{enc}}{4\pi r^2 \varepsilon_0}\, a_r \quad , \frac{D}{E} = \varepsilon_0$$

$$D = \varepsilon_0 E \quad - in\ free\ space \qquad D = \varepsilon\, E \quad - in\ homogenous\ iso\ topic\ medium$$

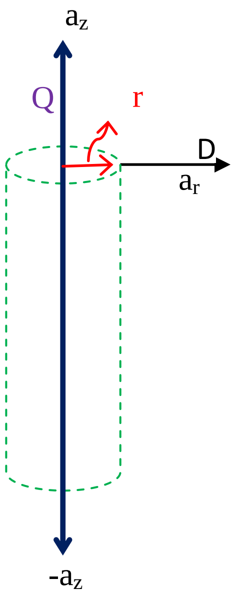
$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_r \equiv Dielectric\ constant$$
$$\varepsilon_0 = permitivity\ of\ free\ space$$
$$\varepsilon = permitivity\ of the\ medium$$

Example: Find D of a uniform finite line charge.

Sol/

The appropriate Gauss surface for a line as a clyndrical



$$\int D \cdot ds = \varphi_{enc}$$

$$\int Da_r \cdot ds\, a_z + \int Da_r \cdot ds\, a_r + \int Da_r \cdot ds\,(-a_z) = \varphi_{enc}$$

$$Dr\int_0^{2\pi} d\varphi \int_0^{l} dz = \varphi_{enc}$$

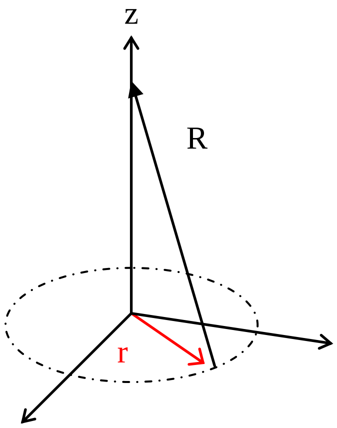
$$\therefore D = \frac{\varphi_{enc}}{2\pi r l}$$

Q/ Find D and E due to a uniform finite plane charge.
Sol./

$$\int D \cdot ds = \varphi = D \cdot rdrd\varphi = D\int_0^{r} rdr \int_0^{2\pi} d\varphi = D\frac{r^2}{2}2\pi = D\pi r^2$$

$$D = \frac{\varphi}{\pi r^2}$$

$$D = \varepsilon_0 E \quad \rightarrow \quad E = \frac{D}{\varepsilon_0} \qquad \therefore E = \frac{\varphi}{\pi r^2 \varepsilon_0}$$



Maxwell's First equation:

$$\int D \cdot ds = \varphi_{enc} = \int \rho_v dv \quad \dots\dots\dots\dots\dots\dots \quad Gauss\ Law$$

$$\oint D \cdot ds = \int (\nabla \cdot D) dv \quad \dots\dots\dots \quad Divergence\ theorm$$

$$\int_v (\nabla \cdot D) dv = \int_v \rho_v dv$$

$$[\nabla \cdot D = \rho]\ The\ point\ form\ of\ Gauss's\ Law\ or\ Maxwell\ 1st\ equation$$

$$D = \varepsilon_0 E \quad \dots\dots\dots\dots \quad in\ free\ space$$

$$D = \varepsilon E \quad \dots\dots\dots\dots \quad in\ other\ medium$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad in\ free\ space$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon} \quad \dots\dots\dots\dots in\ isotropic\ homogenoas\ medium$$

Q1\ Given the field $\left(D = (2x+1)y^2 a_x + 2x(x+1)y a_y\right)\frac{c}{m^2}$ compute the total flux crossing the surface defined by:

$$a)\ x = 0 \qquad -2 \le y \le 2 \qquad -2 \le z \le 2$$

$$b)\ y = 2 \qquad -5 \le x \le 5 \qquad -2 \le z \le 2$$

Q2\ The cylindrical surfaces (r = 3,4 and 5)m contain a uniform charge density of (8, -12 , and $\rho_s x)\frac{nc}{m^2}$ respectively.

a) At what volume must $\rho_s x$ be so that D = 0 for r > 5 ?
b) If $(\rho_s x = 2\frac{nc}{m^2})$ calculate and plot D_r versus r for $0 \le r \le 6 \quad m$

Q3\ The spherical surfaces (r = 3,4 and 5)m contain a uniform charge density of (8, -12 , and $\rho_s x)\frac{nc}{m^2}$ respectively.

a) At what volume must $\rho_s x$ be so that D = 0 for r > 5 ?
b) If $(\rho_s x = 2\frac{nc}{m^2})$ calculate and plot D_r versus r for $0 \le r \le 6 \quad m$

Hint:

$$\int D \cdot ds = \varphi_{enc} = \varphi_1 + \varphi_2 + \varphi_3 = \rho_{s_1} ds + \rho_{s_2} ds + \rho_{s_3} ds = 0$$

Energy Density in Electrostatic Field

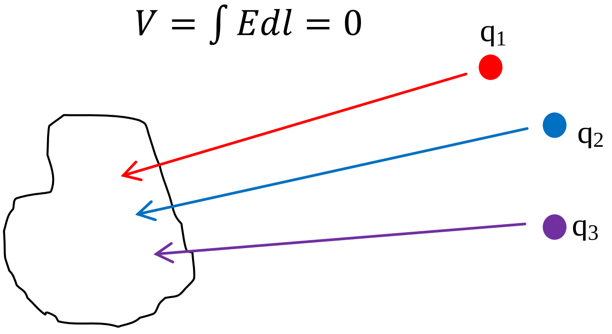
The work required to bring point charge from infinity to point considered in Field (E).

$$W = -\int_{\infty}^{r} Eq.dl = q\left[\int_{\infty}^{r} E.dl\right]$$

$$W = qV$$

Consider the work required to assemble charge by charge a distribution of n=3 point charges

$$V = \int Edl = 0$$



$$W = 0 + q_2V_{2,1} + \left(q_3V_{3,1} + q_3V_{3,2}\right) \ldots\ldots\ldots\ldots\ldots.. \; 1$$

If there charge brought in to place reverse order

$$W = 0 + q_2V_{2,3} + \left(q_1V_{1,3} + q_1V_{1,2}\right) \ldots\ldots\ldots\ldots\ldots. \; 2$$

$$2W = q_1\left(V_{1,3} + V_{1,2}\right) + q_2\left(V_{2,1} + V_{2,3}\right) + q_3\left(V_{3,1} + V_{3,2}\right)$$

$$2W = q_1V_1 + q_2V_2 + q_3V_3$$

$$W = \frac{1}{2}(q_1V_1 + q_2V_2 + q_3V_3)$$

For n-point charge:

$$W = \frac{1}{2}(q_1V_1 + q_2V_2 + \ \ldots\ldots\ldots + q_nV_n)$$

$$W = \frac{1}{2}\sum_{m=1}^{n} q_mV_m \quad \equiv \ for \ \ n - point\ charges\ or\ for\ discrete\ charge$$

For continuous charge distribution:

$$W = \frac{1}{2}\int_{v} \rho_vVdv \quad \rightarrow \text{for continuous charge distribution.} \quad \rho = (\nabla \cdot \mathrm{D})$$

$$\left[W = \frac{1}{2}\int_{v} (\nabla \cdot \mathrm{D})Vdv\right]$$

$$\begin{pmatrix}\nabla\cdot(VD) = (\nabla\cdot D)V + D(\nabla\cdot V)\\ (\nabla\cdot D)V = \nabla\cdot(VD) - D(\nabla\cdot V)\end{pmatrix}\left(\oint_{s} A\cdot ds \ = \int_{v} (\nabla\cdot A)dV \ \ \text{divergence theorem}\right)$$

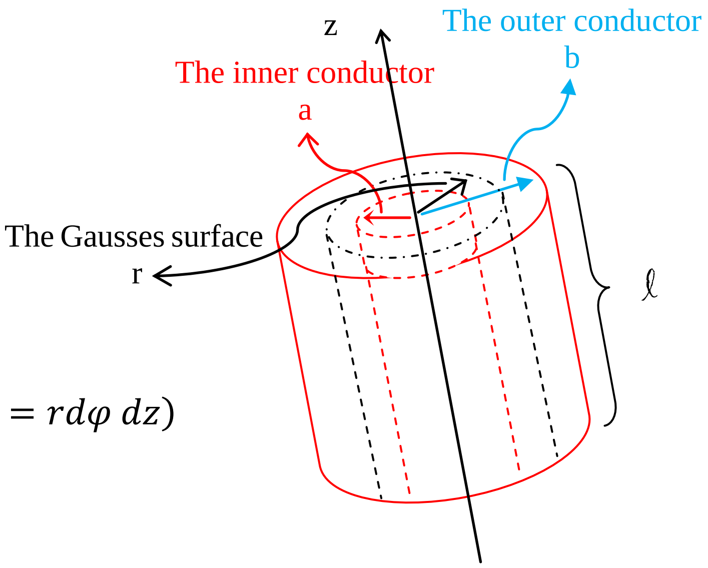
$$W = \frac{1}{2}\int [\nabla \cdot (VD) - D(\nabla \cdot V)]dv = \frac{1}{2}\oint (VD)ds - \frac{1}{2}\int D(\nabla \cdot V)dv$$

$$W = -\frac{1}{2}\int D(-E)dv \qquad wher\ (E = -\nabla V\ and\ D = \varepsilon_0 E)$$

$$\boxed{W = \frac{1}{2}\int \varepsilon_0 E^2 dv \qquad is\ the\ energy\ stored\ in\ the\ electrostatic\ field}$$

Example: Find the energy stored in the electrostatic field of a section of a coaxial cable.

Sol.\



$$\int D \cdot ds = \varphi_{enc} = \int \rho_s ds$$

$$rD\int_0^{2\pi} d\varphi \int_0^l dz = \int \rho_s ds$$

$$= \rho_s a \int_0^{2\pi} d\varphi \int_0^l dz \qquad wher\ (ds = rd\varphi\ dz)$$

$$D.2\pi rl = \rho_s 2\pi al$$

$$D = \frac{\rho_s a}{r} \qquad \therefore E = \frac{\rho_s a}{\varepsilon r} a_r \qquad where\ \ \varepsilon = \varepsilon_0 \varepsilon_r \qquad then:$$

$$W = \frac{1}{2}\varepsilon_0 \int E^2 dv = \frac{1}{2}\int \varepsilon_0 \left(\frac{\rho_s a}{\varepsilon r}\right)^2 rdr\ d\varphi\ dz = \frac{1}{2}\frac{\rho_s{}^2 a^2}{\varepsilon_0 \varepsilon_r^2 r}\int_a^b \frac{dr}{r}\int_0^{2\pi} d\varphi \int_0^l dz$$

$$W = \frac{\rho_s{}^2 a^2}{2\varepsilon_0 \varepsilon_r^2 r}.\ln\frac{b}{a}.2\pi.l \quad \rightarrow \quad W = \frac{\pi \rho_s{}^2 a^2 l}{\varepsilon_0 \varepsilon_r^2 r}.\ln\frac{b}{a} \quad energy\ store$$

The energy stored per unit volume is :

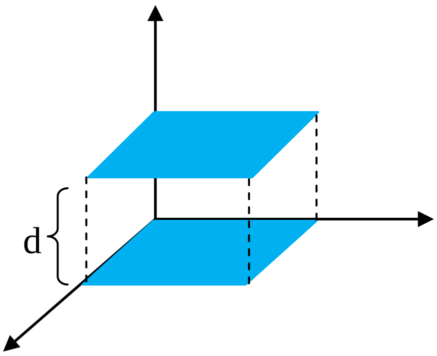
$$u = \frac{W}{V} = \frac{\frac{\pi \rho_s{}^2 a^2 l}{\varepsilon_0 \varepsilon_r^2 r}.\ln\frac{b}{a}}{(\pi b^2 l - \pi a^2 l)} \qquad wher\ \begin{pmatrix} inner\ vol. & outer.vol. \\ \pi a^2 l & \pi b^2 l \end{pmatrix}$$

$$u = \frac{\rho_s{}^2 a^2}{\varepsilon_0 \varepsilon_r^2 r(b^2 - a^2)}.\ln\frac{b}{a}$$

Q1\ Find the energy stored in a parallel plate capacitor.

Sol\

$w$her E=V\d for capacitor then:



$$W = \frac{1}{2}\int \varepsilon_0 E^2 dv$$

$$W = \frac{1}{2}\int \varepsilon_0 \left(\frac{V}{d}\right)^2 dxdydz$$

$$W = \frac{1}{2}\left[\varepsilon_0 \left(\frac{V}{d}\right)^2 Ad\right]$$

$$W = \frac{1}{2}\varepsilon_0 \frac{V^2 A}{d}$$

Q2\Given $E = -5e^{-\frac{r}{a}}\, a_r$ in cylindrical coordinate. Find energy stored in the volume described by $r \leq 2a\ ,\ 0 \leq z \leq 5a.$

Q3\ Find the energy stored in a system of three equal point charges of (2nc) arranged in a line with 0.5cm separation between them.