CHAPTER THREE

# Poisson's and Laplace's Equation

***Poisson's and Laplace's Equation:***

Poisson's and Laplace's Equation provides a method where by the potential function V can be determined.

$$\nabla \cdot D = \rho \qquad Mxweel\ first\ equation$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon} \qquad is\ the\ medium\ is\ homogenous\ and\ isotropic$$

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\varepsilon}$$

$$\nabla^2 \mathrm{V} = \frac{-\rho}{\varepsilon} \quad \ldots\ldots\ldots \quad Poisson's\ eq.$$

If the region of interest contains charge in a known distribution, Poisson's equation used to determent the function.

Very often the region is charge free that is i.e.,$(\rho = 0) \rightarrow \nabla^2 \mathrm{V} = 0$ , Laplace's eq.

$$\left.\begin{aligned} \nabla^2 \mathrm{V} &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ \nabla \mathrm{V} &= \frac{\partial \mathrm{V}}{\partial \mathrm{x}}\mathrm{i} + \frac{\partial \mathrm{V}}{\partial \mathrm{y}}\mathrm{j} + \frac{\partial \mathrm{V}}{\partial \mathrm{z}}\mathrm{k} \end{aligned}\right\} \text{Cartesian coordinate}$$

$$\left.\begin{aligned} \nabla^2 \mathrm{V} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \\ \nabla \mathrm{V} &= \frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \varphi}a_\varphi + \frac{\partial V}{\partial z}a_z \end{aligned}\right\} \text{cylindrical coordinate}$$

$$\left.\begin{aligned} \nabla^2 V &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} \\ \nabla \mathrm{V} &= \frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}a_\varphi \end{aligned}\right\} \text{sphericalcoordinate}$$

## Cartesian sol. in one variable:

$$V = V(x, y, z)$$

$$if\ V = V(x) or\ V = V(y)\ or\ V = V(z)\quad then:$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} \quad or \quad \nabla^2 V = \frac{\partial^2 V}{\partial y^2} \quad or \quad \nabla^2 V = \frac{\partial^2 V}{\partial z^2}$$

$$If\ V = V(x)\ only$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0 \quad \rightarrow \quad \int \frac{d}{dx}\left(\frac{dV}{dx}\right) = \int 0 \quad \rightarrow \quad \frac{dV}{dx} = A \quad \equiv in\ one\ variable$$

$$dV = Adx \quad \rightarrow \quad \int dV = \int A\, dx \quad \rightarrow \quad \therefore\ V = Ax + B$$

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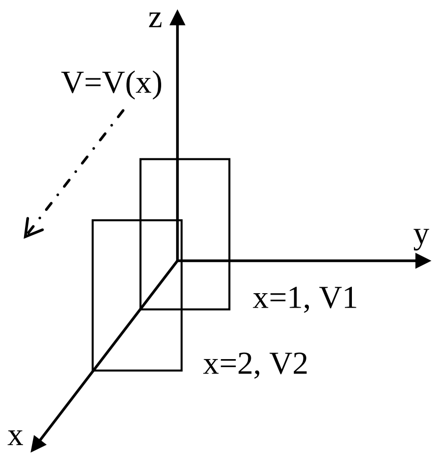
For a parallel conductor

Let: $V = V_1 \quad at \quad x = x_1\ , \quad V = V_2 \quad at \quad x = x_2$

$$\left.\begin{matrix} V_1 = Ax_1 + B \\ V_2 = Ax_2 + B \end{matrix}\right\} \quad subtract\ them \quad \rightarrow \quad A = \frac{V_1 - V_2}{x_1 - x_2}$$

$$B = V_1 - Ax_1 = V_1 - \left(\frac{V_1 - V_2}{x_1 - x_2}\right)x_1 = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

$$V = Ax + B = \left(\frac{V_1 - V_2}{x_1 - x_2}\right)x + \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$




$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2} \quad \ldots\ldots\ldots \quad General\ solution$$

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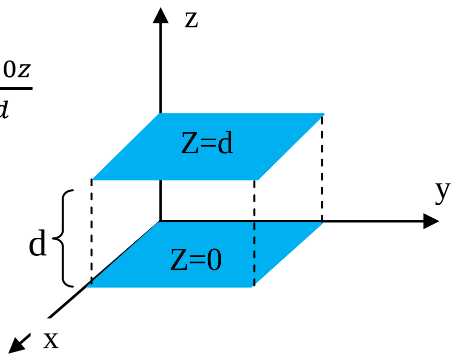
Example: Consider the parallel conductors where *V=0 at Z=0 , V=100 volt at Z=d.*

Sol\ $V = \frac{V_1(z-z_2) - V_2(z-z_1)}{z_1 - z_2} = \frac{V(z-d) - 100(z-0)}{0-d} = \frac{100z}{d}$

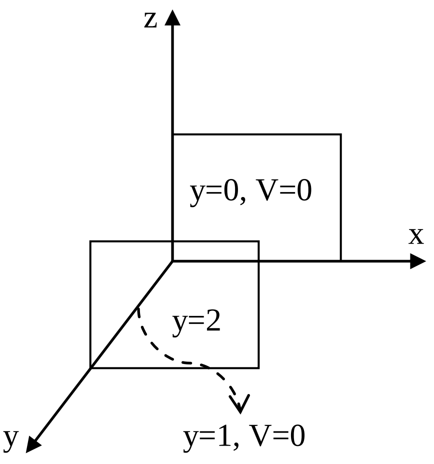
$$E = -\nabla V = -\frac{dV}{dz}a_z = -\frac{d}{dz}\left(\frac{100z}{d}\right) = \frac{100}{d}$$

$$D = \varepsilon_0 E = \frac{100}{d}\varepsilon_0$$



Ex.: Two Parallel conducting planes in free space are at y=0 and y=2 cm, and the zero voltage reference is at y=1 cm, if$D = 2.53j\,\frac{nc}{m^2}$ between the two conductors. Determine the conductor voltage.

Sol.\ $\nabla^2 V = \frac{\partial^2 V}{\partial y^2} = 0$




$$\int \frac{d}{dy}\left(\frac{dV}{dy}\right) = \int 0 \quad \rightarrow \quad V = Ay + B$$

$$E = -\nabla V \quad \rightarrow \quad E = -Aj \;\ldots\ldots\ldots..1$$

$$D = \varepsilon_0 E \quad \rightarrow \quad E = \frac{D}{\varepsilon_0} \quad in\ free\ space$$

$$E = \frac{253 \times 10^{-9}}{8.8542 \times 10^{-12}} = 2.26 \times 10^4 \frac{V}{m}$$

From eq. 1 we get valueof A

$$A = -2.86 \times 10^4 \frac{V}{m} \quad , \quad \therefore V = -2.86 \times 10^4\, y + B$$

Boundary conditionat y=1cm

$$0 = -2.86 \times 10^4 \times 10^{-2} + B \qquad \therefore B = 286$$

$$V = -2.86 \times 10^4\, y + 286$$

at y=0

$$V_1 = -2.86 \times 10^4 \times 0 + 286 \quad \rightarrow \quad V_1 = 286\ volt$$

at y=2 cm

$$V_2 = -2.86 \times 10^4 \times 2 \times 10^{-2} + 286 \quad \rightarrow \quad V_2 = -286\ volt$$

## Cylindrical solution in one variable:

$$V = V(r,\varphi,z)$$

$$V = V(r) \quad or \quad V = V(\varphi) \quad or \quad V = V(z)$$

1- $For \quad V = V(r)$only

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

$$\int \frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \quad \rightarrow \quad r\frac{dV}{dr} = A$$

$$\int dV = A\int \frac{dr}{r} \quad \rightarrow \quad V = A\ln r + B$$

As a Example:

For boundary condition $\begin{matrix} V = V_0 & at & r = a \\ V = 0 & at & r = b \end{matrix}$

Equipotential surface in concentric cylindrical.

$$\left.\begin{matrix} V_0 = A\ln a + B \\ 0 = A\ln b + B \end{matrix}\right\} \; from\ them\ we\ get \quad A = \frac{V_0}{\ln a - \ln b} = \frac{V_0}{\ln\frac{a}{b}}$$

$$B = -A\ln b = \frac{-\ln b}{\ln\frac{a}{b}}\mathrm{V}_0$$

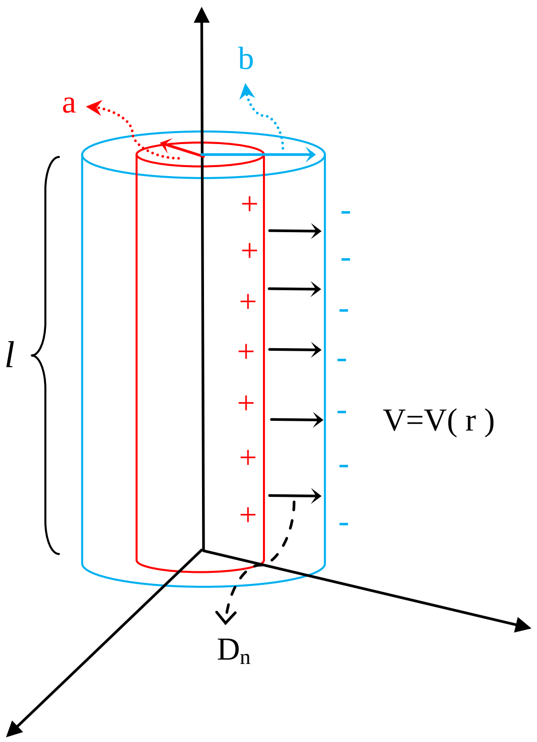
$$\therefore \quad \mathrm{V} = \frac{V_0}{\ln\frac{a}{b}}\ln \mathrm{r} - \frac{\ln b}{\ln\frac{a}{b}}\mathrm{V}_0$$

$$\mathrm{V} = \mathrm{V}_0\frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}$$

$$E = -\nabla V = -\frac{dV}{dr}\mathrm{a_r}$$

$$= -[\frac{\mathrm{V}_0}{\ln\frac{a}{b}}\frac{\mathrm{d}}{\mathrm{dr}}(\ln \mathrm{r} - \ln \mathrm{b})] = \frac{\mathrm{V}_0}{\mathrm{r}\ln\frac{b}{a}}$$

$$\mathrm{D} = \varepsilon_0 \mathrm{E} = \frac{\varepsilon_0 \mathrm{V}_0}{\mathrm{r}\ln\frac{b}{a}}$$



At r =a

$$D_n = \frac{\varepsilon_0 V_0}{r \ln\frac{b}{a}} D_n = flux = \frac{\varphi}{A} = \rho_s = \frac{\varphi}{2\pi a l}$$

$$\frac{\varphi}{2\pi a l} = \frac{\varepsilon_0 V_0}{a \ln\frac{b}{a}} \qquad \therefore\ \varphi = \frac{\varepsilon_0 V_0}{a \ln\frac{b}{a}} \times 2\pi a l = \frac{\varepsilon_0 V_0}{\ln\frac{b}{a}} 2\pi l$$

$$C = \frac{\varphi}{V_0} = \frac{\varepsilon_0}{\ln\frac{b}{a}} 2\pi l$$

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Q/ Find V,E& D for the region between two concentric cylinders where V=0 at r = 1mm, & V=150volt at r =20mm.

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2- For $V = V(\varphi)$ only, and the boundary condition might be:

$$V = 0 \quad at \quad \varphi = 0$$
$$V = V_0 \quad at \quad \varphi = \alpha$$

$$\nabla^2 V = \frac{1}{r^2}\frac{d^2V}{d\varphi^2} = 0 \qquad \int \frac{d}{d\varphi}\left(\frac{dV}{d\varphi}\right) = \int 0 \quad , \quad \frac{dV}{d\varphi} = A$$

$$dV = A d\varphi \qquad V = A\varphi + B$$

$$at\ V = 0\ ,\ \varphi = 0\ , \qquad and\ \ V = V_0\ ,\ \ \varphi = \alpha$$

$$\left.\begin{array}{r} 0 = 0 + B \\ V_0 = A \cdot \alpha + B \end{array}\right] \begin{array}{l} B = 0 \\ A = \dfrac{V_0}{\alpha} \end{array}$$

$$\therefore\ V = \frac{V_0}{\alpha}\varphi + 0 \qquad \nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \varphi} a_\varphi + \frac{\partial V}{\partial z} a_z$$

$$E = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \varphi} a_\varphi = -\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{V_0}{\alpha}\varphi\right) a_\varphi = -\frac{1}{r}\frac{V_0}{\alpha} a_\varphi$$

$$E = \frac{V_0}{r\alpha}(-a_\varphi)$$

$$D = \varepsilon_0 E = \frac{\varepsilon_0}{r}\frac{V_0}{\alpha}(-a_\varphi)$$

Example: In cylindrical coordinate there are two plane charged ($\varphi = constant$), then planes are insulated along z-axis. Find the expression for D between the planes where: $V = 100v \quad at \quad \varphi = \alpha \quad and \quad V = 0 \quad at \quad \varphi = 0$

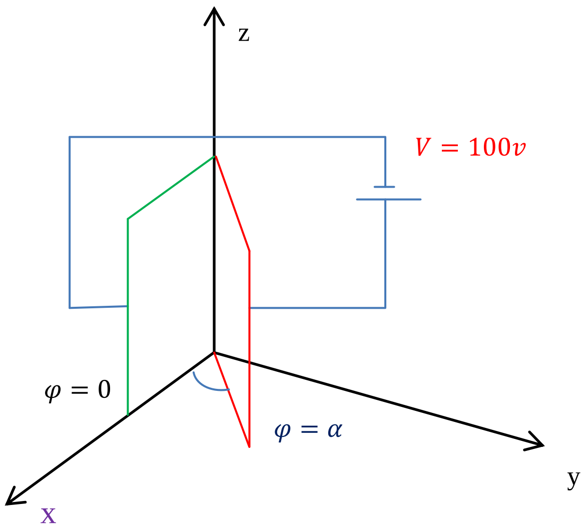
Sol/ $V = A\varphi + B$

$$\left.\begin{aligned} 100 &= A\alpha + B \\ 0 &= 0 + B \end{aligned}\right\} \therefore \quad \begin{aligned} B &= 0 \\ A &= \frac{100}{\alpha} \end{aligned}$$

$$V = \frac{100\varphi}{\alpha}$$

$$E = -\nabla V = -\frac{1}{r}\frac{dV}{d\varphi}a_\varphi = \frac{-100}{r\alpha}a_\varphi$$

$$D = \frac{-100\varepsilon_0}{r\alpha}a_\varphi$$




*The equipotential surfaces aresemicircul radial planes*, $V =$

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3- For V=V(z) only

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \quad \rightarrow \quad V = Az + B$$

## Spherical solution in one variable

$$V = V(r,\theta,\varphi)$$

1- For V=V(r) only

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

$$r^2\frac{dV}{dr} = A \qquad \rightarrow \qquad dV = A\frac{dr}{r^2}$$

$$V = -\frac{A}{r} + B$$

Example: In spherical coordinates $V = 0\ at\ r = 0.1m\ and\ V = 100v\ at\ r = 2m$ . Find E and D.

Sol./

$$V = -\frac{A}{r} + B$$

$$\left.\begin{aligned} 0 &= -\frac{A}{0.1} + B \\ 100 &= -\frac{A}{2} + B \end{aligned}\right\} \; 100 = -\frac{A}{2} + \frac{A}{0.1} = A\left(\frac{20-1}{2}\right) \;\rightarrow\; A = 9.5$$

$$B = \frac{A}{0.1} = \frac{9.5}{0.1} = 95$$

$$\therefore\ V = \frac{-9.5}{r} + 95$$

2- For V=V(θ) only

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

$$\int \frac{d}{d\theta}\left(sin\,\theta\frac{dV}{d\theta}\right) = \int 0$$

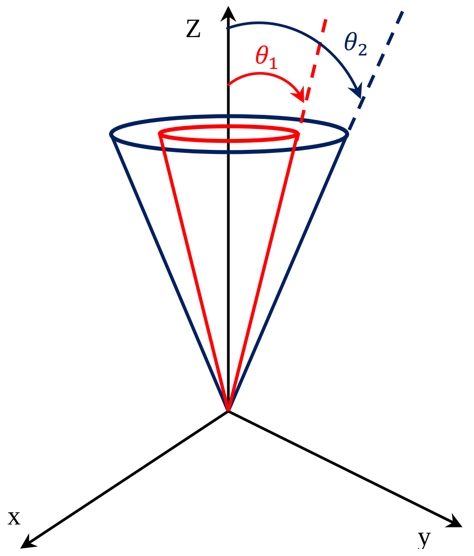
$$sin\,\theta\frac{dV}{d\theta} = A \qquad \rightarrow \qquad \int dV = A\int\frac{d\theta}{\sin\theta}$$

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B$$

Example: For the region between the two co-axial cones, the cones vertex are insulates at [r=0]. Find the potential function for $V = V_1 \ at \ \theta = \theta_1 \ and \ V = 0 \ at \ \theta = \theta_2$ .

Sol./

$$V = A \ln\left(\tan\frac{\theta}{2}\right) + B$$




*The equipotential surfaces in spherical coordinates at $V = V(\theta)$ are coaxial cones*

$$\left.\begin{aligned} V_1 &= A\, ln\left(tan\frac{\theta_1}{2}\right) + B \\ 0 &= A\, ln\left(tan\frac{\theta_2}{2}\right) + B \end{aligned}\right\}$$

$$V_1 = A\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)$$

$$A = \frac{V_1}{\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)}$$

$$B = \frac{-V_1\, ln\left(tan\frac{\theta_2}{2}\right)}{\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)}$$

Then from above equations we get:

$$V = \frac{V_1\, ln\left(tan\frac{\theta}{2}\right)}{\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)} - \frac{V_1\, ln\left(tan\frac{\theta_2}{2}\right)}{\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)}$$

$$V = V_1 \frac{ln\left(tan\frac{\theta}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)}{\left(ln\left(tan\frac{\theta_1}{2}\right) - ln\left(tan\frac{\theta_2}{2}\right)\right)}$$

3- For $V = V(\varphi)$ only

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d^2V}{d\varphi^2} = 0$$

$$\frac{d^2V}{d\varphi^2} = 0 \quad \rightarrow \quad \int \frac{d}{d\varphi}\left(\frac{dV}{d\varphi}\right) = \int 0 \quad \rightarrow \quad \frac{dV}{d\varphi} = A$$

$$\int dV = \int A d\varphi$$

$$V = A\varphi + B$$

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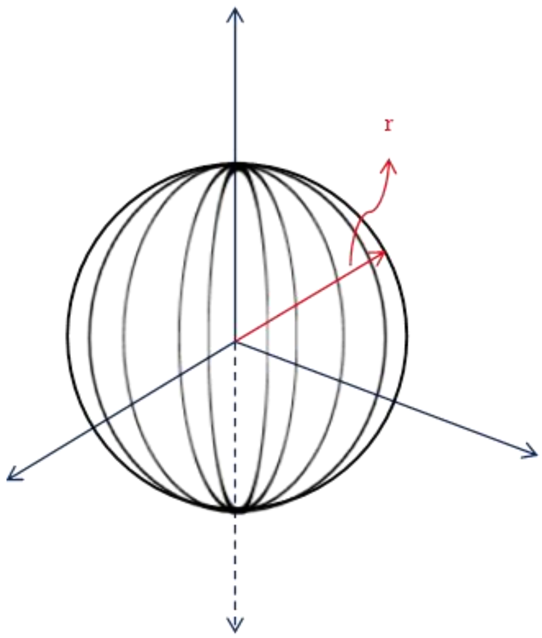
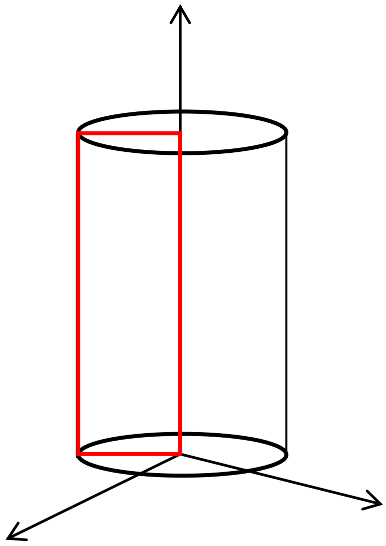



*The equipotential surfaces aresemicircul radial planes*,

$V = V(\varphi)$



*In clyndircal coordinate,*

$V = V(\varphi)$

*The equipotential surfaces are Rectangular radial plane*

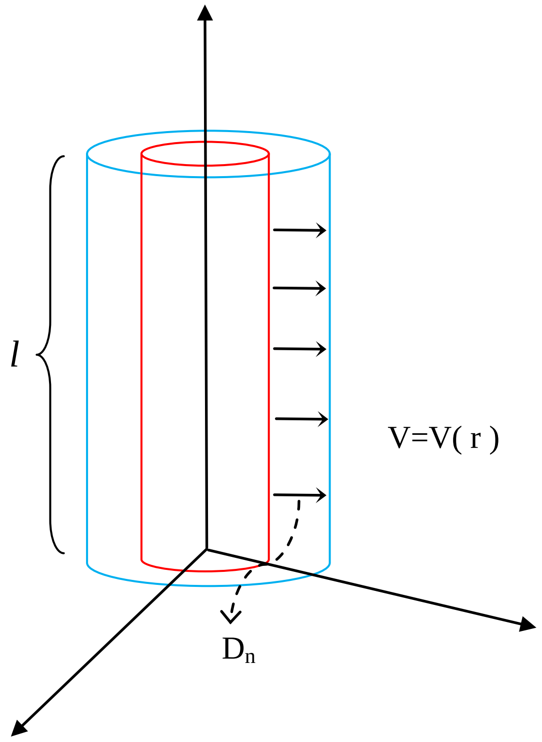
Solution of Poisson's eq.:

$$\nabla^2 V = \frac{-\rho}{\varepsilon}$$

Example: The region between two concentric cylindrical contains a uniform charge density ($\rho$).

Sol./ For concentriccylindrical equipotential surfaces.$V = V(r)$

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = \frac{-\rho}{\varepsilon}$$




$$\int d\left(r\frac{dV}{dr}\right) = \int \frac{-\rho}{\varepsilon} r dr$$

$$r\frac{dV}{dr} = \frac{-\rho}{2\varepsilon} r^2 + A$$

$$\therefore \quad \frac{dV}{dr} = \frac{-\rho r}{2\varepsilon} + \frac{A}{r}$$

$$\int dV = \int \left(\frac{-\rho r}{2\varepsilon} + \frac{A}{r}\right) dr$$

$$V = -\frac{\rho r^2}{4\varepsilon} + A\ln r + B$$

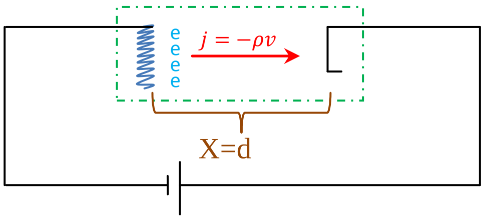
$$E = -\nabla V = -\frac{dV}{dr} a_r$$

$$\therefore \quad E = \frac{\rho r}{2\varepsilon} - \frac{A}{r}$$

$$D = \varepsilon E = \varepsilon\left(\frac{\rho r}{2\varepsilon} - \frac{A}{r}\right) = \frac{\rho r}{2} - \frac{\varepsilon A}{r}$$

Example: A parallel plate Diode, operating condition x-axis, during the prose's space charge build up in the region between the electrons. Find the un expression for the potential function between the plates if the current density $j = -\rho v$ where: $\begin{matrix} V = 0 & at\ x = 0 \\ V = V_0 & at\ x = d \end{matrix}$

Sol./ charge density $\rho = -\frac{j}{v}$




$$eV = \frac{1}{2}mv^2 \quad \rightarrow \quad v = \sqrt{\frac{2eV}{m}}$$

$$\rho = \frac{-j}{\sqrt{\frac{2eV}{m}}}$$

$$\nabla^2 V = \frac{-\rho}{\varepsilon} \quad \rightarrow \quad \frac{d^2V}{dx^2} = \frac{1}{\varepsilon}\frac{+j}{\sqrt{\frac{2eV}{m}}} = \frac{j}{\varepsilon}\left(\frac{2e}{m}\right)^{\frac{-1}{2}} V^{\frac{-1}{2}}$$

$$\frac{d^2V}{dx^2} = k\, V^{\frac{-1}{2}} \qquad where\ \ k = \frac{j}{\varepsilon}\left(\frac{2e}{m}\right)^{\frac{-1}{2}}$$

$$2\left(\frac{dV}{dx}\right)\cdot\frac{d^2V}{dx^2} = k\, V^{\frac{-1}{2}}\ \cdot\ 2\left(\frac{dV}{dx}\right)$$

$$\int \frac{d}{dx}\left(\frac{dV}{dx}\right)^2 dx = \int 2k\ \left(\frac{dV}{dx}\right) V^{\frac{-1}{2}}\, dx$$

$$\left(\frac{dV}{dx}\right)^2 = 4kV^{\frac{+1}{2}} \quad \rightarrow \quad \frac{dV}{dx} = 2\sqrt{kV^{\frac{+1}{2}}}$$

$$\int V^{-\frac{1}{4}}\, dV = 2\sqrt{k}\ dx$$

$$\frac{4}{3}V^{\frac{3}{4}} = 2\sqrt{k}\, x + A$$

First boundary condition: $V = 0 \quad at\ \ x = 0$

$$\frac{4}{3}(0)^{\frac{3}{4}} = 2\sqrt{k}\ 0 + A \quad \rightarrow \quad A = 0$$

$$\therefore \quad \frac{4}{3}V^{\frac{3}{4}} = 2\sqrt{k}\, x \quad \dots\dots\dots \ 1$$

Second boundary condition: $V = V_0 \quad at\ \ x = d$

$$\frac{4}{3}(V_0)^{\frac{3}{4}} = 2\sqrt{k}\ d \quad \dots\dots\dots \ 2$$

From eq. 1 &2

$$\frac{\frac{4}{3}V^{\frac{3}{4}}}{\frac{4}{3}(V_0)^{\frac{3}{4}}} = \frac{2\sqrt{k}\, x}{2\sqrt{k}\, d} \quad \rightarrow \quad \frac{V^{\frac{3}{4}}}{(V_0)^{\frac{3}{4}}} = \frac{x}{d}$$

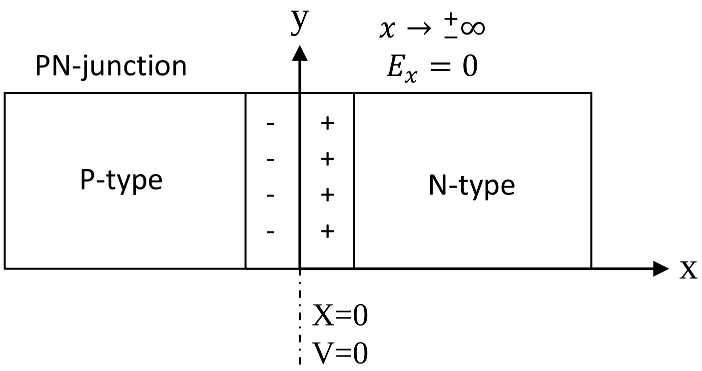
$$\therefore \quad V = V_0\left(\frac{x}{d}\right)^{\frac{4}{3}}$$

Q1/

A p-n junction between two halves of a semiconductor bar extended in the x-direction assume that the region for $x < 0$ is p-type and that the region for $x > 0$ is n-type. A charge distribution of this form may be approximated by the expression

$\left(\rho = 2\rho_0 sech\frac{x}{a}\ tanh\frac{x}{a}\right)$, with

$\rho_{max} = \rho_0 \quad where\ \rho_{max} \equiv maximum\ charge\ density.$ Find V.

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$$\nabla^2 V = -\frac{\rho_0}{\varepsilon}$$

$$\nabla^2 V = \frac{2\rho_0 sech\frac{x}{a}\ tanh\frac{x}{a}}{\varepsilon}$$

$$\frac{dV}{dx} = \frac{2a\rho_0\ sech\frac{x}{a}}{\varepsilon} + C_1$$

$$E_x = \frac{2a\rho_0\ sech\frac{x}{a}}{\varepsilon} + C_1$$

$$at\ \ x \to \pm\infty \qquad E_x = 0\ , \qquad \therefore C_1 = 0$$

$$E_x = \frac{dV}{dx} = \frac{2a\rho_0\ sech\frac{x}{a}}{\varepsilon}$$

Integrating again:

$$V = \frac{4a^2\rho_0}{\varepsilon}\tan^{-1} e^{\frac{x}{a}} + C_2$$

Let us arbitrarily select our zero reference of potential at the center of the junction, $= 0$ ;

$$0 = \frac{4a^2\rho_0}{\varepsilon}\frac{\pi}{4} + C_2 \qquad\qquad \therefore\ C_2 = -\frac{4a^2\rho_0}{\varepsilon}\frac{\pi}{4}$$

$$V = \frac{4a^2\rho_0}{\varepsilon}\left(\tan^{-1} e^{\frac{x}{a}} - \frac{\pi}{4}\right)$$

Q2/ Find E in the region between the two cones, where $\begin{Bmatrix} V = V_1 \;\; at \;\; \theta_1 = 20^0 \\ V = 0 \;\; at \;\; \theta_1 = 160^0 \end{Bmatrix}$.

Q3/ coaxial conducting cylindrical are located at r=4m and 15cm, the value of E is $\left(E = 20a_r \frac{kV}{m}\right) \; at \; r = 6cm$, and the potential of the more positive conductors is 200V.

Find a- potential difference between two conductors.

b- the capacitance of the system if $\varepsilon_r = 2.7$ for the medium between the two cylinders