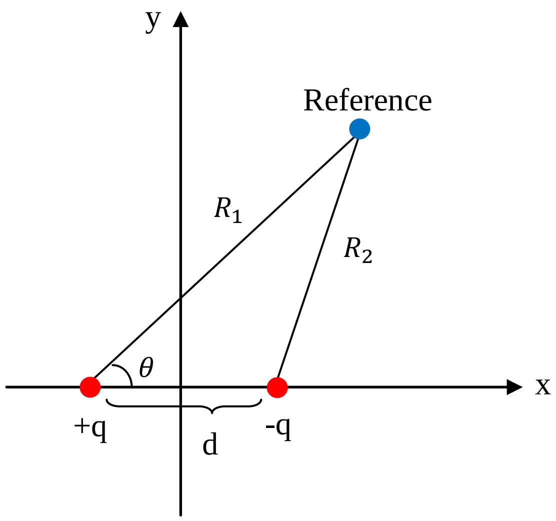
CHAPTER FOURE

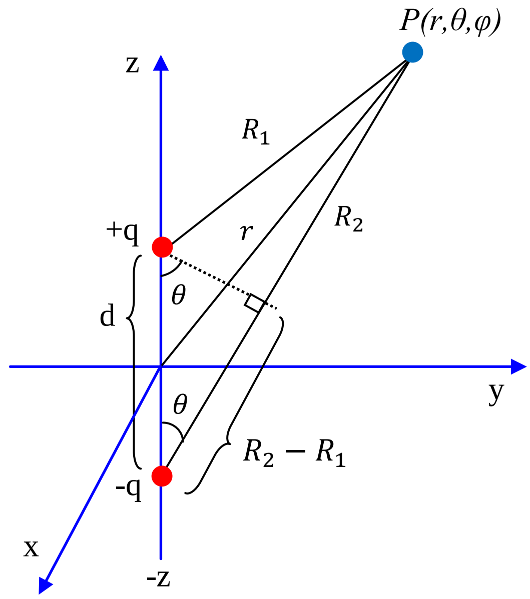
Dipole, Conductors, Dielectrics

The Dipole (Electric Dipole)

Two point charge of equal magnitude and opposite side, separated by a small distance



Potential at point (P)



$$V = V_1 + V_2 = \frac{+q}{4\pi\varepsilon R_1} + \frac{-q}{4\pi\varepsilon R_2}$$

$$V = \frac{q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{q}{4\pi\varepsilon}\left(\frac{R_2 - R_1}{R_1 R_2}\right)$$

$$For \quad r >> d$$

$$R_1 = R_2 = r$$

$$R_2 - R_1 = d\cos\theta$$

$$V = \frac{q}{4\pi\varepsilon}\left(\frac{d\cos\theta}{r^2}\right)$$

$$E = -\nabla V = -\left(\underbrace{\frac{dV}{dr}a_r}_{E_r} + \underbrace{\frac{1}{r}\frac{dV}{d\theta}a_\theta}_{E_\theta} + \underbrace{\frac{1}{r\sin\theta}\frac{dV}{d\varphi}a_\varphi}_{E_\varphi = 0}\right)$$

$$E_r = -\frac{dV}{dr} = -\frac{d}{dr}\left[\frac{q}{4\pi\varepsilon}\left(\frac{d\cos\theta}{r^2}\right)\right]$$

$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon r^3}\, a_r$$

$$E_\theta = -\frac{dV}{d\theta} = -\frac{d}{d\theta}\left[\frac{q}{4\pi\varepsilon}\left(\frac{d\cos\theta}{r^2}\right)\right]$$

$$E_\theta = \frac{p\sin\theta}{4\pi\varepsilon r^3}\ a_\theta$$

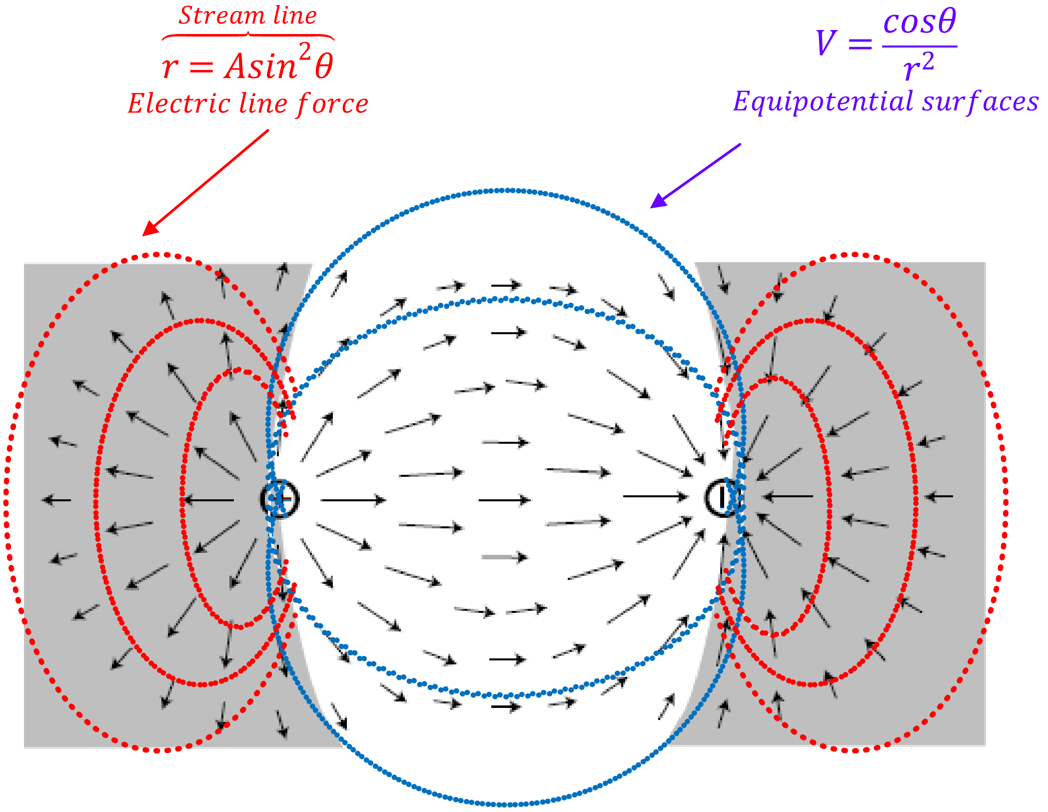
$$E = E_r + E_\theta = \frac{2p\cos\theta}{4\pi\varepsilon r^3}\ a_r + \frac{p\sin\theta}{4\pi\varepsilon r^3}\ a_\theta$$

$$= \frac{p}{4\pi\varepsilon r^3}\ (2\cos\theta a_r + \sin\theta\ a_\theta)$$

$$V = \frac{qd\cos\theta}{4\pi\varepsilon r^2} = V_0\frac{\cos\theta}{r^2}$$

V_0 = Is the maximum potential

Let V_0 =1 volt $\quad V = \frac{\cos\theta}{r^2}$



Conductors, Dielectrics and Capacitance

The fundamental electromagnetic principles on which resistance and capacitance depends, are the subject of this chapter:

1- Current and current density.
2- Continuity equation of current.
3- Metallic conductors and ohms low.
4- Conductor boundary conditions.
5- Polarization of Dielectric material.
6- The Capacitance.

1- Current and current density:

The motion of the electric charge consisted a current. The unit of current is "Ampere".

Ampere: One coulomb per second crossing given reference point or area.

$$I = \frac{d\varphi}{dt} \quad scalar$$

We will defined the concept of current density

$$\overbrace{J = \frac{\Delta I}{\Delta s} \quad current\ density}^{motion\ charge}$$

$$I = \int J \cdot ds$$

$$\overbrace{\varphi = \psi}^{Static\ charge}$$

$$D = \frac{\Delta \psi}{\Delta s}$$

$$\psi = D \cdot ds$$

Current density (J) can be related to the velocity and charge density

$$J_c = \frac{\Delta I}{\Delta s} = \frac{\Delta \varphi}{\Delta s \Delta t} = \frac{\Delta \varphi}{\Delta s \Delta L} \cdot \frac{\Delta L}{\Delta t} = \frac{\Delta \varphi}{\Delta V} \cdot \frac{\Delta L}{\Delta t}$$

$$J_c = \rho v \qquad For\ free\ charge$$

The relation is true if we treat the charge as free gas of charges.

$$J_c - \ Conduction\ current = \sigma E$$

$$J - \ Convection\ current = \rho v$$

$$J_d - \ Displacement\ current = \frac{\partial D}{\partial t}$$

$$But \qquad v = \mu E \qquad escape\ velosity$$

$$\mu \equiv mobility\ of\ charge\ carrers$$

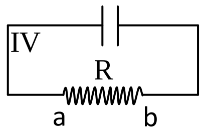
$$\therefore \quad J = \rho\mu E \qquad\qquad \rho\mu = \sigma = conductivity$$

$$J_c = \sigma E$$

$$\sigma = \frac{J_c}{E} = \frac{\left(\frac{I}{S}\right)}{\left(\frac{V}{L}\right)} = \frac{I}{S}\cdot\frac{L}{V} = \frac{L}{S}\cdot\frac{I}{V} = \frac{L}{S\cdot R} \quad ---- \quad \sigma = \frac{L}{S\cdot R}$$

$$J_c = \frac{I}{S} = \sigma E \quad \rightarrow \quad \frac{I}{S} = \frac{L}{S\cdot R}\cdot\frac{V}{L} \quad \rightarrow \quad R = \frac{V}{I}$$

$$R = \frac{V_{ba}}{I}$$



$$R = \frac{-\int_a^b E\cdot dl}{\int J\cdot ds}$$

$$R = \frac{-\int_a^b E\cdot dl}{\sigma\int E\cdot ds} \qquad Integral\ form\ of\ Ohm's\ law$$

Example:Find the current in the circular wire if $[J = 15(1 - e^{-1000r})a_z]\ at\ r = 2mm.$

$$I = \int J.ds = \int J.ds\ \cos\theta \qquad , \quad \theta = 0$$

$$I = \int 15\,(1 - e^{-1000r})a_z.rdrd\varphi a_z$$

$$I = 15\int_0^{2\pi} d\varphi \int_0^r (1 - e^{-1000r})rdr$$

$$I = 1.33\times 10^{-4} Amp$$

Example:Find Resistance between the inner and outer surface for a block of Ag metal in cylindrical coordinates- where$(0 \leq z \leq 5cm\,, 0.2 \leq r \leq 3m)$.

If you know $\sigma = 6.17 \times 10^{7}\Omega^{-1}m^{-1}$, $J = \frac{k}{r}a_r$

$$R = \frac{-\int_a^b Edl}{\sigma \int Eds} \qquad J = \frac{k}{r}a_r,\ J = \sigma E \qquad \therefore E = \frac{J}{\sigma} = \frac{k}{\sigma r}a_r$$

$$R = \frac{-\int_{0.2}^{3} \frac{k}{\sigma r} a_r dr a_r}{\sigma \int_0^5 \frac{k}{\sigma r} a_r\, r d\varphi dz a_r} = \frac{-\int_{0.2}^{3} \frac{1}{\sigma r} dr}{r \int_0^{2\pi} d\varphi \int_0^5 dz} =$$

Continuity of current

The concepts of the current follow is followed by:

1- Conservative of charge.
2- Continuity equation.

1- The principle of Conservative of charge states simply that charges can be neither created nor destroyed, +ve and –ve charge may be simultaneously created, obtained by separation , destroyed or lost by recombination.

2- Continuity equation of current:

$$I = \int J.ds \qquad at\ any\ refres\ poiter\ or\ area$$

For closed surface

$$I = \oint J.ds \qquad flow\ of\ positve\ charge$$

Out word flow of +ve charge from a closed surface.

This must be balances by decrease of +ve charge or increased of –ve charge.

If the charge inside the closed surface is denoted by (Q_i) then the decrease rate of +ve charge is $\left(-\frac{dQ_i}{dt}\right)$

$$I = \oint J.ds = -\frac{dQ_i}{dt}$$

$$using\ divergen\ theorm\ \oint J.ds = \int (\nabla.J)dv$$

$$\int (\nabla.J)dv = -\frac{dQ_i}{dt}$$

$$\int (\nabla.J)dv = -\frac{d}{dt}\int \rho dv = -\int \frac{d\rho}{dt}dv$$

$$\therefore \quad \nabla . J = -\frac{d\rho}{dt} \qquad Equation\ of\ continuity$$

$$\nabla . J \equiv out\ word\ flow\ of\ +ve\ charge$$

$$-\frac{d\rho}{dt} \equiv rate\ of\ decrease\ of\ +ve\ charge$$

$$J = \sigma E \qquad\qquad \nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot J = \sigma(\nabla \cdot E)$$

$$-\frac{d\rho}{dt} = \sigma \frac{\rho}{\varepsilon_0}$$

$$\frac{d\rho}{dt} + \frac{\sigma\rho}{\varepsilon_0} = 0$$

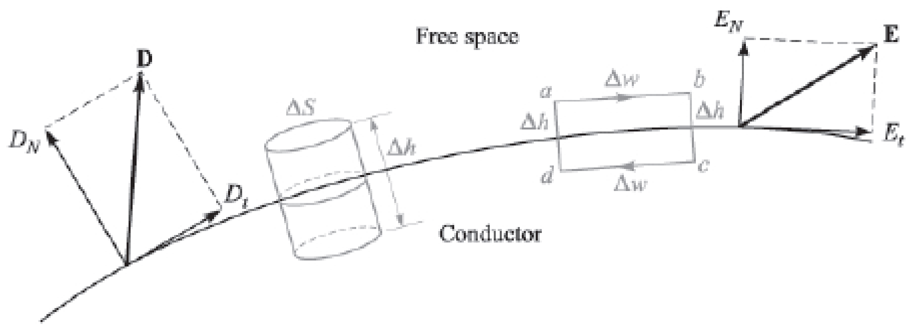
$$\int \frac{d\rho}{\rho} = \int -\frac{\sigma}{\varepsilon_0} dt$$

$$\rho = \rho_0 e^{-\frac{t}{\tau}} \qquad\qquad \tau = \frac{\varepsilon_0}{\sigma} \equiv relaxation\ time$$

Conductor property and boundary condition

1- $\rho = o\ and\ \rho_s$ reside on the exterior surface.
2- $E = o$ within the conductors.

We study the fields external to the conductor and we wish to relate the external field to the charge on the surface.



We prove $E_t = D_t = 0$

$$\oint E.dl = 0 \qquad static\ (conservative\ field)$$

$$\int_a^b E_t.\Delta w + \int_b^c (-E_n).\frac{\Delta h}{2} + \int_c^d (-E_t).\Delta w + \int_d^a E_n.\frac{\Delta h}{2} = 0$$

$$\int_a^b E_t.\Delta w = 0 \qquad \rightarrow \quad E_t.\Delta w \qquad \therefore\ E_t = 0$$

$$D_t = \varepsilon_0 E_t = 0$$

$$\therefore \quad E_t = D_t = 0$$

2nd we will find the relation between $D_n\ and\ \rho_s$ for the cylindrical Gausses surface between conduction – free space boundary

$$\int_{top} D_n.ds + \int_{side} D_t.ds + \int_{bottom} D_n.ds = Q_{enc}$$

$$\int D_n.ds = \int \rho_s.ds$$

$$D_n = \rho_s$$

The desired boundary conditions for conductor – free space boundary, in electrostatic field are:

1- $E_t = D_t = 0$
2- $D_n = \rho_s = \varepsilon_0 E_n$

Example:Tow concentric cylinder ($r_a = 0.01m$, $r_b = 0.08m$)where$\left(\rho_{sa} = 40\,\frac{pc}{m^2}\right)$, $D\&E$ exst between the cylinders, and equal to zero elsewhere. Find (ρ_{sb}).

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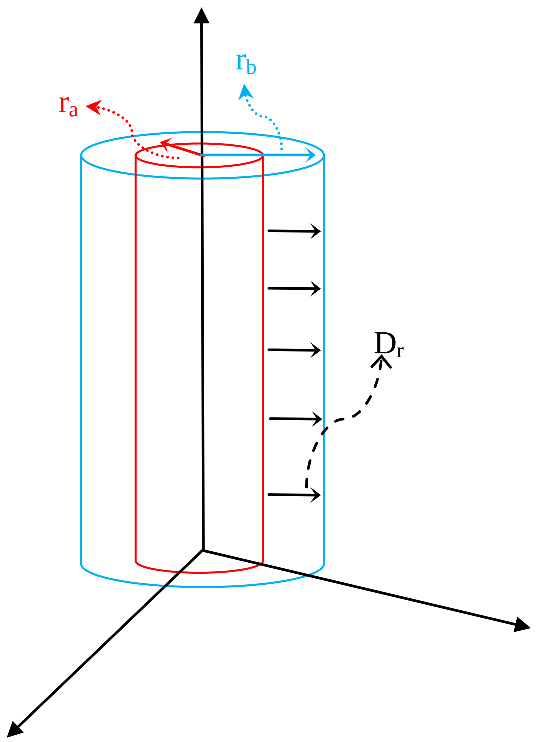


Between cylinders $\rho = 0$

$$\nabla . D = \rho = 0$$

$$\nabla . D = \frac{1}{r}\frac{d}{dr}(rD_r) = 0$$

$$\int d\,(rD_r) = 0$$

$$(rD_r) = A$$

$$\therefore\ D_a = \frac{A}{r_a} = \rho_{sa} \qquad \rightarrow\ A = 0.4\,\frac{pc}{m}$$

$$\rho_{sb} = D_b = \frac{A}{r_b} = \frac{0.4\,\frac{pc}{m}}{0.08m} = 5\,\frac{pc}{m^2}$$

Dielectric Material:

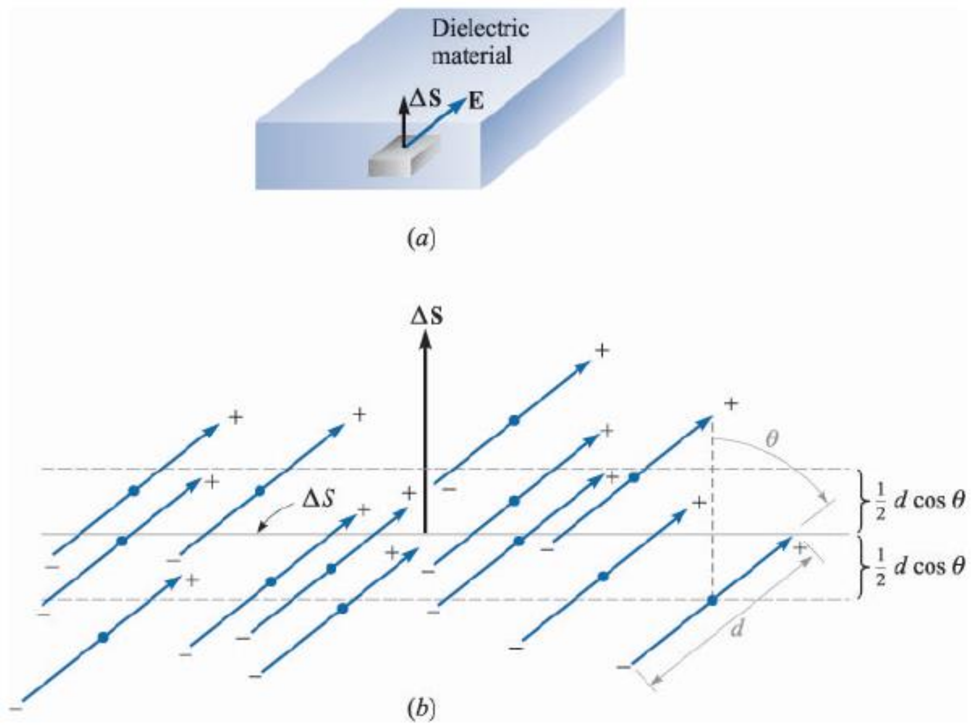
Dielectric material are composed of –ve and +ve charges, whose center don't quite coincide. This are not free charges and they con not contribute to the condition processes, rather thy are bound in place by atomic and nuclear forces.

The action of external field can only shift positions slightly. They are called bound charges.

Dielectric Material	
Polar	Non-Polar
-Have permanent displacement between +veand –ve charges. -So they have permanent dipoles: $$p = Qd$$ -This dipoles are randomly distribute so: $$p_{total} = 0$$ -The action of external field is to align thisdipoles. -The additional of strong field produce additional displacement.	-Doesn't have permanent dipoles. -But the external field will produce a displacement between +ve and –ve charges.(shift the +ve and –ve charges in opposite direction) -This shift will produce a dipole moment: $$p = qd$$ $$p_{total} = \sum_{i=1}^{n\Delta v} p_i$$ $$\Delta v = a\ small\ element\ of\ volume$$ $$n = no.\ of\ dipoles\ per\ unit\ volume$$ -Now we can define the polarization (P): the dipole moment per unit volume. $$P = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} p_i$$ and we show that the bound volume charge density acts like $\rho \equiv$ free volume charge density

For a dielectric material:

The net charge crosses Δs in normal director:



(a) An incremental surface element ΔS is shown in the interior of a dielectric in which an electric field **E** is present. (b) The nonpolar molecules form dipole moments **p** and a polarization **P**. There is a net transfer of bound charge across Δ**S**.

$\Delta Q_b = nQd.\Delta s$

$\Delta Q_b = P.\Delta s$

For close surface: $Q_b = -\oint P.\Delta s$

By divergent theorem we get:

$$\oint P.\Delta s = \int (\nabla \cdot P) dv$$

$$Q_b = \int \rho_s dv \qquad \therefore \nabla \cdot P = -\rho_b$$

The Gausses law in terms of $(\varepsilon_0 E\ ) and\ (Q_T)$

$$Q_T = \oint \varepsilon_0 E_0 ds \qquad Q_T = Q + Q_b \quad ; Q_b = -\oint \rho_v dv$$

$$Q + Q_b = \int \varepsilon_0 E_0 ds$$

$$Q = \oint (\varepsilon_0 E + P) ds \qquad [Free\ charge]$$

$$Q = \oint D.ds = \varepsilon_0 E + P \qquad [polarization]$$

P = This as the added term to D, where the polarized material is present.

$$Q_b = -\oint \rho_b dv \qquad \nabla \cdot P = -\rho_b \ \ldots\ldots\ldots\ldots..1$$

$$Q = -\oint \rho dv \qquad \nabla \cdot D = \rho \ \ldots\ldots\ldots\ldots\ldots.. \ 2$$

$$Q_T = -\oint \rho_T dv \quad \nabla \cdot (\varepsilon_0 E) = \rho_T \ \ldots\ldots\ldots\ldots.. \ 3$$

- Homogenous: (ρ) is not change from point to another point.
- Isotropic: $(\rho, \sigma, \varepsilon, \mu)$ not depend on the direction of the external field.
- Isotropic materials E and P are parallel.
- Most of engineering materials (dielectric) are isotropic at intermediate.
- Ferroelectric (E,P) has a nonlinear relation and shows hysteresis loop effects.

For Linear isotopic materials:

$$P = X_e \varepsilon_0 E \qquad X_e = Electric\ suceptibility$$

$$D = \varepsilon_0 E + P$$

$$D = \varepsilon_0 E + X_e \varepsilon_0 E$$

$$D = (1 + X_e)\varepsilon_0 E \qquad \varepsilon_r = 1 + X_e$$

$$D = \varepsilon_r \varepsilon_0 E$$

Example: Find the polarization within a material which has;

$a)\ D = \frac{1.5\mu c}{m^2}, E = 15\frac{kv}{m}$

$b)\ D = \frac{2.8\mu c}{m^2}, X_e = 1.7$

$c) n = 10^{20}\frac{molecular}{m^3}, \quad p = 1.5 \times 10^{-26}\ cm\ and\ E = 10^5\frac{v}{m}$

$D)\ E = 50\frac{kv}{m}, \varepsilon_r = 4.4$

Sol/

a- $D = \varepsilon_0 E + P$

$$P = D - \varepsilon_0 E = 1.5 \times 10^{-6} - 8.8442 \times 10^{-12} \times 15000 = 1.367 \times 10^{-6}\frac{c}{m^2}$$

b- $P = X_e \varepsilon_0 E$

$$D = \varepsilon_r \varepsilon_0 E$$

$$P = \frac{X_e \varepsilon_0 D}{\varepsilon_0 (1 + X_e)} \quad D = (1 + X_e)\varepsilon_0 E$$

$$E = \frac{D}{(1 + X_e)\varepsilon_0}$$

$$\therefore\ P = \frac{X_e D}{(1 + X_e)} = 1.763 \times 10^{-6}\frac{c}{m^2}$$

c- $P = nqd = np = 10^{20} \times 1.5 \times 10^{-26} = 1.5 \times 10^{-6}\frac{c}{m^2}$

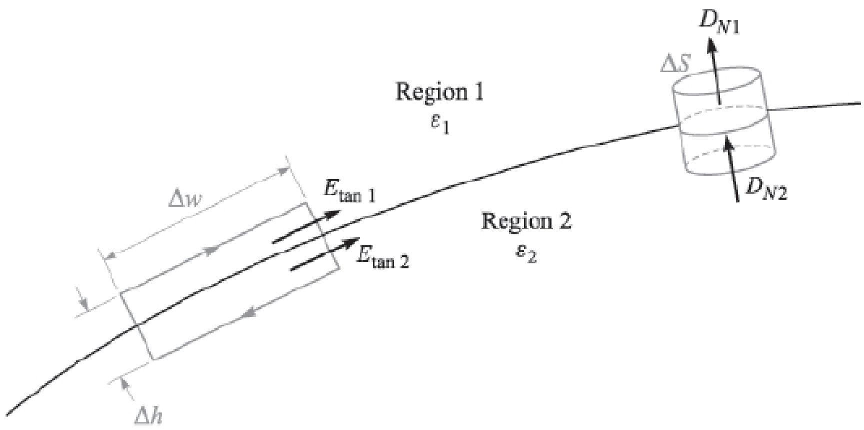
d- $P = X_e \varepsilon_0 E \qquad P = D - \varepsilon_0 E = \varepsilon_r \varepsilon_0 E - \varepsilon_0 E = \varepsilon_0 E(\varepsilon_r - 1)$

$$P = (\varepsilon_r - 1)\varepsilon_0 E = 1.505 \times 10^{-6}\frac{c}{m^2}$$

Boundary condition for perfect dielectrics:

1- Interface between two dielectric having permittivities of $\varepsilon_1 \& \varepsilon_2$ and occupying the regions 1 and 2, as shown in figure below. We first examine the tangential components by using:

$$\oint E \cdot dl = 0$$



$$\int E_{t1} \cdot \Delta w - \int E_{n1} \cdot \frac{1}{2}\Delta h - \int E_{n2} \cdot \frac{1}{2}\Delta h - \int E_{t2} \cdot \Delta w + \int E_{n2} \cdot \frac{1}{2}\Delta h + \int E_{n1} \cdot \frac{1}{2}\Delta h = 0$$

$for\ \ \Delta h \to 0$

$$\int E_{t1} \cdot \Delta w = \int E_{t2} \cdot \Delta w$$

$$E_{t1} = E_{t2} \qquad The\ tangential\ component\ of\ E\ are\ cotinous. V_{ab} = V_{dc}$$

$$E_{t1} = \frac{D_{t1}}{\varepsilon_1}$$

$$E_{t2} = \frac{D_{t2}}{\varepsilon_2} = \frac{D_{t1}}{\varepsilon_1}$$

$$\frac{D_{t1}}{D_{t2}} = \frac{\varepsilon_1}{\varepsilon_2} \qquad The\ tangential\ component\ of\ D\ are\ not\ cotinous$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "PILL BOX" shown at the right in figure above. The sides are again very short, and the flux leaving the top and the bottom surfaces is the difference:

$$Q_{enc} = D_{n1} \cdot \Delta s - D_{n2} \cdot \Delta s = \rho_s \Delta s$$

$$D_{n1} - D_{n2} = \rho_s$$

What is this surface charge density?

1- It cannot be a bound surface charge density, because we take to account the polarization
2- It cannot be a free surface charge density, because we take to account perfect dielectric.

Except for this special case, then, we may assume ρ_s is zero on the interface and

$$D_{n1} = D_{n2} \qquad \textit{The normal companent of D is continuous.}$$

Then:

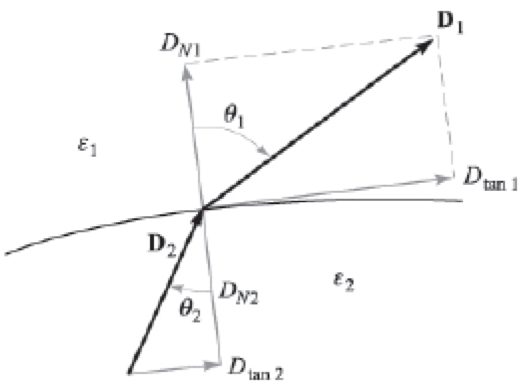
$$D_{n1} = \varepsilon_1 E_{n1}$$

$$D_{n2} = \varepsilon_2 E_{n2}$$

$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$\frac{E_{n1}}{E_{n2}} = \frac{\varepsilon_2}{\varepsilon_1} \qquad \textit{The normal component of E are not cotinous}$$

These conditions may be combined to show the change in vector (E,D) at thesurface:



$$D_{n1} = D_1 \cos\theta_1 \quad , \quad D_{n2} = D_2 \cos\theta_2$$

$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$

$$\frac{D_1}{D_2} = \frac{\cos\theta_2}{\cos\theta_1}$$

$$D_{t1} = D_1 \sin\theta_1 \qquad , \qquad D_{t2} = D_2 \sin\theta_2$$

$$\frac{D_1 \sin\theta_1}{D_2\, sin\,\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\frac{\cos\theta_2}{\cos\theta_1} \cdot \frac{\sin\theta_1}{sin\,\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Q/Prove :1- $E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1}$

2-$D_2 = D_1\sqrt{cos^2\,\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}$

Interface between dielectric and conductor

The conductor free space boundary are valid for the conductor dielectric interface. If we replace $\varepsilon_0 \rightarrow \varepsilon$

$$D_t = E_t = 0$$

$$D_n = \varepsilon E_n = \rho_s$$

Example: A slab of Teflon extending from(x=0) to (x=a) with free space on both sides of it, and external field $E_{ex} = E_0 a_x \& D_{ex} = \varepsilon_0 E_0 a_x \quad and \quad P_{ex} = 0$. Find E ,D , P in the side the material.

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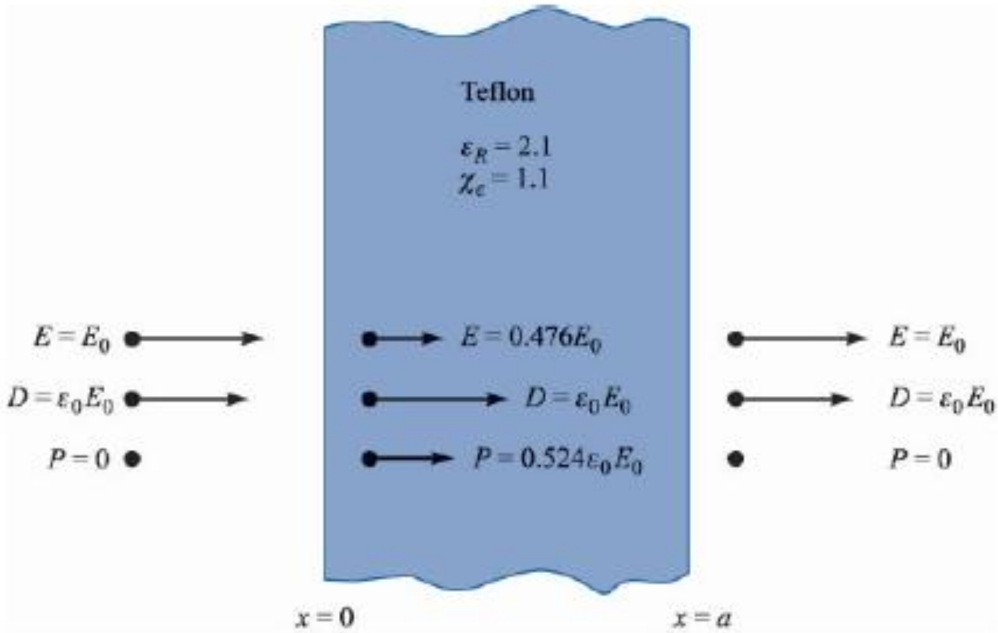

$E_t = D_t = 0$

$D_n = \rho_s = \varepsilon_0 E_n = \varepsilon_0 E_0\, a_x$

$D_{n1} = D_{n2} \quad \rightarrow \quad D_{ex} = D_{in} = \varepsilon_0 E_0\, a_x$

$$E_{in} = \frac{D_{in}}{\varepsilon_0 \varepsilon_R} = \frac{D_{in}}{\varepsilon} = \frac{\varepsilon_0 E_0\, a_x}{\varepsilon_0 \times 2.1} = 0.476 E_o a_x$$

$P = D_{in} - \varepsilon_0 E_{in} = \varepsilon_0 E_0 - \varepsilon_0 \times 0.476 E_o a_x$

$P = 0.524 \varepsilon_0 E_0\, a_x$