CHAPTER SEVEN

TIME-VARYING FIELDS AND MAXWELL'S EQUATIONS

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Two new concepts will be introduced

1- The electric field (E) produced by a changing magnetic field.
2- The magnetic field (H) produced by a changing electric field.

FARADAY'S LAW

Time-varying magnetic field produces an electromotive(Electromagnetic) force (emf) which may establish a current in a suitable closed circuit.

An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields, and we shall define it below. Faraday's law is customarily stated as

$$emf = -\frac{d\varphi}{dt}(volt) \qquad \textit{This eqution implies a closed path}$$

A nonzero value of $\left(\frac{d\varphi}{dt}\right)$may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The minus sign (-) is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement that the induced voltage acts to produce an opposing flux is known as Lenz's law.

If the closed path is consist of a number (N) of turns:

$$emf = -N\frac{d\varphi}{dt}$$

But for a closed path in a time changing fields "non conservating field"

$$V = \oint E.dl = emf$$

$$\oint E.dl = -\frac{d\varphi}{dt}$$

This investigation can be divided in to two parts:

1- Electromagnetic force (emf)made be a changing magnetic field within a stationary path(transformer emf).
2- emf net by moving path with in a stationary or constant magnetic field.(Motional, or Generator, emf).

1- For stationary path:

$$emf = \oint E.dl = -\frac{d\varphi}{dt}$$

$$\oint E.dl = \int (\nabla \times E)_n \cdot ds$$

$$-\frac{d\varphi}{dt} = -\frac{d}{dt}\int B \cdot ds$$

$$\int (\nabla \times E)_n \cdot ds = -\int \frac{dB}{dt} \cdot ds$$

$$\nabla \times E = -\frac{dB}{dt} \qquad Maxwell\ \ equation$$

Example: If $B = B_0 e^{kt} a_z$in cylindrical region $r < b$, find E using faraday's law and Maxwell equations

$$\oint E.dl = -\frac{d\varphi}{dt}$$

$$\oint E.dl = -\frac{dB}{dt} \cdot ds$$

$$E \int_0^{2\pi} r d\varphi = -k\, B_0 e^{kt} \int_0^r \int_0^{2\pi} r d\varphi dr$$

$$E_\varphi \cdot 2\pi r = -kB_0 e^{kt} \cdot \frac{r^2}{2} 2\pi$$

$$E_\varphi = -\frac{r}{2} k B_0 e^{kt}$$

$$(\nabla \times E)_n = -\frac{dB}{dt}$$

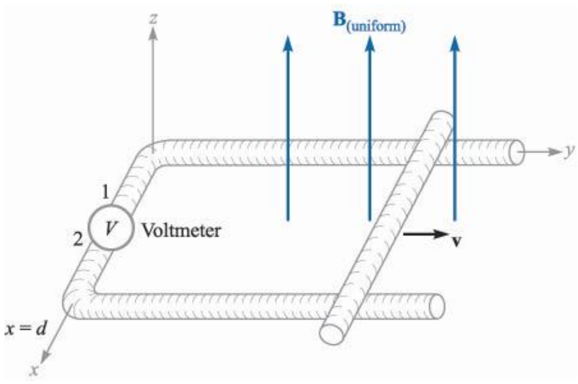
$$(\nabla \times E)_z = -\frac{dB}{dt}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\varphi\right) - \frac{\partial E_r}{\partial \varphi}\right] a_z = -kB_0 e^{kt}$$

$$\partial\left(rE_\varphi\right) = -kB_0 e^{kt} r dr$$

$$rE_\varphi = -\frac{r^2}{2} kB_0 e^{kt} \qquad \rightarrow \qquad E_\varphi = -\frac{r}{2} kB_0 e^{kt}$$

2) Now let us consider the case of a time-constant flux and a moving closed path.



B is the magnetic flux density, constant (in space and time) and is normal to the plane (d*y) containing the closed path.
The flux passing through the surface within the closed path at any time t is then

$$\varphi = \int B \cdot ds = B \cdot d \cdot y$$

From faraday $$emf = -\frac{d\varphi}{dt} = -\frac{d}{dt}(B \cdot d \cdot y)$$

$$emf = -Bd\frac{dy}{dt} = -Bdv$$

$$emf = \oint E \cdot dl = -Bdv$$

The last equation can be obtained by other way:

The force on charge q moving at velocity (v) in a magnetic field (B) is:

$$F = qv \times B \qquad v \perp B$$

$$\frac{F}{q} = v \times B = vB \sin\theta \qquad \theta \equiv \perp \equiv 1$$

$$\frac{F}{q} = vB$$

$$E_m = \frac{F}{q} [motional\ electric\ field\ intensity]$$

$$E_m = v \times B$$

$$\oint E_m \cdot dl = \oint_0^d (v \times B) dl$$

$$emf = \oint E_m \cdot dl = -Bdv$$

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3) The combination of the two:

$$\left[emf = \oint E \cdot dl = -\int \frac{dB}{dt} \cdot ds + \oint (v \times B) \cdot dl \right]$$

This expression is equivalent to the faraday law.

$$emf = -\frac{d\varphi}{dt}$$

Now let us turn attention to the time varying field and consider Maxwell's Eq.

$$[\nabla \times H = J] \quad \ldots\ldots\ldots \quad \text{Maxwell's Eq}$$

and show its inadequacy for time-varying conditions by taking the divergence of each side,

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J$$
$$0 = \nabla \cdot J$$

$$But \quad \nabla \cdot J = -\frac{\partial \rho}{\partial t} \qquad [equation\ of\ continuity]$$

Then shows us that $(\nabla \times H = J)$ can be true only if $\left(\frac{\partial \rho}{\partial t} = 0\right)$. This is an unrealistic limitation, and $(\nabla \times H = J)$ must be amended before we can accept it for time varying fields. Suppose we add an unknown term G to $(\nabla \times H = J)$,

$$\nabla \times H = J + G$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J + \nabla \cdot G$$

$$0 = \nabla \cdot J + \nabla \cdot G \qquad \rightarrow \qquad 0 = -\frac{\partial \rho}{\partial t} + \nabla \cdot G$$

$$\nabla \cdot G = +\frac{\partial \rho}{\partial t} \qquad [Max. 1^{st}\ eq.\ \ \rho = \nabla \cdot D]$$

$$\nabla \cdot G = \frac{\partial}{\partial t}(\nabla \cdot D) \qquad \rightarrow \qquad \nabla \cdot G = \nabla \cdot \frac{\partial D}{\partial t}$$

$$\therefore\ G = \frac{\partial D}{\partial t}$$

The amended Maxwell 2^{nd} eq. is for time varying field is:

$$\left[\nabla \times H = J + \frac{\partial D}{\partial t}\right]$$

$$\nabla \times H = J_c + J_d \quad \begin{matrix} (J_c = \sigma E) \ \ conduction\ current \\ \left(J_d = \frac{\partial D}{\partial t}\right) \ \ displacement\ current \\ (J = \rho v) \ \ convection\ current \end{matrix}$$

Point form	Integral form
$\nabla \cdot D = \rho$ $\nabla \times H = J + \frac{\partial D}{\partial t}$ $\nabla \times E = -\frac{\partial B}{\partial t}$ $\nabla \cdot B = 0$	$\oint D \cdot ds = \int \rho dv \quad (Gauss\ law)$ $\oint H \cdot dl = \int J \cdot ds + \int \frac{\partial D}{\partial t} \cdot ds \quad (\text{Amp. law})$ $\oint E \cdot dl = -\int \frac{\partial B}{\partial t} \cdot ds \quad (Faraday\ law)$ $\oint B \cdot ds = 0$

Static electrical field	Study magnetic field
$V = \int \frac{\rho dv}{4\pi\varepsilon r}$ $\nabla^2 V = -\frac{\rho}{\varepsilon} \quad or \ \nabla^2 V = 0$ $E = -\nabla V$	$A = \int \frac{\mu J dv}{4\pi r}$ $\nabla^2 A = -\mu J \quad or \ \ \nabla^2 A = 0$ $B = \nabla \times A \ \ , \ \nabla \cdot A = 0$

The inadequacy in eq. ($E = -\nabla V$) for time varying field is obvious:

$$\nabla \times E = -\nabla \times \nabla V$$

By adding the term N in eq.($E = -\nabla V$) we get:

$$E = -\nabla V + N \quad then \quad \nabla \times E = -\nabla \times \nabla V + \nabla \times N$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad Max.\, 2^{nd}\ eq.$$

$$\nabla \times \nabla V = 0 \qquad V \equiv scalar$$

$$B = \nabla \times A$$

$$\therefore \quad -\frac{\partial B}{\partial t} = \nabla \times N$$

$$-\frac{\partial(\nabla \times A)}{\partial t} = \nabla \times N \quad \Rightarrow \quad -\nabla \times \frac{\partial A}{\partial t} = \nabla \times N$$

$$N = -\frac{\partial A}{\partial t}$$

The amended eq. for E in time varying field is:

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

Under study or dc condition we know ($\nabla \cdot A = 0$). In time varying field there is a general relation liking (A) and (V) together is called "Lorentz" condition.

$$\nabla \cdot A + \epsilon\mu \frac{\partial v}{\partial t} = 0 \qquad \text{Lorentz condition}$$

The non-Homogenous equations for V and A

The differential equation satisfied for V and A.

static	Steady
$\nabla^2 V = -\frac{\rho}{\varepsilon}$ $\nabla \cdot D = 0$ $\nabla \cdot E = \frac{\rho}{\varepsilon}$ $\nabla \cdot \left(-\nabla V - \frac{\partial A}{\partial t}\right) = \frac{\rho}{\varepsilon}$ $\nabla^2 V + \frac{\partial(\nabla \cdot A)}{\partial t} = -\frac{\rho}{\varepsilon}$ $\nabla \cdot A = -\epsilon\mu \frac{\partial V}{\partial t} \quad \mathrm{L.C}$ $\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$ $\nabla^2 V - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$ $Where \;\; \frac{1}{v^2} = \epsilon\mu$	$\nabla^2 A = -\mu J$ $\nabla \times H = J + \frac{\partial D}{\partial t}$ $\frac{1}{\mu}\nabla \times B = J + \varepsilon \frac{\partial E}{\partial t}$ $\nabla \times \nabla \times A = \mu J + \mu\varepsilon \frac{\partial}{\partial t}(-\nabla V - \frac{\partial A}{\partial t})$ $\nabla(\nabla \cdot A) - \nabla^2 A = \mu J - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 A}{\partial t^2}$ $\nabla \cdot \left[\nabla \cdot A + \mu\varepsilon \frac{\partial V}{\partial t}\right] - \nabla^2 A = \mu J - \mu\varepsilon \frac{\partial^2 A}{\partial t^2}$ $0^{\Updownarrow} \; from \; L.c \uparrow$ $\nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu J$