Mathematica Statistcal

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Lecture notes
2021-2022

Mathematical Statistics

Course outline

Date	Topics	Notes
	Chapter One Introduction to Probability and Basic Statistical review	
First week	- Introduction - Probability and Basic Statistical / REVIEW	
Second week	- Random Variable - Types of Random Variables - Probability (Mass & Density) Function	
Third week	- The Cumulative Distribution Function (c.d.f) - The Mode	
Fourth week	- The Median - Exam in Chapter One	
	Chapter Two Mathematical Expectation & Moment	
Fifth week	- Mathematical Expectation	
Sixth week	- Moment (Non – Central & Central) - Coefficient of Skewnes	
Seventh week	- Coefficient of Kurtosis - Cheby Shev's Inequality	
Eighth week	- The Moment Generating Function	
	Chapter Three Joint &Conditional Probability	
Ninth week	- Joint Probability Density Function - Joint Probability Mass Function - Joint Cumulative Distribution Function	
Tenth week	- Marginal Probability Distribution Function - Expectation Joint Mathematical Function	
Eleventh week	- Covariance & Correlation Coefficient - Joint moment generation function	
Twelfth week	- Stochastic Independence	
Thirteenth week	- Conditional Probability Distribution Function - Conditional Probability Cumulative Distribution	

Date	Topics	Notes
Fourteenth week	- Conditional Expectation & Variance - Exam	
Chapter Four Discrete Distribution		
Fifteenth week	- Discrete Uniform Distribution - Bernoulli Distribution	
Sixteenth week	- Binomial Distribution - Poisson Distribution	
Seventeenth week	- Geometric Distribution - Negative Binomial Distribution	
Eighteenth week	- Hyper Geometric Distribution - Other Distribution	
Chapter Five Continuous Distribution		
Nineteenth week	- Continuous Uniform Distribution - Beta Distribution	
Twentieth week	- Gamma Distribution - Exponential Distribution - Chi – Square Distribution	
Twenty-First week	- Weibull Distribution - Other Distribution	
Twenty-second week	- Normal Distribution	
Twenty-third week	- Exam	
Chapter Six Transformation		
Twenty-fourth week	- Transformation of Discrete type	
Twenty-fifth week	- Transformation of Continuous type	
Twenty-sixth week	- Order Statistics	
Twenty-seventh week	- Univariate Probability distribution function	
Twenty-eight week	- Sample Median	
Twenty-nine week	- Bivariate p.d.f	
Thirteenth week	- Exam and Review of year	

The most important thing that the students should keep the subject under control, we should take this point into consideration.

1. The important of integration in the first stage, students should review the basic rules.
2. Memorizing or recognizing statistical rules which are (26) basic rules that we always take them into consideration.
3. Students should make a connection between the previous subject and current one.
4. While displaying important points students should write them down because these notes are crucial for solving the questions.
5. Following up those questions that are left unsolved students should do their best to solve them.

References:

1- Introduction to Mathematical Statistics, 6th Ed 2004. by Hogg, McKean, and Craig.

2- Mathematical Statistics, 2nd Ed 2003, by Jun Shao,

3- Introduction to Mathematical Statistics, 3rd Ed 1970. by Hogg, McKean, and Craig.

4- Introduction to Mathematical Statistics, 4th Ed 1983. by Hogg, McKean, and Craig.

"If you can't explain it simply, you don't understand it well enough."

- Albert Einstein

Same Laws about the Statistical Parameters

$1-(e^{-\infty}=0),(e^{\infty}=\infty),(e^{0}=1),((\infty)^{\#}=\infty),((\infty)^{-\#}=0),(\frac{1}{\infty}=0)$
$2-\sum_{x=0}^{\infty} r^x = 1+r+r^2+r^3+\ldots=\frac{1}{1-r}, 0<r<1$, Where r is constant
$3-\sum_{x=1}^{\infty} r^x = r+r^2+r^3+\ldots=\frac{r}{1-r}, 0<r<1$
$4-\sum_{x=n}^{\infty} r^x = r^n+r^{n+1}+r^{n+2}+\ldots=\frac{r^n}{1-r}, 0<r<1$
$5-\sum_{x=0}^{n} r^x = 1+r+r^2+\ldots=\frac{1-r^{n+1}}{1-r}, 0<r<1$
$6-\sum_{x=1}^{n} r^x = r+r2+\ldots=\frac{r(1-r^n)}{1-r}, 0<r<1$
$7-\sum_{x=1}^{n} r = nr$ Where r is constant
$8-\sum_{x=m}^{n} r = r(n-m+1)$ Where m is a real number
$9-\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$ $\&-\sum_{x=1}^{n} x^3 = \left(\frac{n(n+1)}{2}\right)^2$
$10-\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$
$11-\sum_{x=0}^{n} C_x^n a^x b^{n-x} = (a+b)^n$
$12-\sum_{x=0}^{\infty} \frac{r^x}{x!} = e^r$
$13-\sum_{x=0}^{\infty} C_{k-1}^{x+k-1} r^x = (\frac{1}{1-r})^k$
$13-\sum_{x=0}^{\infty} C_{k-1}^{x+k-1} r^x = (\frac{1}{1-r})^k$

$14-\sum_{x=0}^{k} C_x^n C_{k-x}^m = C_k^{n+m}\ ,\ \sum_{x=0}^{k} C_x^n C_{k-x}^{n-m} = C_k^n$
$15-\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{z-m}{\sigma})^2} dx = \sqrt{2\pi} - \infty < x < \infty, z > 0$
$16-\int_0^{\infty} x^{r-1} e^{-x} dx = \Gamma(r) \quad Where\ \Gamma(r) = (r-1)!$
$17-\int_0^{\infty} x^{r-1} e^{-Bx} dx = \frac{\Gamma(r)}{B^r}$
$18-\int_0^{\infty} x^{r-1} e^{-x/B} dx = \Gamma(r) B^r \quad r, B > 0$
$19-\int_0^{1} x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
$20-\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
$21-Ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots..$
22 –The Permutation of tossing [for coin (n) times is S=2^n] &[for the affair dice is S=6^n]
$23-(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$
$24-(1-x)^{-2} = \sum_{k=0}^{\infty} (k+1) x^k = 1 + 2x + 3x^2 + 4x^3 + \ldots$
25-$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2} dz = 1$
26 $C_x^{x+r-1} = (-1)^x C_x^{-r} \qquad C_r^x = C_{x-r}^x$

Chapter One
An Introduction to Probability and Basic Statistical Review

- ***1 – 1 :Random Variable (r.v)***

Def' : - A random variable X is a real valued function defined on the sample space, which takes values from the sample space (S) to a space real numbers R is a random experiment

$X^{f_n}: S \rightarrow R R(x) = \{x: x(w), x, w \in S\} \subseteq R$

Example (1): let a coin be tossed three times, and let (x) be a head observed (occur) then;

Solution/

Let: H=head and T=tail

Sample space=8= {HHH, HHT, THH, HTH, TTH, THT, HTT, TTT}

Thus x= {0, 1,2,3}

Probability= $\frac{part}{sample\ space}$

P(0)=$\frac{p(TTT)}{sample\ space} = \frac{1}{8}$

P(1)=$\frac{p(HTT\ or\ THT\ or\ TTH)}{sample\ space} = \frac{3}{8}$

P(2)= $\frac{p(HTT\ or\ THT\ or\ TTH)}{sample\ space} = \frac{3}{8}$

P(3)= $\frac{p(HHH)}{sample\ space} = \frac{3}{8}$

Example (2): let a pair of fair dices are tossed one time, and let (x) be the *sum* of two numbers that occur, then:

Solution/

S.space=$how\ many\ face\ dice^{how\ many\ tossed} = 6^2 = 36$

P(2)= $\frac{1}{36}$ = (1 ,1)

P(3)= $\frac{2}{36}$ = (1,2) (2,1)

P(4)= $\frac{3}{36}$ = (1,3)(3,1)(2,2)

P(5)= $\frac{4}{36}$ =(2,3)(3,2)(1,4)(4,1)

p(6)= $\frac{5}{36}$ =(3,3)(2,4)(4,2)(1,5)(5,1)

p(7)= $\frac{6}{36}$ =(3,4)(4,3)(5,2)(2,5)(6,1)(1,6)

p(8)= $\frac{5}{36}$ =(4,4)(3,5)(5,3)(2,6)(6,2)

P(9) =$\frac{4}{36}$ =(5,4)(4,5)(3,6)(6,3)

P(10)= $\frac{3}{36}$ =(5,5)(6,4)(4,6)

P(11)= $\frac{2}{36}$ =(5,6)(6,5)

P(12)= $\frac{1}{36}$ =(6,6)

- ***1 – 2 : Probability (mass & density) function***

Def. (1): if (x) be a discrete r.v. with different values ($x_1, x_2, \ldots, x_n$), then the function

$$P(x) = \begin{cases} P(X = x) \ if \quad i = 1,2,3,\ldots,n,\ldots \\ 0 \quad o.w(otherwise) \end{cases}$$

is defined to be probability mass function of x (p.m.f).
and it is a real – value function, and satisfies following properties: -

1. $0 \le P(x) \le 1, \ For \quad all \quad x$
2. $\sum_{R_{(x)}} P(x) = 1$
3. $P(x \in A) = \sum_{x \in A} P(x) = \sum_{x \in A} P(X = x_i)$

Where (A) is any subset of the space of (x)

Def. (2) : if (x) be a continuous r.v., defined on an interval, then the function f(x) is defined to be probability density function (P.d.f)of (x), and it is a mathematical f^n, and satisfies, the following properties : -

1. $0 \le f(x) \le 1 \qquad x \in R$
2. $\int_{R_x} f(x)dx = 1$
3. $P(x \in A) = \int f(x)dx,$

$whereAisanysubset(aneventon)thespaceofx.$

Example(3) : - *let a coin be tossed twice, and let x be the number of head that occur, Then :*

1 – Find P(x) of each value of (x).
2 – Check P(x) is (p.m.f) of (x).
3 – Graph P(x).

Solution/

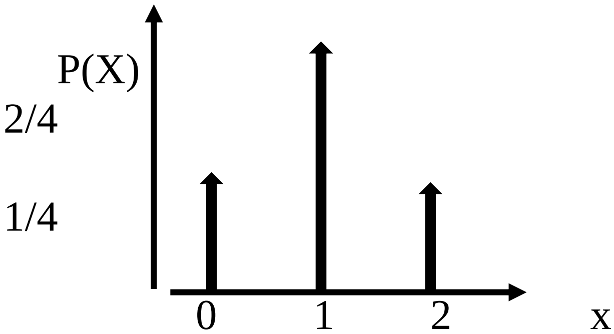
1 – Find P(x) of each value of (x).
S. Space=$2^2 = 4$ S={HH,HT,TH,TT}
P(0)=$\frac{1}{4} = (TT)$ P(1)= $\frac{2}{4}$=(TH)(HT) P(2)= $\frac{1}{4}$ =(HH)

2 – Check P(x) is (p.m.f) of (x).

$$\sum_{0}^{2} p(X) = 1 \ \ Thus \quad P(0) + p(1) + p(2) = 1 \qquad \frac{1}{2} + \frac{2}{2} + \frac{1}{2} = 1$$

That means the p(x) is a p.m.f of (x).

3 – Graph P(x).



***Example(4)** : - let (x) be a r.v. with p.m.f, P(x) where :*

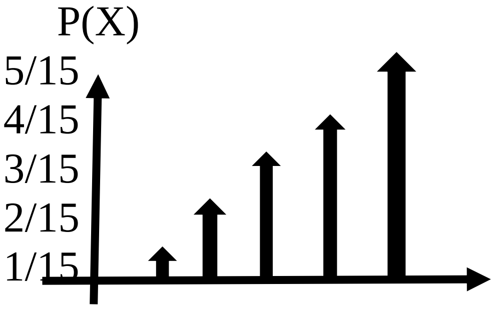
$$P(x)\begin{cases} \frac{x}{15} & x = 1,2,3,4,5 \\ 0 & o.w \end{cases}$$

1 – find the p(x) of each value of (x) .
2 – check P(x) is (p.m.f) of (x) and graph it.
Solution/
1/p(1)= $\frac{1}{15}$ P(2)= $\frac{2}{15}$ P(3)= $\frac{3}{15}$ P(4)= $\frac{4}{15}$ P(5)= $\frac{5}{15}$
2 – check P(x) is (P.M.F) of (x) and graph it.
Solution:

$$\sum_{1}^{5} p(X) = 1 \ \ Thus \quad p(1) + p(2) + p(3) + p(4) + p(5) = 1$$

$$\frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = 1$$

Thus it is PMF



1 2 3 4 5 x

***Example (5):** - let x be a r.v. with p.d.f f(x) is :*

$$f(x) = \begin{cases} \frac{1}{x^2} & 1 < x < \infty \\ 0 & o.w \end{cases}$$

Let A1={ x : 1< x < 2} , A_2={ x : 4< x < 5}

Find check f(x) is a (p.d.f) of (x).
$P(A_1)$, $P(A_2)$, $P(A_1 U A_2)$

Solution:

1) $\int_1^\infty \frac{1}{x^2} dx = \int_1^\infty x^{-2} dx = \frac{-1}{x} I_1^\infty = \frac{-1}{\infty} - \left(\frac{-1}{1}\right) = 0 + \frac{1}{1} = 1$

2) P(A1)=$\int_1^2 \frac{1}{x^2} dx = \int_1^2 x^{-2} dx = \frac{-1}{x} I_1^2 = \frac{-1}{2} - \left(\frac{-1}{1}\right) = \frac{-1}{2} + \frac{1}{1} = \frac{1}{2}$

P(A2)=$\int_4^5 \frac{1}{x^2} dx = \int_4^5 x^{-2} dx = \frac{-1}{x} I_4^5 = \frac{-1}{5} - \left(\frac{-1}{4}\right) = \frac{-1}{5} + \frac{1}{4} = \frac{1}{20}$

$$P(A1 \cup A2) = p(A1) + p(A2) - p(A1 \cap A2)$$

$$P(A1 \cup A2) = \frac{1}{2} + \frac{1}{20} - 0 = \frac{22}{40}$$

***Example (6) :** - let x be a r.v. with p.m.f.*

$$P(x) = \begin{cases} c\left(\frac{2}{3}\right)^x & x = 1,2,3,\ldots \\ 0 & o.w \end{cases}$$

$find\ c?$

Solution:

c$\left[\frac{\frac{2}{3}}{1-\frac{2}{3}}\right]$ =1 thus $\frac{2c}{1} = 1\ and\ c$ =1/2

$$\sum_1^\infty r^x = \frac{r}{1-r}$$

Example (7) : - let x be a r.v. with p.d.f. f(x) ;

$$f(x) = \begin{cases} \frac{3x^2}{8} & 0 < x < 2 \\ 0 & o.w \end{cases}$$

$$let A_1 = \{x: 0 < x < 1\}, A_2 = \{x: \frac{1}{2} < x < 2\}$$

$find P(A_1), P(A_2), P(A_1 \cup A_2)$ and check f(x) is ($p.d.f$) of (x).

Solution/

P(A1)=$\int_0^1 \frac{3x^2}{8} dx = \frac{x^3}{8} I_0^1 = \frac{1^3}{8} - 0 = \frac{1}{8}$

P(A2)=$\int_{\frac{1}{2}}^2 \frac{3x^2}{8} dx = \frac{x^3}{8} I_{\frac{1}{2}}^2 = \frac{2^3}{8} - \frac{\left(\frac{1}{2}\right)^3}{8} = \frac{8}{8} - \frac{1}{64} = \frac{63}{64}$

$$P(A1 \cap A2) = \int_{\frac{1}{2}}^1 \frac{3x^2}{8} dx = \frac{x^3}{8} I_{\frac{1}{2}}^2 = \frac{1^3}{8} - \frac{\left(\frac{1}{2}\right)^3}{8} = \frac{8}{8} - \frac{1}{64} = \frac{7}{64}$$

$$P(A1 \cup A2) = p(A1) + p(A2) - p(A1 \cap A2)$$

$$P(A1 \cup A2) = \frac{1}{8} + \frac{63}{64} - \frac{7}{64} = \frac{64}{64} = 1$$

***Example (8)** : - let x be a r.v. with p.m.f f(x):*

$$P(x) = \begin{cases} \frac{x}{15} & x = 1,2,3,4,5 \\ 0 & o.w \end{cases}$$

$find: 1) P(x = 1 or 2)$

$2) P(\frac{1}{2} < X < \frac{5}{2})$

$3) P(1 \le X \le 2)$

$4) P(X > 3)$

Solution>>>>>>>>

1)p(x=1 or 2)=p(x=1)+p(x=2)=$\frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

2)p($\frac{1}{2} < x < \frac{5}{2}$) = p(x = 1) + p(x = 2) = $\frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

3)p($1 \le x \le 2$) = p(x = 1) + p(x = 2) = $\frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

4)p(x>3)=p(x=4)+p(x=5)=$\frac{4}{15} + \frac{5}{15} = \frac{9}{15} = \frac{3}{5}$

Example (9) : - let x be a r.v. with p.d.f of f(x) :

$$f(x) = \begin{cases} \frac{x+2}{18} & -2 < x < 4 \\ 0 & o.w \end{cases}$$

$$find: 1)\ Pr(x^2 < 9) \quad 2)\ Pr(-2 < x < 3) \quad 3)\ Pr(-4 < x < 3)$$

Solution/

1- $p(x^2 < 9) = p(-3 < x < 3) = \int_{-3}^{-2} \frac{x+2}{18} + \int_{-2}^{3} \frac{x+2}{18} = 0 + \frac{1}{18}(\frac{x^2}{2} + 2x)I_{-2}^{3} =$
$\frac{1}{18}\left(\left(\frac{3^2}{2} + 2(3)\right) - \left(\frac{-2^2}{2} + 2(-2)\right)\right) = \frac{25}{36}$

2- p(-2<x<3)=$\int_{-2}^{3} \frac{x+2}{18} = \frac{1}{18}\left(\left(\frac{3^2}{2} + 2(3)\right) - \left(\frac{-2^2}{2} + 2(-2)\right)\right) = \frac{25}{36}$

3- p(-4<x<3)=$\int_{-4}^{-2} f(x)dx + \int_{-2}^{3} \frac{x+2}{18} dx = 0 + \frac{1}{18}(\frac{x^2}{2} + 2x)I_{-2}^{3} = \frac{25}{36}$

Example (10): - let x be a r.v. with p.d.f. of f(x) then ;

$$f(x) = \begin{cases} \frac{x^2}{18} - 3 < x < 3 \\ 0 \quad o.w \end{cases}$$

$$find: 1)\ Pr(|x| < 1) \qquad 2)\ Pr(x^2 < 9)$$

Solution /

1)p(|x|<1)=p(-1<x<1)=$\int_{-1}^{1} \frac{x^2}{18} dx = \frac{x^3}{54} I_{-1}^{1} = \frac{1^3}{54} - \frac{(-1)^3}{54} = \frac{2}{54}$

2)$p(x^2 < 9) = p(-3 < x < 3) = \int_{-3}^{3} \frac{x^2}{18} = \frac{x^3}{54} I_{-3}^{3} = \frac{3^3}{54} - \frac{(-3)^3}{54} = \frac{54}{54} = 1$

Example (11): - let x be a r.v. with p.m.f. of P(x)

$$P(x) = \begin{cases} \left(\frac{2}{c}\right)^{-x} x = 1,2,3,\dots \\ 0 o.w \end{cases}$$

$$when c > 0 and c < 2 (0 < c < 2)$$

$then; find the value of the constant\ c.$

Solution /

$$\sum_{1}^{\infty} (\frac{c}{2})^x = 1$$

So/$\frac{\frac{c}{2}}{1-\frac{c}{2}} = 1$ ➡ $\frac{c}{2} + \frac{c}{2} = 1$ ➡ $c = 1$

1 – 3 The Cumulative Distribution Function (CDF)

If x is a r.v. with p.d.f. ; then the function F(x) is called the (c.d.f) of x, and is defined as ;

$$F(x) = Pr(X \le x)$$

$$F(x) = \begin{cases} \sum_{u=-\infty}^{x} p(u) \; if x is a discrete r.v. \\ \\ \int_{-\infty}^{x} f(u)du \; if x is a continuous r.v. \end{cases}$$

Which have the following properties : -

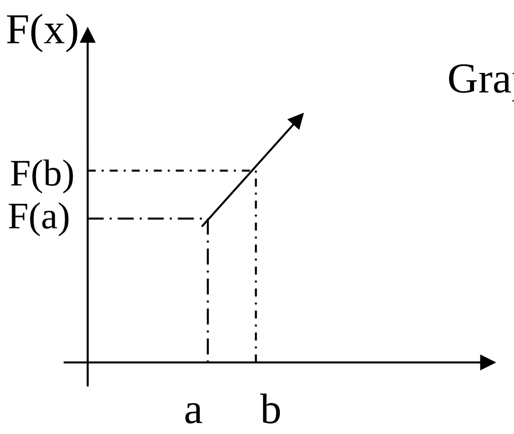
$1 - 0 \le F(x) \le 1, -\infty \le x \le \infty$

$2 - \quad F(\infty) = 1 \to F(\infty) = Pr(X < \infty) = \int_{-\infty}^{\infty} f(u)du = 1$

$F(-\infty) = 0 \to F(-\infty) = Pr(X < -\infty) = \int_{-\infty}^{-\infty} f(u)du = zero$

$3 - If a \le b \to F(b) \ge F(a)$

$i.e\ F(x) is\ a\ non - descrea\sin g\ f^n.$



Graphical of F(x) function

4 – If we want to use F(x) [c.d.f] to find f(x) [p.d.f] we will depend on the following definition : -

Def. :- let x be a r.v. having p.d.f f(x), and c.d.f [F(x)] then ;

$$\boldsymbol{f(x_i) = F(x_i) - F(x_i - 1)} \; if \;\; x \;\; is \;\; a discrete \;\; random\ varibale$$

$$\boldsymbol{f(x) = \frac{\partial}{\partial x}} \;\; F(x) \; if \;\; x \;\; is\ a\ continuous\ random\ varibale$$

5 – If the distribution f ^{n}. F(x) is continuous at x = b, then ; P(x = b) = zero

$$p(x=b)=\int_b^b f(x)dx=F(x=b)|_b^b=F(x=b)-F(x=b)=0$$

and

$$Pr(a\le x\le b)=\int_a^b f(x)dx=\int_{-\infty}^b f(x)dx-\int_{-\infty}^a f(x)dx=F(b)-F(a)$$

6 – If x be a discrete r.v. then;

$$a-Pr(a\le x\le b)=Pr(x\le b)-Pr(x\le a)=F(b)-F(a-1)$$

$$b-Pr(a<x\le b)=Pr(x\le b)-pr(x\le a-1)=F(b)-F(a)$$

$$c-Pr(a<x<b)=Pr(x\le b-1)-Pr(x\le a-1)=F(b-1)-F(a)$$

$$d-Pr(a\le x<b)=Pr(x\le b-1)-Pr(x\le a)=F(b-1)-F(a-1)$$

$$8-F(x,y)=\begin{cases}\int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv cont.\\ \sum_{-\infty}^{x}\sum_{-\infty}^{y} P(x,y)\end{cases}$$

$$9-F(x,y)exist\rightarrow f(x,y)\frac{\partial^2 F(x,y)}{\partial x\partial y}$$

Example (12): -

$$P(x) = \begin{cases} \frac{1}{6} & x = 1,2,3,4,5,6 \\ 0 & o.w \end{cases}$$

$Find \quad CDF.$

Solution /

$P(x)=\frac{1}{6} \qquad x = 1,2,3,4,5,6$

$$F(x)=\begin{bmatrix} 0 & x < 1 \\ \frac{1}{6} & 1 \le x < 2 \\ \frac{2}{6} & 2 \le x < 3 \\ \frac{3}{6} & 3 \le x < 4 \\ \frac{4}{6} & 4 \le x < 5 \\ \frac{5}{6} & 5 \le x < 6 \\ 1 & 6 \le x \end{bmatrix}$$

Example (13): - Let x be a r.v. with p.d.f. ;

$$f(x) = \begin{cases} \frac{x}{10} & x = 1,2,3,4 \\ 0 & o.w \end{cases}$$

$Find \quad CDF?$

Solution /

$$F(x)=\begin{bmatrix} 0 & x < 1 \\ \frac{1}{10} & 1 \le x < 2 \\ \frac{3}{10} & 2 \le x < 3 \\ \frac{6}{10} & 3 \le x < 4 \\ 1 & x \ge 4 \end{bmatrix}$$

Example (14) : - let x be a r.v. with p.d.f.

$$f(x) = \begin{cases} \frac{2}{x^3} & 1 < x < \infty \\ 0 & 0.w \end{cases}$$

$$find \quad the \quad CDF?$$

Solution /

$$\int_1^x \frac{2}{x^3} dx = \quad 2 \times \frac{1}{-2x^2} I_1^x = 1 - \frac{1}{X^2}$$

F(x)= $\begin{bmatrix} 0 & x < 1 \\ 1 - \frac{1}{x^2} & 1 \leq x < \infty \\ 1 & x \rightarrow \infty \end{bmatrix}$

Example (15) : - let

$$F(x) = 1 - e^{-x} \qquad x \geq 0$$

$$Find\ the\ f(x)?\ or\ p.d.f?$$

Solution /

f(x)= $F' = e^{-x}$

f(x)= $\begin{bmatrix} e^{-x} & 0 < x < \infty \\ 0 & o.w \end{bmatrix}$

Example(16) : - let the r.v. have a p.d.f. ;

$$f(x) = \begin{cases} 2xe^{-x^2} & x > 0 \\ 0 & o.w \end{cases}$$

$Find \quad the \quad CDF?$

Solution /

$$\int_0^x 2xe^{-x^2} dx = -e^{-x^2} I_0^x = 1 - e^{-x^2}$$

F(x)= $\begin{bmatrix} 0 & x < 0 \\ 1 - e^{-x^2} & 0 \leq x < \infty \\ 1 & x \quad \infty \end{bmatrix}$

Example (17) : - Let x be a r.v. having p.d.f .

$$f(x)=\begin{cases}\frac{1}{2\sqrt{x}}\, 0<x<\frac{1}{4}\\ \frac{1}{4\sqrt{x}}\,\frac{1}{4}<x<\frac{9}{4}\\ 0 o.w\end{cases}$$

$$find the c.d.f of x. \& Pr(\frac{1}{16}<x<\frac{1}{2})$$

Solution /

1)F(x1)=$\int_0^x \frac{1}{2\sqrt{x}}dx=\frac{1}{2}*2\sqrt{x}I_0^x=\sqrt{x}$

F(x)=$\int_0^{\frac{1}{4}} \frac{1}{2\sqrt{x}}dx+\int_{\frac{1}{4}}^x \frac{1}{4\sqrt{x}}dx=\frac{\sqrt{x}}{2}+\frac{1}{4}$

F(x)=$\begin{bmatrix} 0 & x<0\\ \sqrt{x} & 0\le x<\frac{1}{4}\\ \frac{\sqrt{x}}{2}+\frac{1}{4} & \frac{1}{4}\le x<\frac{9}{4}\\ 1 & x\ge\frac{9}{4}\end{bmatrix}$

Pr($\frac{1}{16}<x<\frac{1}{2}$) $=F\left(\frac{1}{2}\right)-F\left(\frac{1}{16}\right)=\frac{1}{2}\sqrt{\frac{1}{2}}+\frac{1}{4}-\sqrt{\frac{1}{16}}=\frac{1}{2\sqrt{2}}$

Example (18) : - Let

$$f(x)=\begin{cases}\frac{1}{3} & x=-1,0,1\\ 0 & o.w\end{cases}$$

$find\ the;$

1)$c.d.f.$

2) $Pr(-1<x\le -0.5)$

3) $Pr(0\le x<1.5)$

4) $Pr(-1\le x\le 1)$

Solution /

1) cdf?

$$F(x)=\begin{bmatrix} 0 & x<-1 \\ \frac{1}{3} & -1\le x<0 \\ \frac{2}{3} & 0\le x<1 \\ 1 & x\ge 1 \end{bmatrix}$$

2)Pr($-1< x \le -0.5) = F(-0.5) - F(-1) = \frac{1}{3} - \frac{1}{3} = 0$

3)Pr($0\le x < 1.5) = F(1.5-1) - F(0-1) = \frac{2}{3} - \frac{1}{3} = 0$

4)Pr($-1\le x \le 1) = 1 - 0 = 1$

5)Pr($0\le x < 3) = F(3-1) - F(0-1) = 1 - \frac{1}{3} = \frac{2}{3}$

Example (19): - let

$$f(x)=\begin{cases} 1 & x=0 \\ 0 & o.w \end{cases}$$

Solution /

Find the cdf?

$$F(x)=\begin{bmatrix} 0 & x<0 \\ 1 & 0\le x \end{bmatrix}$$

Point & interval (Discrete r.v.)

Without point & interval

$Pr(a \le x \le b) = F(b) - F(a - 1)$

$Pr(a < x \le b) = F(b) - F(a)$

$Pr(a \le x < b) = F(b - 1) - F(a-1)$

$Pr(a < x < b) = F(b - 1) - F(a)$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{15} & 1 \le x < 2 \\ \frac{3}{15} & 2 \le x < 3 \\ \frac{6}{15} & 3 \le x < 4 \\ \frac{10}{15} & 4 \le x < 5 \\ 1 & 5 \le x \end{cases}$$

$$Pr(2 \le x \le 4) = F(4) - F(1) = \frac{10}{15} - \frac{1}{15} = \frac{9}{15}$$

$$Pr(2 < x \le 4) = F(4) - F(2) = \frac{10}{15} - \frac{3}{15} = \frac{7}{15}$$

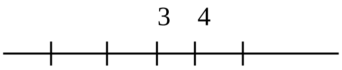
$$Pr(2 \le x < 4) = F(3) - F(1) = \frac{6}{15} - \frac{1}{15} = \frac{5}{15}$$

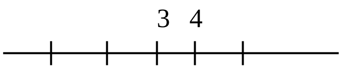
$$Pr(2 < x < 4) = F(3) - F(2) = \frac{6}{\ } - \frac{3}{\ } - \frac{3}{\ }$$

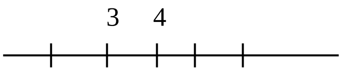
With point & interval

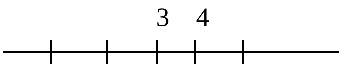
$$\left.\begin{array}{l} Pr(a + h \le x \le b + h) \\ Pr(a + h < x \le b + h) \\ Pr(a + h \le x < b + h) \\ Pr(a + h < x < b + h) \end{array}\right\} = F(b + h) - F(a + h)$$

where $h > 0$

$Pr(2.5 \le x \le 4.5) =$ 

$Pr(2.5 < x \le 4.5) =$ 

$Pr(2.5 \le x < 4.5) =$ 

$Pr(2.5 < x < 4.5) =$ 

$$F(4.5) - F(2.5) = \frac{10}{15} - \frac{3}{15} = \frac{7}{15}$$

Without point & equal

$Pr(x = a) = F(a) - F(a - 1)$

$Pr(x = 4) = F(4) - F(3)$

$$= \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

with point & equal

$Pr(x = a + h) = zero$

$Pr(x = 2.5) = 0$

point & interval (Continuous r.v.)

For Example:

$$f(x)=\begin{cases}2x0<x<1\\0o.w\end{cases}$$

$$*\ p_r(\frac{1}{4}<x<\frac{1}{2})=\int_{\frac{1}{4}}^{\frac{1}{2}}2xdx=x^2\Big]_{\frac{1}{4}}^{\frac{1}{2}}=(\frac{1}{2})^2-(\frac{1}{4})^2=\frac{3}{16}$$

By C.d.f

$$*\ p_r(X<x)=\int_0^X 2XdX=X^2\Big]_0^x=(x)^2-(0)^2=x^2$$

$$f(x)=\begin{cases}o & x<0\\x^2 & 0\le x<1\\1 & x\ge 1\end{cases}$$

$$p_r(\frac{1}{4}<x<\frac{1}{2})=F(\frac{1}{2})-F(\frac{1}{4})=(\frac{1}{2})^2-(\frac{1}{4})^2=\frac{3}{16}$$

$$\left.\begin{matrix}p_r(a\le x\le b)\\p_r(a\le x<b)\\p_r(a<x\le b)\\p_r(a<x<b)\end{matrix}\right\}=\int_a^b f(x)dx=F(b)-F(a)$$

$$p(x=a)=0\quad\Rightarrow\int_a^a f(x)dx=F(a)-F(a)=0$$

$$p_r(x>a)=\int_a^\infty f(x)dx=F(\infty)-F(a)=1-F(a)$$

$$p_r(x<a)=\int_{-\infty(\min interval)}^{a} f(x)dx=F(a)-F(-\infty)=F(a)-0=F(a)$$

Example (20) : - let

$$f(x)=\begin{cases}\frac{x}{6} & x=1,2,3\\ 0 & o.w\end{cases}$$

$y=x^2 \rightarrow Find \quad CDF \;\; of\; y.$

Solution /

$$y=x^2 \qquad \therefore \sqrt{y}=x \qquad =pr(-\sqrt{y}\le y\le \sqrt{y})$$

F(y)=$\sum_{1}^{\sqrt{y}}\frac{x}{6}=\frac{\sqrt{y}(\sqrt{y}+1)}{6*2} \qquad using\ \sum_{1}^{n}x=\frac{n(n+1)}{2}$

F(y)=$\begin{bmatrix} 0 & y<1\\ \frac{\sqrt{y}(\sqrt{y}+1)}{12} & 1\le y<9\\ 1 & y\ge 9\end{bmatrix}$

Example (21) : - Let

$$f(x)=\begin{cases}2x & 0<x<1\\ o & o.w\end{cases}$$

$$y=\sqrt{x} then; find f(y)?$$

Solution /

y=$\sqrt{x} \qquad y^2=x \quad then; find\ \ f(y)?$

F(y)=$\int_0^{y^2} 2xdx = X^2 I_0^{y^2} = y^4$

f(y)=F'(y)=4y^3

f(y)=$\begin{bmatrix} 4y^3 & 0<y<1\\ 0 & o.w\end{bmatrix}$

MATHEMATICS
is not about
numbers, equations,
computations, or
algorithms:
it is about
UNDERSTANDING.

William Paul Thurston

1 – 3 The Mode

A mode of a distribution of one a r.v. x of the continuous or discrete type is the value of x that maximize the p.d.f. f(x) .
Such that:

a) If x is discrete r.v. ; then the value which be greater that is mode of dist.

Find the mode of the following distribution $f(x) = \begin{cases} \frac{2^x e^{-2}}{x!} & x = 0,1,2,\dots \\ 0 & o.w \end{cases}$

Solution:

X	0	1	2	3	4	5	6 ……..
f(x)	e^{-2}	$2e^{-2}$	$2e^{-2}$	$\frac{4}{3}e^{-2}$	$\frac{2}{3}e^{-2}$	$\frac{4}{15}e^{-2}$	………

We see that $f(0) < f(1) = f(2) > f(3) > f(4) > \dots$.. Then the distribution has two modes $x=1$ and $x=2$ (Bimodal).

b) If x is continuous r.v. ; then we must take the second derivation of p.d.f.; then if this derivative less than zero that is the mode of dist.

Find the mode of the following distribution $f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & o.w \end{cases}$

Solution:

$f'(x) = 6(1-2x) \rightarrow f'(x) = 0 \rightarrow 6(1-2x) = 0 \rightarrow x = \frac{1}{2}$

$f''(x) = -12 < 0$ when $x = \frac{1}{2}$

$\therefore$ Mode for this distribution is $x = \frac{1}{2}$

Remark:
In case of continuous *r.v.* Searching about a value of *x* that maximizes *f(x)* is as follows: $f'(x) = 0$ if $f''(x) < 0$.

"Learn from yesterday, live for today, hope for tomorrow. The important thing is not to stop questioning."

Albert Einstein

Example (22) Find the mode for the following distribution.

$$f(x)=\frac{1}{4}\left(\frac{3}{4}\right)^{x-1} \qquad ,x=1,2,3,\ldots$$

Solution /

f(x)=$\frac{1}{4}\left(\frac{3}{4}\right)^{x-1} \qquad x=1,2,3,\cdots$

f(1)=$\frac{1}{4}\left(\frac{3}{4}\right)^{1-1}=0.25 \;\; .f(2)=\frac{1}{4}\left(\frac{3}{4}\right)^{2-1}\; 0.1875$

until x is increase that is result decrease

thus x=1 is mode.

Example (23) Find the mode for the following distribution.

$$f(x)=\frac{1}{2}x^2 e^{-x} \qquad 0<x<\infty$$

Solution /

f(x)=$\int_0^\infty \frac{1}{2}x^2e^{-x}$ *and* $\frac{dy}{dx}=xe^{-x}-\frac{1}{2}x^2e^{-x} \;\; xe^{-x}\left(1-\frac{1}{2}\right)=0$

$$xe^{-x}=0 \quad or\ 1-\frac{1}{2}x= \quad thus\ x=2$$

$$\frac{d^2y}{d^2x}=e^{-x}-xe^{-x}-xe^{-x}+\frac{1}{2}x^2e^{-x}=f^2(0)=1\ \ or\ f^2(2)$$

$$=-0.135335 \qquad \therefore x=2\ it\ is\ mode$$

Example (24) Find the mode for the following distribution.

$$f(x)=12x^2(1-x) \qquad 0<x<1$$

Solution /

f(x)=$12x^2(1-x) \qquad 0<x<1$

f(x)=$12x^2-12x^3 \quad f'=24x-36x^2=0 \;\; x=0\ \ or\ x=\frac{2}{3}$

f"$24-72x \quad .\ f(0)$=24>0 or f$\left(\frac{2}{3}\right)=24-72\ f\left(\frac{2}{3}\right)=-24$

x=$\frac{2}{3}$ *is mode*

1 – 4 The Median

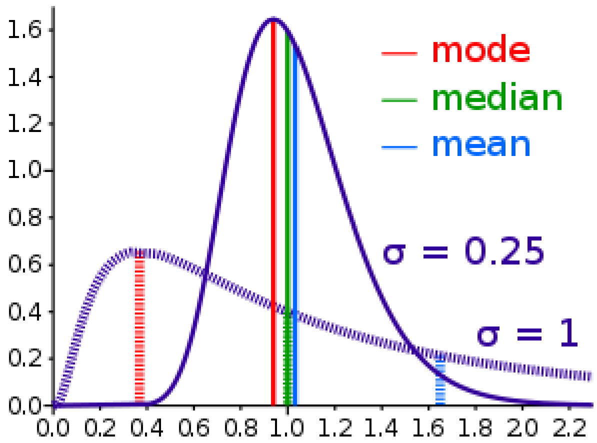
A median of a distribution of one r.v. of the continuous or discrete type. Is a value of x that satisfies the following conditions;

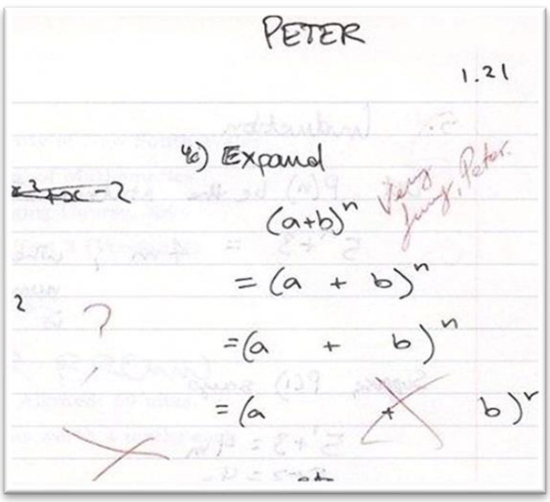
$$1 - Pr(x < m) = \sum_{x<m} P(x) \leq \frac{1}{2} for a disc. r. v.$$

$$2 - Pr(x \leq m) = \sum_{x \leq m} P(x) \geq \frac{1}{2}$$

and;

$$3 - Fm = P(X \leq m) = \int_{-\infty}^{m} f(x)dx = \frac{1}{2} for cont. r. v.$$





Example (25) Find the median for the following distribution.

$f(x)=\frac{x}{15}$ $\quad ,x=1,2,3,4,5$

Solution /

$f(x)=\frac{x}{15}$ $\quad x = 1,2,3,4,5$

$p(x < m) \leq \frac{1}{2}$	$p(x \leq m) \geq \frac{1}{2}$
$p(x < 1) = 0$ ✓	$p(x \leq 1) = \frac{1}{15}$ ☒
$p(x < 2) = \frac{1}{15}$ ✓	$p(x \leq 2) = \frac{3}{15}$ ☒
$p(x < 3) = \frac{3}{15}$ ✓	$p(x \leq 3) = \frac{6}{15}$ ☒
$p(x < 4) = \frac{6}{15}$ ✓	$p(x \leq 4) = \frac{10}{15}$ ✓
$p(x < 5) = \frac{10}{15}$ ☒	$p(x \leq 5) = 1$ ✓

X=4 is median

Example (26) Find the median for the following distribution.

$f(x)=\frac{1}{4}\left(\frac{3}{4}\right)^{x-1}$ $\quad x=1,2,3,\ldots$

Solution /

$f(x)=\frac{1}{4}\left(\frac{3}{4}\right)^{x-1}$ $\quad x = 1,2,3 \cdots$

$p(x < m) \leq \frac{1}{2}$	$p(x \leq m) \geq \frac{1}{2}$
$p(x < 1) = 0$ ✓	$p(x \leq 1) = \frac{1}{4}$ ☒
$p(x < 2) = \frac{1}{4}$ ✓	$p(x \leq 2) = \frac{7}{16}$ ☒
$p(x < 3) = \frac{7}{16}$ ✓	$p(x \leq 3) = \frac{37}{64}$ ✓
$p(x < 4) = \frac{37}{64}$ ☒	$\vdots$ ✓
$\vdots$ ☒	

Thus x=3 is median

Example (27) Find the median for the following distribution.

$f(x) = 3x^2 \qquad 0 < x < 1$

Solution /

$\int_0^m 3x^2 = \frac{1}{2} \rightarrow x^3 I_0^m = \frac{1}{2} \rightarrow m^3 - 0 = \frac{1}{2} \Rightarrow m = \frac{1}{\sqrt[3]{2}}$

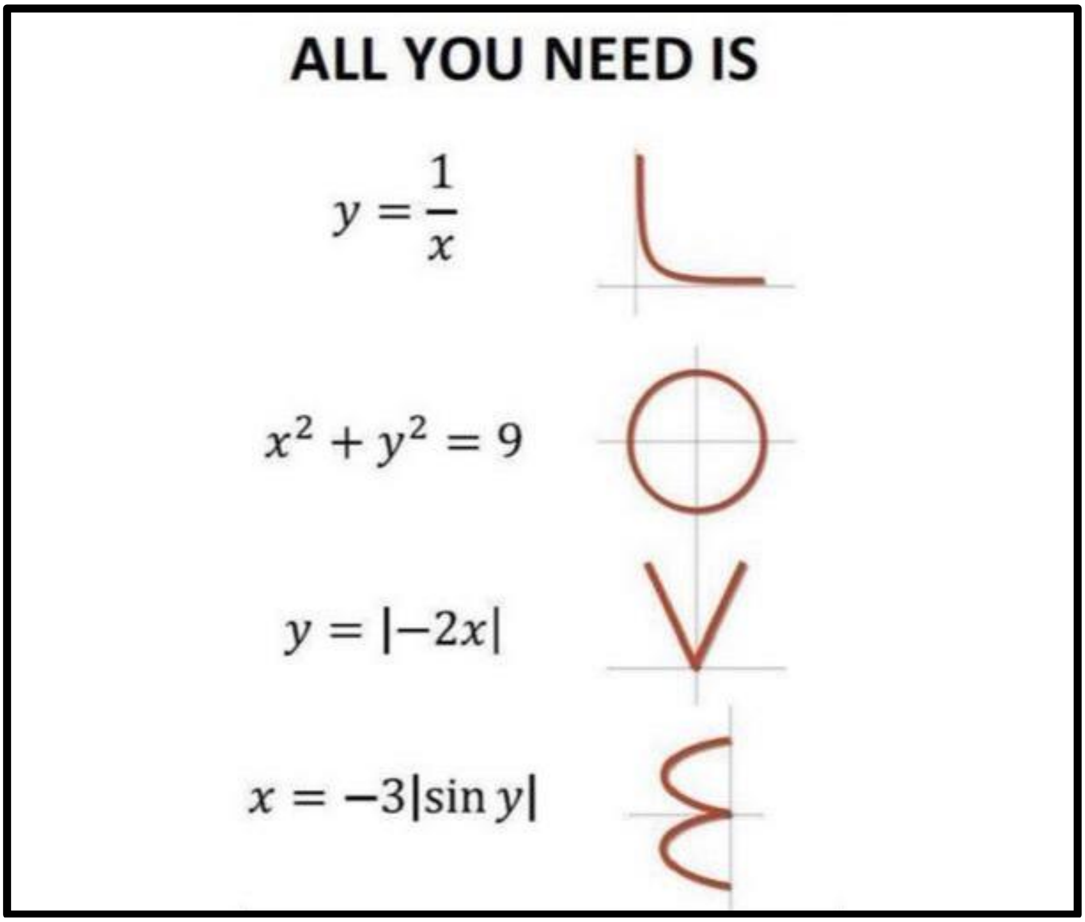
Example (28) Find the median for the following distribution.

$f(x) = 3(1-x)^2 \qquad 0 < x < 1$

Solution /

$$\int_0^m 3(1-x)^2 dx = \frac{1}{2} \qquad \frac{-3(1-x)^3 I_0^m}{3} = \frac{1}{2}$$

$$-(1-m)^3 - 1 = \frac{1}{2} \qquad m = 1 - \frac{1}{\sqrt[3]{2}}$$



Exercises of Chapter One

Probability and random variable exercises.

Exercises (1) : - let x be a r.v. with p.m.f. of x;

$$P(x) = \begin{cases} \frac{c}{x!(4-x)!}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{4-x} & x = 0,1,2,3,4 \\ 0 & o.w \end{cases}$$

find the value of the constant c.

Result / $c = 4!$

Exercises (2) : - let x be a r.v. with p.m.f. of P(x);

$$P(x) = \begin{cases} \frac{(cx+1)}{100} & x = 0,1,2,\dots,10 \\ 0 & o.w \end{cases} \qquad find\ c?$$

Exercises (3) : - let x be a r.v. with p.d.f. of f(x);

$$f(x) = \begin{cases} \frac{c}{m} & 0 < x < m \\ 0 & o.w \end{cases}$$

Find the value of the constant c?

Result / $c = 1$

Exercises (4) : - let x be a r.v. with p.m.f. of P(x);

$$P(x) = \begin{cases} \left(\frac{1}{2}\right)^{x+1} & x = 0,1,2,\dots \\ 0 & o.w \end{cases}$$

1) check $P(x)$ is a p.m.f.
2) find $Pr(x \geq 2)$.
3) $Pr(0 < x < 3)$.
4) $Pr(2 < x \leq 4)$.
5) $Pr(2 \leq x \leq 5)$.

Result /

1) $\sum_0^\infty \left(\frac{1}{2}\right)^{x+1} = \frac{1}{2}\sum_0^\infty \left(\frac{1}{2}\right)^{x} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1 \quad \therefore p(x)\ is\ a\ p.m.f$

2) $p(x \geq 2) = \frac{1}{4}$ **3)** $p(0 < x < 3) = \frac{3}{8}$ **4)** $p(2 \leq x \leq 5) = \frac{15}{64}$ **5)** $p(2 < x \leq 4) = \frac{3}{32}$

Exercises (5) : - Let

$$f(x) = \begin{cases} e^{-x} 0 < x < \infty \\ 0 \qquad\qquad o.w \end{cases}$$

1) $check\ f(x) is\ a\ p.d.f.of x.$
2) $find\ Pr(0 < x < 1).$
3) $Pr(x = 3).$
4) $Pr(3 \le x \le 5).$

Result /

1) $\int_0^\infty e^{-x} dx = 1$
2) $p(0 < x < 1) = 1 - e^{-1}$
3) $p(x = 3) = 0$
4) $p(3 \le x \le 5) = e^{-3} - e^{-5}$

Exercises (6): - let x be a r.v. with p.m.f of p(x):

$$P(x) = c\left(\frac{3}{4}\right)^x \quad x = 0,1,2,\dots$$

$$= 0 o.w$$

$find\ the\ value\ of\ the\ constant\ of$ c?

Result /$c = \frac{1}{4}$

C.d.f exercises.

Exercises (7) : - Let

$$f(x) = \begin{cases} \frac{x}{15} & x = 1,2,3,4,5 \\ 0 & o.w \end{cases}$$

$find\ the\ c.d.f.\ \&\ Graphical\ of\ F(x)$

Result F(x)=
$$\begin{bmatrix} 0 & x < 1 \\ \frac{1}{15} & 1 \le x < 2 \\ \frac{3}{15} & 2 \le x < 3 \\ \frac{6}{15} & 3 \le x < 4 \\ \frac{10}{15} & 4 \le x < 5 \\ 1 & x \ge 5 \end{bmatrix}$$

Exercises (8) : - let

$$f(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$find\ the\ c.d.f.\ \&\ graphical.$

Result $F(x)=\begin{bmatrix} 0 & x<0 \\ 1-(1-x)^3 & 0\le x<1 \\ 1 & x\ge 1 \end{bmatrix}$

Exercises (9) : - let x have c.d.f given by:

$$F(x)=\begin{cases} 0 x<0 \\ 1-e^{-x}-xe^{-x} 0<x<\infty \\ 1 x\to\infty \end{cases}$$

1) $find the p.d.f. of x.$
2) $Pr(1<x<3)$.
3) $Pr(x>3)$.

To Find p.d.f. from c.d.f.

f(x) = F(x) – F(x-1)

Result /

1)find pdf of x?

f(x)=xe^{-x}

2)Pr(1<x<3)= 0.537

3)P(x>3)= $2e$

Mode exercises

Exercises (10) Find the mode for the following distribution.

$$f(x)=C_x^5\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x} \quad ,x=0,1,2,3,4,5$$

Result /mode is x=2 and 3

Exercises (11) Find the mode for the following distribution.

$$f(x)=\frac{x}{6} \quad ,x=1,2,3$$

Result / $x=3$ *it is mode*

Exercises (12) Find the mode for the following distribution.

$f(x)=e^{-x} \qquad x>0$

Result /

$$x=\infty$$

Have not mode because result =∞

Median exercises

Exercises (13) Find the median for the following distribution.

$f(x)=\frac{x}{6} \qquad ,x=1,2,3$

Result / X=2,3 is median

Exercises (14) Find the median for the following distribution.

$f(x)=C_x^4\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} \qquad x=0,1,2,3,4$

Result / X=1 is median

Exercises (15) Find the median for the following distribution.

$f(x)=e^{-x} \qquad x>0 \;\; Or \;\; 0<x<\infty$

Result / $m=0.69$

Exercises (16) Find the median for the following distribution.

let $x \sim uniform(\theta_1,\theta_2)$ *another words*

$f(x)=\frac{1}{\theta_2-\theta_1} \qquad \theta_1<x<\theta_2$

Result / $m=\frac{\theta 2+\theta 1}{2}$

CHAPTER (2)

(Mathematical Expectation & Moment)

(Discrete Distribution)

Subjects
2 – 1 Mathematical Expectation.
2 – 2 Moment.
2 – 3 Cheby SHev's Inquality.
2 – 4 The Moment Generating Function (m.g.f)

2 – 1 Mathematical Expectation

Let x be a random variables and let Q(x) be a function of x, then the mathematical expectation of Q(x) is denoted by E[Q(x)],

$$E[Q(x)] = \begin{cases} \sum_{R_x} Q(x)P(x) \, if x is discrete r.v. \\ \int Q(x)f(x)dx if x is continuous r.v. \end{cases}$$