* Properties of Expectation: -

1) $E\,a = a$ if a is a constant (real number), then

2) If a is a constant, and $Q(x)$ is a fun. of x, then
$E[a\{Q(x)\}] = a\,E[Q(x)]$

3) If $a_1 \& a_2$ are two constants, then;
$E(a_1 x + a_2) = E\,a_1 . E(x) + E\,a_2$
$E(a_1 x + a_2) = a_1\,E\,x + a_2$

4) If $a_1 \& a_2$ are two constants, and $Q_1(x) \& Q_2(x)$ are two functions of x then;
a) $E[Q_1(x) \mp Q_2(x)] = E[Q_1(x)] \mp E[Q_2(x)]$
b) $E[a Q_1(x) \mp b Q_2(x)] = a\,E[Q_1(x)] \mp b\,E[Q_2(x)]$

5) If $y = x_1 + x_2 + x_3 \Rightarrow Ey = Ex_1 + Ex_2 + Ex_3$

6) If x and y are two random variables then;
$E(xy) = E(x).E(y)$ if $x \& y$ are independent
$E(xy) \neq E(x).E(y)$ if $x \& y$ are dependent

7) Let $y = \sum x_i \Rightarrow E(y) = \sum E(x_i) \Rightarrow E(y) = nE(x)$

8) $\sum_{\forall x} |u(x)| p(x) < \infty$

9) $E(\overline{x}) = M$

10) $E(x^2) \geq [E(x)]^2$

2 – 2 Moment

1 – Non – Central Moment: -

In special case of non – central moment is the moment about the origin i.e (when a=0, then the r-th non – central moment about origin is a=0)

$$Mr = E(x)^r = \begin{cases} \sum x^r p(x) xis disc.r.v. \\ \int x^r f(x) dx xis cont.r.v. \end{cases}$$

2 – Central Moment :

Def [n] : let x be a r.v. with p.d.f f(x), then the r-th central moment of x about mean (M) is denoted by Mr', and is defined as ;

$$Mr' = E(x-m)^r = \begin{cases} \sum_{\forall x} (x-m)^r p(x) for dis.r.v. \\ \int_{R_x} (x-m)^r f(x) dx for cont.r.v. \end{cases}$$

$$r = 2 \rightarrow M_2{}' = E(x-M)^2 = \sigma_x^2 \Rightarrow var\,i\,ance$$

$$r = 3 \rightarrow M_3{}' = E(x-M)^3 = Skewnes$$

$$r = 4 \rightarrow M_4{}' = E(x-M)^4 = kurtosis$$

Then;

$$Coefficient of Skewnes = \frac{E(x-M)^3}{\sigma^3} = \frac{M_3{}'}{\sigma^3} = \alpha_3$$

$$Coefficient of Kurtosis = \frac{E(x-M)^4}{\sigma^4} = \frac{M_4{}'}{\sigma^4} = \alpha_4$$

Relation between non – central moment (Mr) & central moment (Mr')

$$r = 2 \Rightarrow M_2{}' = E(x - M)^2 = \sigma^2$$

$$M_2{}' = E(x)^2 - 2ME(x) + M^2$$

$$M_2{}' = E(x)^2 - M^2$$

$$M_2{}' = M_2 - M_1^2$$

$$\therefore M_2 = M_2{}' + M_1^2$$

Example (1) : - Let x have the p.d.f

$$f(x) = \frac{x+2}{18} - 2 < x < 4$$

$$= 0 o.w$$

$$find: 1) E(x) 2) E[(x+2)^3] 3) E[(6x - 2(x+2)^3]$$

Result /

1)E(x)=?

E(x)=$\int_{-2}^{4} x f(x) dx = \int_{-2}^{4} x \left(\frac{x+2}{18}\right) dx = \frac{x^3}{54} + \frac{x^2}{18} I_{-2}^{4} = \left(\frac{4^3}{54} + \frac{4^2}{18}\right) - \left(\frac{-2^3}{54} + \frac{-2^2}{18}\right) = 2$

2)E(x+2)$^3 = \int_{-2}^{4} (x+2)^3 f(x)\, dx = \int_{-2}^{4} \frac{(x+2)^4}{18} dx = \frac{(x+2)^5}{18*5} I_{-2}^{4} = \frac{6^5}{90} = 86.4$

3)E(6x-2(x+2)$^3 = 6E(x) - 2E(x+2)^3 = 6 * 2 - 2 * 86.4 = -160.8$

Example(2) : - Let x be a r.v. with

$$f(x) = 2(1 - x) 0 < x < 1$$

$$= 0 o.w$$

$$find: 1) M_1 \& M_2 and E[(6x + 3x^2)] and \sigma^2$$

Result

1)M1&M2?

M1=E(x)=$\int_0^1 x(2(1-x))dx = \int_0^1 (2x - 2x^2)dx = x^2 - \frac{2x^3}{3} I_0^1 = 1 - \frac{2}{3} - 0 = \frac{1}{3}$

M2=E(x^2) $= \int_0^1 x^2$f(x)dx=$\int_0^1 (2x^2 - 2x^3)dx = \frac{2x^3}{3} - \frac{2x^4}{4} I_0^1 = \frac{2}{3} - \frac{2}{4} - (0) = \frac{1}{6}$

3)E(6x+3x^2)=6E(x)+3E(x^2)=6*$\frac{1}{3}+3*\frac{1}{6}=\frac{5}{2}$

4)$\sigma^2=v(x)=E(x^2)$-(E(x))^2=$\frac{1}{6}-(\frac{1}{3})^2$=$\frac{1}{6}-\frac{1}{9}=\frac{3}{54}=\frac{1}{18}$

Example(3) : - Let x be a r.v. with p.d.f of ;

$$f(x)=\begin{cases}\frac{1}{2m} -m<x<m\\ 0 o.w\end{cases}$$

$$Find the\ variance of dist. of x.$$

Result

1)V(x)=E(x^2)-{E(x)}^{2}=$\frac{m^2}{3}-0=\frac{m^2}{3}$

E(x)=$\frac{1}{2m}\int_{-m}^{m} xdx=\frac{1}{2m}*\frac{x^2}{2}I_{-m}^{m}=\frac{1}{2m}\left(\frac{m^2}{2}-\frac{(-m)^2}{2}\right)=\frac{1}{2m}(0)=0$

E(x^2)=$\frac{1}{2m}\int_{-m}^{m}x^2$ dx=$\frac{1}{2m}\left(\frac{x^3}{3}\right)I_{-m}^{m}=\frac{1}{2m}\left(\frac{m^3}{3}+\frac{m^3}{3}\right)=\frac{1}{2m}*\frac{2m^3}{3}=\frac{m^2}{3}$

Example(4) : - let

$$f(x)=\begin{cases}3e^{-3x} 0<x<\infty\\ 0 o.w\end{cases}$$

$$find the mean and the\ variance of dist. of x.$$

Result

E(x)=$\int_0^\infty 3xe^{-3x}$dx=3($\frac{xe^{-3}}{3}-\frac{e^{-3x}}{9}$)$I_0^\infty=3*\frac{1}{9}=\frac{1}{3}$

E(x^2)=$\int_0^\infty 3x^2e^{-3x}dx=3\left(\frac{-x^2e^{-3x}}{3}-\frac{2xe^{-3x}}{9}-\frac{2e^{-3x}}{27}\right)I_0^\infty=\frac{2}{9}$

V(x)=E(x^2)-(E(x))^2=$\frac{2}{9}-\left(\frac{1}{3}\right)^2=\frac{1}{9}$

Example(5) : - Let

$$f(x)=\begin{cases}2x0<x<1\\ \\0o.w\end{cases}$$

$$bethep.d.fofxcomputeE(\sqrt{x})\&p.d.fofy=\sqrt{x}$$

Result

be the p d f of x compute E($\sqrt{x}$)& $pdf\ of\ y=\sqrt{x}$
thus $y^2=x$

E($\sqrt{x}$) $=\int_0^1 2x*\sqrt{x}=\frac{4x^{\frac{5}{2}}}{5}I_0^1=\frac{4}{5}-0=\frac{4}{5}$

F(Y) =P (Y$\leq y)=pr(\sqrt{x\leq y)=p(x\leq y^2}$

F(y)=$\int_0^{y^2}2x=x^2I_0^{y^2}=y^4$

f(y)=$4y^3$

f(y)=$\begin{bmatrix}4y^3 & 0<y<1\\ 0 & o.w\end{bmatrix}$

Example(6) : - for each of the following p.d.f compute ;

$$Pr(\mu-\sqrt{20}\sigma<x<\mu+\sqrt{20}\sigma)$$

$$a)f(x)=\begin{cases}6x(1-x)0<x<1\\ \\0o.w\end{cases}$$

$$b)p(x)=\begin{cases}\frac{1}{66}xx=1,2,3,...,11\\ \\0o.w\end{cases}$$

Result

$$\rightarrow E(x)=\int_0^1 x\ (6x-6x^2)dx=\int_0^1(6x^2-6x^3)dx=2x^3-\frac{3x^4}{2}I_0^1=\frac{6}{12}$$

E(x^2) $=\int_0^1 x^2(6x-6x^2)dx=\int_0^1(6x^3-6x^4)dx=\frac{3x^4}{2}-\frac{6x^5}{5}I_0^1=\frac{6}{20}$

Thus V(x)=$\frac{6}{20}-\left(\frac{1}{2}\right)^2=\frac{1}{20}\rightarrow\sigma=\frac{1}{\sqrt{20}}$

Pr($\frac{1}{2}-\sqrt{20}*\frac{1}{\sqrt{20}}<x<\frac{1}{2}+\sqrt{20}*\frac{1}{\sqrt{20}}$) $=$ Pr $(\frac{-1}{2}<x<\frac{3}{2})$ =?

Pr $(\frac{-1}{2}<x<\frac{3}{2})=\int_{-\frac{1}{2}}^{0}f(x)dx+6\int_{0}^{1}f(x)dx+\int_{1}^{\frac{3}{2}}f(x)dx=0+1+0=1$

Out range all range out range

b)p(x)=$\frac{x}{66}$ $x=1,2,3,\cdots,11$

$\sum x^2=\frac{n(n+1)(2n+1)}{6}$

$\sum x^3=\left(\frac{n(n+1)}{2}\right)^2$

Ex)=$\sum_1^{11}x*\frac{x}{66}=\frac{1}{66}*\frac{11(11+1)(2*11+1)}{6}=7.66$

E(x^2)=$\frac{1}{66}\sum_1^{11}x^3=\left(\frac{11(11+1)}{2}\right)^2=66$

V(x)=E(x^2)-(E(x))^2=66-(7.66)^2=7.32$\rightarrow\sigma=\sqrt{7.32}=2.7$

Pr(7.66-$\sqrt{20}*2.7<x<7.66+\sqrt{20}*2.7)=$ Pr $(-4.44<x<19.96)$ =?

Pr(-4.44<x<19.96)=$\sum_{-4.44}^{1}f(x)+\sum_1^{11}f(x)+\sum_{11}^{19.96}f(x)=0+1+0=1$

Out range all range out range

Example(7) : - Let

$$f(x)=\begin{cases}|1-x| & -1<x<2\\ 0 & o.w\end{cases}$$

$find\ mean\ \&\ variance\ of\ dist.\ of\ x.$

Result

1) E(x) =$\int_{-1}^{1}x(1-x)dx+\int_1^2 x(-1+x)dx=\int_{-1}^{1}x-x^2+$

$\int_1^2-x+x^2=\frac{x^2}{2}-\frac{x^3}{3}\ I_{-1}^{1}+(\frac{-x^2}{2}+\frac{x^3}{3})\ I_1^2=\frac{1}{6}$

$$\rightarrow E(x^2)=\int_{-1}^{1}x^2-x^3+\int_1^2-x^2+x^3=\frac{x^3}{3}-\frac{x^4}{4}\ I_{-1}^{1}\ \frac{-x^3}{3}+\frac{x^4}{4}\ I_1^2$$

$$=\frac{25}{12}$$

2)v(x)=E(x^2) - (E(x))^2 $=\frac{25}{12}-(\frac{1}{6})^2 \quad =\frac{25}{12}-\frac{1}{6}=\frac{37}{18}$

Example(8) : - Find the mean & variance of the dist. that the following c.d.f .

$$F(x)=\begin{cases}0 x<0\\ \frac{x}{8} 0\le x<2\\ \frac{x^2}{16} 2\le x<4\\ 1 x\ge 4\end{cases}$$

Result

$$\text{F (x)}=\begin{bmatrix}0 & x<0\\ \frac{x}{8} & 0\le x<2\\ \frac{x^2}{16} & 2\le x<4\\ 1 & x\ge 4\end{bmatrix}\rightarrow F'(x)=\begin{bmatrix}\frac{1}{8} & 0\le x<2\\ \frac{x}{8} & 2\le x<4\\ 0 & o.w\end{bmatrix}\text{=f(x)}$$

E(x)=$\int_0^2 x*\frac{1}{8}dx+\int_2^4 x*\frac{x}{8}dx=\frac{x^2}{16}I_0^2+\frac{x^3}{24}I_2^4=\frac{2^2}{16}-0+\frac{4^3}{24}-\frac{2^3}{24}=\frac{31}{12}$

E(x^2)=$\int_0^2 x^2*\frac{1}{8}+\int_2^4 x^2*\frac{x}{8}=\frac{x^3}{24}I_0^2+\frac{x^4}{32}I_2^4=\frac{2^4}{24}-0+\frac{4^4}{32}-\frac{2^4}{32}=\frac{47}{6}$

V(x)=E(x^2)-(E(x))^2=$\frac{47}{6}-\left(\frac{31}{12}\right)^2=\frac{167}{144}$

Example (9) : - Let

$$p(x)=\begin{cases}e^{-x} 0<x<\infty\\ 0 o.w\end{cases}$$

$$find: 1)Mean 2)Variance \sigma_x^2$$

Result

f(x)=$\begin{bmatrix}e^{-x} & 0<x<\infty\\ 0 & o.w\end{bmatrix} \qquad \int_0^\infty x^{r-1}e^{-x}=\gamma r$

E(x)=$\int_0^\infty xe^{-x}dx=\gamma 2=1!=1$

E(x^2)=$\int_0^\infty x^2e^{-x}dx=\gamma 3=2!=2$

V(x)=E(x^2)-(E(x))^2=$2-1^2=1$

2 – 3 Cheby Shev's Inequality

Theorem : Let u(x) be a non – negative function of the r.v. (x) [i.e u(x) ≥ 0]

If E[u(x)] exists, then; for every positive constant C, (C > 0), then

$$P[u(x) \geq C] \leq \frac{E[u(x)]}{C}$$

Theorem : Cheby Shev's In quality

Let a r.v. x have probability Distribution function, which we assume only that there is a finite variance (σ^2) and (μ). Then for every K > 0 .

$$P[|x-m| \geq K\sigma] \leq \frac{1}{K^2} \ldots\ldots\ldots\ldots\ldots\ldots upperboundedand;$$

$$P[|x-m| < K\sigma] \geq 1-\frac{1}{K^2} \ldots\ldots\ldots\ldots\ldots lowerbounded$$

Where K is constant (positive number) to be greater than one.$Pr[\,u(x) \geq C] \leq \frac{E[u(x)]}{C}$

$$Pr[\,|x-m| \geq K\sigma] \leq \frac{1}{K^2}$$

1) $Pr[\,|x-m| \geq K\sigma] \leq \frac{1}{K^2} \ldots\ldots\ldots\ldots\ldots\ldots Upperb.$

$$\therefore Pr(\mu + k\sigma \leq x \leq \mu - k\sigma) \leq \frac{1}{k^2} \ldots\ldots\ldots\ldots U.b$$

$$2)\ Pr[\,|x-m|<k]\geq 1-\frac{1}{k^2}\ldots\ldots\ldots\ldots\ldots Lowerb.$$

$$\therefore Pr(\mu-k\sigma<x<\mu+k\sigma)\geq 1-\frac{1}{k^2}\ldots\ldots\ldots L.b.$$

Example(10) : - Let x be a r.v. and E(x) = 3 , $E(x)^2 = 13$ use cheby shev's inequality to determine lower bounded for the probability P(-2< x <8)

Result

$\Pr(\mu-\sigma k<x<\mu+\sigma k)=\Pr(3-\sigma k<x<3+\sigma k)\geq 1-\frac{1}{k^2}$

$V(x)=\sigma^2=E(x^2)-\left(E(x)\right)^2=13-3^2=4\ thus\ \sigma=2$

3-2k=-2 and 3+2k=8

$K=\frac{5}{2}\quad and\quad k=\frac{5}{2}$

$\Pr(-2<x<8)=1-\frac{1}{k^2}=1-\frac{1}{\left(\frac{5}{2}\right)^2}=\frac{21}{25}$

Example(11) : - Let x be a r.v. with p.d.f of f(x) ;

$$f(x)=\begin{cases}\frac{1}{2\sqrt{3}}-\sqrt{3}<x<\sqrt{3}\\ 0o.w\end{cases}$$

$$find the lower bounded for the probability of this\ int\ e\ rval.$$

Result

$E(x)=\int_{-\sqrt{3}}^{\sqrt{3}}x*\frac{1}{2\sqrt{3}}dx=\frac{x^2}{4\sqrt{3}}I_{-\sqrt{3}}^{\sqrt{3}}=0$

$E(x^2)=\int_{-\sqrt{3}}^{\sqrt{3}}x^2*\frac{1}{2\sqrt{3}}dx=\frac{x^3}{6\sqrt{3}}I_{-\sqrt{3}}^{\sqrt{3}}=1$

$\Pr(0-k<x<0+k)\geq 1-\frac{1}{k^2}\quad thus -k=-\sqrt{3}\ and\ k=\sqrt{3}$

Thus $k=\sqrt{3}$

$\Pr(-\sqrt{3}<x<\sqrt{3})=1-\frac{1}{(\sqrt{3})^2}=\frac{2}{3}$

Example(12): - If x be a r.v. with E(x)=μ , such that Pr(x < 0) = 0

Show that $Pr(x\geq 2m)\leq\frac{1}{2}$

$$by eu sin g the first theory$$

Result

$\Pr(u(x) \geq c) \leq \frac{E(u(x))}{c}$ $\therefore U(x) = x \;\; and \; c = 2m$

$\Pr(x \geq 2m) \leq \frac{E(x)}{2m} \rightarrow pr(x \geq 2m) \leq \frac{m}{2m} = \frac{1}{2}$

Example(13) : - Let x be a r.v. with p.d.f

X	-1	0	1
P(x)	$\frac{1}{8}$	$\frac{6}{8}$	$\frac{1}{8}$

Use cheby shev's inequality to determine an upper bounded of [P(x ≥ -1)] &

[P(x ≤ -1)] ⟷ Pr(1 ≤ x ≤ -1)

Result

X: -1 0 1

P(x): $\frac{1}{8}$ $\frac{6}{8}$ $\frac{1}{8}$

Use cheby shev's inequality to determine an upper bounded of [P(x ≥ +1)] & [P(x ≤ -1)]↔ Pr(1 ≤ x ≤ -1)

$E(x)=\sum_{-1}^{1} xP(x) = -1 * \frac{1}{8} + 0 + \frac{1}{8} * 1 = 0$

$E(x^2)=\sum_{-1}^{1} x^2P(x) = -1^2 * \frac{1}{8} + 0 + \frac{1}{8} * 1^2 = \frac{2}{8}$

$V(x)=\frac{2}{8} - 0 = \frac{2}{8} = \frac{1}{4} \rightarrow \sigma = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$\Pr(0+\frac{1}{2}k < x < 0 - \frac{1}{2}k) \leq \frac{1}{k^2}$

$$\frac{1}{2}k = 1 \quad and \; -\frac{1}{2}k = -1 \quad thus \; k = 2$$

$\Pr(1 \leq x \leq -1) = \frac{1}{k^2} = \frac{1}{2^2} = \frac{1}{4}$

2 – 4 The Moment Generating Function (m.g.f)

Let x be a r.v. with p.d.f of f(x), the moment generating function (m.g.f) of x denoted by [$M_x(t)$], and is defined for all real value of $t(-h < x < h)$ [where h is a positive number]. Such that the mathematical expectation Ee^{tx} exist, thus ;

$$M_x(t) = Ee^{tx} = \begin{cases} \sum_{\forall x} e^{tx} p(x) \; if x is disc. r.v. \\ \int_{R_x} e^{tx} f(x) dx \; if x is cont. r.v. \end{cases}$$

Definition : -

Let $M_x(t)$ be the moment generation function of a r.v. x that is existed. Then the moment of r-th order about origin at (t = 0) is define as ;

$$E\,x^r = M_x^r(0) = \frac{\partial^r M_x(t)}{\partial t^r}\Big|_{t=0} = M_r \qquad r = 0,1,2,\ldots$$

$$M_x(t) = E\,e^{tx}$$

Bye using maclaurin's expansion factorial

$$Ee^{tx} = E\left[\sum_{k=0}^{\infty} \frac{(tx)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(x)^k$$

$$Ee^{tx} = 1 + tE(x) + \frac{t^2}{2} E(x)^2 + \frac{t^3}{3!} E(x)^3$$

We can generate the moment of r-th order about origin by find the derivative of r-th order of $M_x(t)$ fun. with respect to t. then we make (t=0)

$$M_r = E\,x^r = \frac{\partial^r M_x(t)}{\partial t^r}\Big|_{t=0}$$

The properties of moment generating function (m.g.f)

1) If x be a r.v. and $M_x(t)$ exist, and let $y_1 = a + bx or y_2 = x + a[a, b \neq 0 constant] then;$

$$M_{y_1}(t) = e^{at} M_x(bt) and M_{y_2}(t) = e^{at} M_x(t)$$

2) Let $y = bx; then M_y(t) = M_x(bt)$

3) Let $y = \frac{x+a}{b}, then M_y(t) = e^{\frac{a}{b}t} M_x(t)$

4) The m.g.f of standard degree Z about the origin is;

$$M_z(t) = e^{\frac{-m}{\sigma}} M_x(\frac{t}{\sigma})$$

5) If $M_x(t)$ exist; then $k_k(t) = ln\, M_x\,(t)$ exist, where $K_x(t)$ is called cumulate generation function, then

$$K_x''(0) = var(x), K_x'(0) = E(x) = M = Mean$$

6) If $M_x(t)$ exist and E(x) defined, then $M_x(t)$ be minimum value at (t=0), and then this function shall have minimum value equal to $[Ee^{0x}]$=1

Example(14) : - Let x be a r.v. with p.d.f

$$f(x) = \begin{cases} xe^{-x} x > 0 \\ 0 o.w \end{cases}$$

$find: 1) M_x(t) 2) M 3) \sigma_x^2$

Result

f(x)= xe^{-x} $\quad 0 < x < \infty \quad \sum x^{r-1} e^{-bx} = \frac{\gamma r}{b^r}$

$$\rightarrow \quad Mx(t) = \int_0^\infty xe^{-x} \,.\, e^{tx} dx$$

$$= \int_0^\infty x \; e^{-x(1-t)} dx = \frac{\sqrt{2}}{(1-t)^2} = \frac{1}{(1-t)^2} \quad I_{t=0} = 1$$

$$\rightarrow \quad M_x^1\,(t) = E(x) = (1-t)^{-2} \; = \frac{2}{(1-t)^3} \; I_{t=0} = 2$$

$$\rightarrow \quad M_x^2(t) \; = \frac{d}{dt} E2(1-t)^{-3} = \frac{6}{(1-t)^4} \quad I_{t=0} = 6$$

$\rightarrow \quad v(x) = 6 - 2^2 = 2$

Example(15): -

$$f(x) = \begin{cases} \frac{1}{m} 0 < x < m \\ \\ 0 o.w \end{cases}$$

$$Find the m.g.f of x.$$

Result

Mx(t)=$\frac{1}{m}\int_0^m e^{tx} dx = \frac{e^{tx}}{mt} I_0^m = \frac{e^{tm}-1}{mt}$ *using law lobital* $\lim_{n\to\infty} \frac{fx'}{gx'}$

=$\frac{e^{tm}-1}{mt} = \frac{me^{tm}}{m} I_{t=0} = 1$

Example(16) : - Let ;

$$f(x) = \begin{cases} e^{-x} 0 < x < \infty \\ \\ 0 o.w \end{cases}$$

$$Find M_x(t) m.g.f of x M_1, M_2, Var(x)$$

Result

Mx(t)=$\int_0^\infty e^{-x} e^{tx} dx = \int_0^\infty e^{-x(1-t)} dx = \frac{e^{-x(1-t)}}{1-t} I_0^\infty = \frac{1}{1-t} I_{t=0} = 1$

Mx'(t)=$(1 - e^t)^{-2} I_{t=0} = 1$

Mx''(t)=$2(1 - t)^{-3} I_{t=0} = 2$

V(x)=2-1^2=1

Example(17) : - Let x be a r.v. with moment generating function

$$M_x(t), (-h < t < h), Pr\,o\,ve that;$$

$$Pr(\,x \geq a) \leq e^{-at} M_x(t) 0 < t < h$$

$$and$$

$$Pr(\,x \leq a) \leq e^{-at} M_x(t) - h < t < 0$$

Result

Pr(x$\leq a) < e^{-at}\ M_x\ (t) \qquad - n < t < 0$
Solved : -
Let u(x)=e^{tx} $\quad and \quad c = e^{at}$
Pr($e^{tx} \geq e^{at}$) $\leq \frac{E(e^{tx})}{e^{ta}} \quad \rightarrow \quad pr\ (tx) \geq at\ ) = \frac{1}{e^{at}}\ M_x(t) = e^{-at}\ Mx(t)$

$if \quad 0 < t < h$
$\therefore \quad pr(tx \geq at\ ) \leq e^{-at} \quad Mx(t)$
$\therefore \quad pR(x \geq a)\ \leq e^{-at} \quad Mx(t)$
$But \quad - h < t < O$
$\rightarrow \quad pr(tx \geq -a) \leq \quad e^{-at} \quad Mx(t)$
$pr(x \leq a) \leq e^{-at} \quad Mx(t)$

Example(18) : - let

$$f(x) = \begin{cases} \frac{1}{2} e^{-|x|} - \infty < x < \infty \\ \\ 0 o.w \end{cases}$$

$$Find the M_x(t) of x.$$

Result

f(x)=$\frac{1}{2}e^{-x} \quad 0 < x < \infty \quad and\ \frac{1}{2}e^{x} \quad -\infty < x < 0$

Mx(t)=$\frac{1}{2}\int_0^\infty e^{-x}e^{tx}dx + \frac{1}{2}\int_{-\infty}^0 e^{x}e^{tx}dx = \frac{1}{2}\left(\frac{e^{-x(1-t)}}{1-t} I_0^\infty\right) + \frac{1}{2}\left(\frac{e^{x(1-t)}}{1-t} I_{-\infty}^0\right) I_{t=0} = 1$

Example(19) : - Let x be a r.v. with p.m.f of p(x);

$$f(x) = \begin{cases} \frac{3^x e^{-3}}{x!} x = 0,1,2,\ldots \\ 0 o.w \end{cases}$$

$$find: 1) M_x(t) 2) K_x(t) 3) Mean \& Variance$$

$$4) if y = 2 - 3x find M_y(t)$$

$$5) find Mean \& Variance of y.$$

Result

1)Mx(t)=?

Mx(t)=$\sum_0^\infty \frac{e^{-3}3^x}{x!} * e^{tx} = e^{-3} \sum_0^\infty \frac{(3e^t)^x}{x!} = e^{-3} * e^{-3e^t} = e^{-3+3e^t} I_{t=0} = 1$

2)Kx(t)=?

Kx(t)=ln$e^{-3+3e^t} = -3 + 3e^t I_{t=0} = 0$

3)Mx'(t)=?

Mx'(t)=$(e^{-3+3e^t})' = 3e^t * (e^{-3+3e^t}) I_{t=0} = 3$

3)Mx"(t)=$(e^{-3+3e^t})" = e^{-3+3e^{3t}} * 3e^t * 3e^t + 3e^t * e^{-3+3e^{3t}} I_{t=0} = 12$

3)V(x)=12-3^2=3

4)if y=2-3x find My(t)=?

My(t)=$e^{2t} Mx(-3t) = e^{2t-3+3e^{-3t}}$

5)find mean and variance y?

My'(t)=$\left(e^{2t-3+3e^{-3t}} \right)' = (2 - 9e^t)\left(e^{2t-3+3e^{-3t}}\right) I_{t=0} = -7$

My"(t)= $\left(e^{2t-3+3e^{-3t}} \right)" = \left(e^{2t-3+3e^{-3t}} \right) * (2 - 9e^t) * (2 - 9e^t) + 9e^t * \left(e^{2t-3+3e^{-3t}} \right) I_{t=0} = 76$

V(x)=76-(-7)^2=27

Exercise of Chapter Two

Exapmle(1) : - let x have a p.m.f of p(x) that is positive x = -1,0,1 and is zero else where

$$a) if\ p(0) = \frac{1}{2}\ find\ E(x)^2$$

$$b) if\ p(0) = \frac{1}{2}\ and\ E(x) = \frac{1}{6}\ find\ p(-1) \& p(1)$$

Example(2) : - Let

$$f(x) = \frac{1}{5}\ x = 1,2,3,4,5$$

$$= 0\ o.w$$

$$Find: 1) E(x)\ 2) E(x)^2\ 3) E(x+2)^2$$

Example(3) : - Let x be a r.v. with p.m.f of p(x);

$$f(x) = \begin{cases} \frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3 & x = 0,1,2,3,\dots \\ 0 & o.w \end{cases}$$

$$Find\ mean \&\ variance\ of\ x.$$

Example(4) : - Let x have the p.m.f :

$$f(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & o.w \end{cases}$$

$$Find\ E(x)^3, E(x-1)$$

Example (5) : -

$$f(x)=\begin{cases}\frac{2}{x^3} 1<x<\infty \\ \\ 0 o.w\end{cases}$$

$$Find: 1) E(x) 2) \sigma^2$$

Example (6) : - let the r.v. x have standard deviation σ_x show that :

$$a) E\left(\frac{x-m}{\sigma}\right)=0 \quad b) E\left(\frac{x-m}{\sigma}\right)^2=1$$

Example (7) : - Let x be a r.v. with p.d.f of f(x);

$$f(x)=\begin{cases}2^x . \ln 2 -\infty<x<0 \\ \\ 0 o.w\end{cases}$$

$$Find mean \& variance of x.$$

Example (8) : - Let x be a r.v. having a p.d.f given by ;

$$f(x)=\begin{cases}\frac{5x^4}{\beta^5} 0<x<\beta \\ \\ 0 o.w\end{cases}$$

$$find the value of \beta such that E(6x-5)=0$$

Example (9) : - Let x be a r.v. such that $E[(x-b)^2]$ exists for all real b. Show that $E[(x-b)^2]$ is

$$a minimum when b=m=E(x).$$

Example (10) : - let x be a r.v. of f(x) is a p.d.f ;

$$f(x) = \frac{1}{2}(x+1) - 1 < x < 1$$

$$Find the mean \& var i ance of this dist. of x.$$

Example (11) : - let x be a r.v. with p.m.f of the p(x) ;

$$p(x) = \begin{cases} \frac{1}{n} x = 1,2,3,\dots,n \\ 0 o.w \end{cases}$$

$$find the mean \& var i ance of the dist.$$

Example (12) : - let

$$f(x) = \begin{cases} \beta x = 0,1,2,\dots,n \\ 0 o.w \end{cases}$$

$$find: 1) \beta ? 2) E(x) 3) E(x)^2$$

Example (13) : - Let x be a r.v. have p.d.f ;

$$p(x) = \begin{cases} p x = -1,1 \\ 1-2p x = 0 \\ 0 o.w \end{cases}$$

$here 0 < p$

$$< \frac{1}{2}. Find the measure of kurtosis as a function P. Deter min e its$$

$$value when p = \frac{1}{2}, p = \frac{1}{5}, p = \frac{1}{10}, and p = \frac{1}{100}$$

Example (14) : - Let

$$p(x) = \begin{cases} \binom{3}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{3-x} & x = 0,1,2,3 \\ 0 & o.w \end{cases}$$

$$find: 1) E(x) 2) E(x(x-1))$$

Example (15) : - Let a coin be tossed three times, let x be a r.v. of a head that occur. Find mean & variance of x.

Example (16) : - Let x be a r.v. of the continuous type that has p.d.f if m is the unique median of the distribution of x and (b) is a real constant, show that ;

$$E(|x-b|) = E(|x-m|) + 2\int_{m}^{b}(b-x)f(x)dx .$$

Example (17) : - Let x be a r.v. with mean and let $E(x-m)^{2k}$ exist, show that C>0, That ;

$$P_r[x-m) \geq C] \leq \frac{E(x-m)^{2k}}{c^{2k}}$$

Example (18) : - Let P(x) ~ P(λ)

Find m.g.f & M_1, M_2,V(x).

Example(19) : - Let x be distributed as ;

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o.w \end{cases}$$

1) Find the m.g.f of x.

2) Use it to find E(x).

3) and to find σ^2.

Example(20) : - Let

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1,2,3,\ldots \\ 0 & o.w \end{cases}$$

$$Find the M \& \sigma_x^2 by using the (m.g.f).$$

Example(21) : - A fair coin is tossed twice, let x be the number of heads that occur, Find ;

1)The m.g.f of x.

2) The mean & variance of x, by using the m.g.f of x.

Name:- Dler Hussein Kadir .... :ناو

ID Exam 000

Code & Group 0 (A, B, C)

Department of Statistics

Monthly Exam

Chapter Two

math3stat@gmail.com

Sub.: Mathematical Statistics

Date:- 23 – Dec.- 2010

Time: 100 minutes

Q1) ***Let x be a r.v having the p.d.f of x.*** **25 Marks**

$$f(x)=\begin{cases}\dfrac{2}{x^2} & 1<x<2\\ 0 & o.w\end{cases}$$

Show that $p_r(2x \geq \ln 4) \leq 2$?

Q2) ***Let X be ar.v having p.m.f of x and*** **25 Marks**

$$M_x(t)=\frac{e^t}{(2-e^t)}$$

Use Cheby-Shev's inequality to determine lower bounded for the $P_r(-2<x<6)$?

Q3) ***Let x be a r.v having the p.d.f of x*** **25 Marks**

$$f(x)=\begin{cases}e^{-x} & 0<x<\infty\\ 0 & o.w\end{cases}$$

Find:

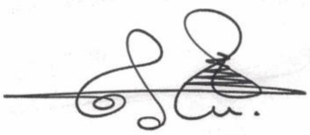
(1) $M_x(t)$?

(2) $K_x(t)$?

(3) If y=3-2x find $M_y(t)$?

(4) E(y)=? & v(y)=?

Q4) *Let a coin be tossed twice .Let x be the number of heads that Occur; Find the coefficient of skewnes(α_3)?* **25 Marks**



Dler Hussein Kadir

The examiner

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100

Best of Luck

Solution all equation

Q1) $f(x)=\frac{2}{x^2} \qquad 1<x<2$

$$p_r(2x \geq \ln 4) \leq 2$$

$$prof: p_r(u(x) \geq C) \leq \frac{E(u(x))}{C}$$

$$Let\, u(x)=2x \quad and \quad C=Ln4$$

$$p_r(2x \geq \ln 4) \leq \frac{E(2x)}{Ln4}$$

$$But \;\; E(2x)=2E(x)=2\int_1^2 xf(x)=2\int_1^2 x\frac{2}{x^2}dx=2\int_1^2 \frac{2}{x}dx=4Ln(x)\Big|_1^2$$

$$=4[Ln(2)-Ln(1)]=4[Ln(2)-0]=4Ln(2)$$

$$p_r(2x \geq \ln 4) \leq \frac{4Ln(2)}{Ln4} \leq \frac{4Ln(2)}{Ln2^2} \leq \frac{4Ln(2)}{2Ln2} \leq 2$$

****

Q2) $M_x(t)=\frac{e^t}{2-e^t}$

$$\because K_x(t)=Ln(M_x(t))=Ln(\frac{e^t}{2-e^t})=Ln(e^t)-Ln(2-e^t)$$

$$=tLn(e)-Ln(2-e^t)=t-Ln(2-e^t)$$

$$P_r(m-k\sigma<X<m+k\sigma) \geq 1-\frac{1}{K^2}$$

$$m=E(x)=K_x'(t)=1-\frac{-e^t}{2-e^t}\,|\,(t=0)=1+\frac{1}{2-1}=2$$

$$v(x)=K_x''(t)=0+\frac{(2-e^t).e^t+e^{2t}}{(2-e^t)^2}\,|\,(t=0)=0+\frac{(2-1).1+1}{(2-1)^2}=2$$

$$\therefore \sigma=\sqrt{2}$$

$$\boxed{P_r(m-k\sigma<X<m+k\sigma) \geq 1-\frac{1}{K^2}}$$

$$P_r(-2<X<6) \geq 1-\frac{1}{K^2}$$

$$P_r(2-k\sqrt{2}<X<2+k\sqrt{2}) \geq 1-\frac{1}{K^2}$$

$$2-k\sqrt{2}=-2 \quad Or \quad 2+k\sqrt{2}=6 \;\Rightarrow k=\frac{4}{\sqrt{2}} \therefore k^2=(\frac{4}{\sqrt{2}})^2=8$$

$$P_r(-2<X<6) \geq 1-\frac{1}{K^2}$$

$$\geq 1-\frac{1}{8} \geq \frac{8-1}{8} \geq \frac{7}{8} \geq \boxed{87.5\%}$$

Q3) *Let x be a r.v having the* p.d.f of x

$$f(x)=e^{-x} \qquad 0<x<\infty$$

$$1-M_x(t)=Ee^{tx}=\int_0^\infty e^{tx}f(x)=\int_0^\infty e^{(t-1)x}dx=\frac{1}{t-1}e^{(t-1)x}\Big|_0^\infty$$

$$=\frac{1}{t-1}e^{-(1-t)x}\Big|_0^\infty=\frac{1}{t-1}(e^{-\infty}-e^0)=\frac{-1}{t-1}\therefore \boxed{M_x(t)=\frac{1}{1-t}}$$

$$2-\boxed{K_x(t)}=Ln\,M_x(t)=Ln\frac{1}{1-t}=\boxed{-Ln(1-t)}$$

$$3-\boxed{M_y(t)}=Ee^{ty}=Ee^{t(3-2x)}=Ee^{3t}e^{-2tx}=e^{3t}Ee^{-2tx}=e^{3t}M_x(-2t)$$

$$=e^{3t}\frac{1}{1-(-2t)}=\boxed{\frac{e^{3t}}{1+2t}}$$

$$\therefore \boxed{\boxed{K_y(t)}=Ln\,M_y(t)=Ln\frac{e^{3t}}{1+2t}=Ln(e^{3t})-Ln(1+2t)=\boxed{3t-Ln(1+2t)}}$$

$$4-\boxed{E(y)}=K_y'(t)=3-\frac{2}{1+2t}=3-2=\boxed{1}\ \&\ \boxed{V(y)}=K_y'(t)=0-\frac{-2(2)}{(1+2t)}=\boxed{4}$$

****

Q4)

$\therefore S=\{HH,HT,TH,TT\} \quad X=0,1,2$

X	0	1	2
$P(x)$	1/4	2/4	1/4

$$\alpha_3=\frac{E(X-M)^3}{\sigma_x^{\ 3}} \quad and \quad E(X-M)^3=E(X^3-3X^2M+3M^2X-M^3)$$

$$E(X-M)^3=EX^3-3MEX^2+3M^2EX-M^3$$

$$\boxed{E(X)}=\sum_{x=0}^{2}xp(x)=(0)(1/4)+(1)(2/4)+(2)(1/4)=\boxed{1}\ \Leftrightarrow \boxed{M_1=E(X)}$$

$$\boxed{E(X^2)}=\sum_{x=0}^{2}xp(x)=(0)^2(1/4)+(1)^2(2/4)+(2)^2(1/4)=\boxed{6/4}$$

$$\boxed{E(X^3)}=\sum_{x=0}^{2}xp(x)=(0)^3(1/4)+(1)^3(2/4)+(2)^3(1/4)=\boxed{10/4}$$

$$\boxed{E(X-M)^3}=\frac{10}{4}-3.(1)\frac{6}{4}+3.(1)^2.(1)-(1)^3=-2+2=\boxed{0}\rightarrow \alpha_3=\frac{0}{\sigma_x^{\ 3}}=0$$

Q3) *Let x be a r.v having the* p.d.f of x

$$f(x)=e^{-x} \qquad 0<x<\infty$$

$$1-M_x(t)=Ee^{tx}=\int_0^\infty e^{tx}f(x)=\int_0^\infty e^{(t-1)x}dx=\frac{1}{t-1}e^{(t-1)x}\Big|_0^\infty$$

$$=\frac{1}{t-1}e^{-(1-t)x}\Big|_0^\infty=\frac{1}{t-1}(e^{-\infty}-e^0)=\frac{-1}{t-1}\therefore \boxed{M_x(t)=\frac{1}{1-t}}$$

$$2-\boxed{K_x(t)}=Ln\,M_x(t)=Ln\frac{1}{1-t}=\boxed{-Ln(1-t)}$$

$$3-\boxed{M_y(t)}=Ee^{ty}=Ee^{t(3-2x)}=Ee^{3t}e^{-2tx}=e^{3t}Ee^{-2tx}=e^{3t}M_x(-2t)$$

$$=e^{3t}\frac{1}{1-(-2t)}=\boxed{\frac{e^{3t}}{1+2t}}$$

$$\therefore \boxed{K_y(t)}=Ln\,M_y(t)=Ln\frac{e^{3t}}{1+2t}=Ln(e^{3t})-Ln(1+2t)=\boxed{3t-Ln(1+2t)}$$

$$4-\boxed{E(y)}=K'_y(t)=3-\frac{2}{1+2t}=3-2=\boxed{1}\ \&\ \boxed{V(y)}=K'_y(t)=0-\frac{-2(2)}{(1+2t)}=\boxed{4}$$

****

Q4)

$\therefore S=\{HH,HT,TH,TT\} \quad X=0,1,2$

X	0	1	2
$P(x)$	1/4	2/4	1/4

$$\alpha_3=\frac{E(X-M)^3}{\sigma_x^{\ 3}} \quad and \quad E(X-M)^3=E(X^3-3X^2M+3M^2X-M^3)$$

$$E(X-M)^3=EX^3-3MEX^2+3M^2EX-M^3$$

$$\boxed{E(X)}=\sum_{x=0}^{2}xp(x)=(0)(1/4)+(1)(2/4)+(2)(1/4)=\boxed{1}\Leftrightarrow\boxed{M_1=E(X)}$$

$$\boxed{E(X^2)}=\sum_{x=0}^{2}xp(x)=(0)^2(1/4)+(1)^2(2/4)+(2)^2(1/4)=\boxed{6/4}$$

$$\boxed{E(X^3)}=\sum_{x=0}^{2}xp(x)=(0)^3(1/4)+(1)^3(2/4)+(2)^3(1/4)=\boxed{10/4}$$

$$\boxed{E(X-M)^3}=\frac{10}{4}-3.(1)\frac{6}{4}+3.(1)^2.(1)-(1)^3=-2+2=\boxed{0}\rightarrow\alpha_3=\frac{0}{\sigma_x^{\ 3}}=0$$