CHAPTER (3)

(Joint, Marginal &Conditional) distribution of Random Variable

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3 – 1 Joint Distribution:

IF $(x_1, x_2, \ldots, x_p)$ are r.v's defined on the sample probability space , then the $(x_1, x_2, \ldots, x_p)$ is called p-dimensional r.v's .

* Joint probability density function [J.p.d.f]

Def[n]: IF $(x_1, x_2, \ldots, x_p)$ are a p-dimensional r.v's ; Then the Joint pro.density fun. Of $(x_1, x_2, \ldots, x_p)$ is defined to be:

$$f(x1, x2, \ldots, xp) = p\{X_1 = x_1, X_2 = x_2, \ldots, X_p = x_p\}$$

Or

$$f(x1, x2, \ldots, xp) = \begin{cases} \sum_{\forall x1}\sum_{\forall x2}\cdots\sum_{\forall xp} f(x_1, x_2, \ldots, x_p) & \text{for dist r.v's} \\ \int_{Rx1}\int_{Rx2}\cdots\int_{Rxp} f(x_1, x_2, \ldots, x_p) d_{xp}, \ldots d_{x2}, d_{x1} & \text{for cont. r.v's} \\ 0 & o.w \end{cases}$$

* The properties of the joint probability density function.

1- $0 \le f(x_1, x_2, \ldots, x_{xp} \le 1) \quad \forall x_i \Rightarrow i = 1,2,\ldots,p$

$p = No.$ of variables

2- $$\sum_{\forall x1}\sum_{\forall x2}\cdots\sum_{\forall xp} f(x_1, x_2, \ldots, x_p) = 1$$

$$\int_{Rx1}\int_{Rx2}\cdots\int_{Rxp} f(x_1, x_2, \ldots, x_p) d_{xp}, \ldots d_{x2}, d_{x1} = 1$$

3-

$$\text{IF } p = 2 \text{ ; Then } p(a < x_1 < b, c < x_2 < d) = \begin{cases} \int_c^d\int_a^b f(x_1, x_2)dx_1dx_2 & cont. \\ \sum_a^b\sum_d^c f(x, y) & disc. \end{cases}$$

4- I p=1 ; Then the j.p.d.f is called (univariate p.d.f) if p=2 ; Then the j.p.d.f is a fun. of tow r.v's and it is called (Bivariate) p.d.f if(p>2) ;Then the j.p.d.f is called (Multivariate p.d.f)

Def[n].: IF p=2 , let x and y are two r.v's then

$$f(x,y)=p(X=x \quad ,Y=y)$$
$$=\sum_{\forall x}\sum_{\forall y} p(x,y) \qquad if \quad x\,\&\,y \quad are \quad discrete \quad r.v's$$
$$=\int_{Rx}\int_{Ry} f(x,y)dydx \qquad if \quad x\,\&\,y \quad are \quad continuous \quad r.v's$$

1) $0\le f(x,y)\le 1 \qquad \forall \quad x,y$

2) $\sum_{\forall x}\sum_{\forall y} f(x,y)=1$

$$\int_{Rx}\int_{Ry} f(x,y)dydx=1$$

Example (1)

$\boxed{Example(1)}: letthej.p.d.foftwor.v's x_1 \& x_2 isdefinded$

$$f(x_1,x_2)=\begin{cases}\frac{x_1+x_2}{21} x_1=1,2,3\\ x_2=1,2\\ 0 o.w\end{cases}$$

$1) p(x_1=1,x_2=2)$

$2) provep(x_1,x_2) isj.p.d.fof x_1,x_2$

Result:

$p_r(X_1=1,X_2=2)=\frac{X_1+X_2}{21}=\frac{1+2}{21}=\frac{3}{21}=\frac{1}{7}$

2) prove $p(x_1,x_2)$ is j.p.f of x_1 & x_2?

$\sum_1^3\sum_1^2\frac{X_1+X_2}{21}=\frac{1+x_2}{21}+\frac{2+x_2}{21}+\frac{3+x_2}{21}=\frac{1+1}{21}+\frac{2+1}{21}+\frac{3+1}{21}+\frac{1+2}{21}+\frac{2+2}{21}+\frac{3+2}{21}=$

$\frac{21}{21}=1$

Example (2) et

$$f(x_1,x_2)=e^{-(x_1+x_2)} 0<x_1,x_2<\infty$$

$= 0 o.w$

$1) p(x_1 = 2, x_2 = 3)$

$2) p(1 < x_1 < 2, 0 < x_2 < 2)$

$3) check the j.p.d.f of x_1, x_2$

Result:

1) $P_r(x_1=0, x_2=1)=0$

2) $P_r(1<x_1<2, 0<x_2<2)$?

$P_r(1<x_1<2, 0<x_2<2) = \int_1^2 \int_0^2 e^{-(x_1+x_2)} dx_1 dx_2 = \int_1^2 e^{-x_1} dx_1 * \int_0^2 e^{-x_2} dx_2 = -e^{x_1} I_1^2 * -e^{x_2} I_0^2 = (-e^{-2} - (-e^{-1}) * (-e^{-2} - (-e^{0})) = (-e^{-2} + e^{-1}) * (-e^{-2} + 1)$

3) Check the j.p.d.f x_1, x2?

Solved/

$= \int_0^\infty \int_0^\infty e^{-(x_1+x_2)} dx_1 dx_2 = \int_0^\infty e^{-x_1} dx_1 * \int_0^\infty e^{-x_2} dx_2 = -e^{x_1} I_0^\infty * -e^{x_2} I_0^\infty = (-e^{-\infty} - (-e^{-0}) * (-e^{-\infty} - (-e^{0})) = (0+1) * (0+1) = 1 * 1 = 1$

Example (3) :*Three coins are tossed ,let x denoted the coins the number of heads that occur on the first two coins ; let y denoted the number of tails that occur on the last two coins.and the j.p.d.f of x & y [i.e Find f(x,y)] show that f(x,y) is a j.p.d.f of x & y.*

Result:

Let h=head and t=tail

Sample space=[(hhh), (ttt), (htt), (thh), (tht), (hth), (tth), (hht)]

y \ x	0	1	2	Sum y
0	…..	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	……..	$\frac{2}{8}$
Sum x	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	1

$\sum_{rang\, y} \sum_{rang\, x} f(x,y) = ?$

f(0,0)+f(0,1)+f(0,2)+f(1,0)+f(1,1)+f(1,2)+f(2,0)+f(2,1)+f(2,2)=

$=0+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{2}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+0=1$

Example (4) *et*

$$f(x,y)=6x^2y0<x<1,0<y<1$$

$$Find: p(0<x<\frac{3}{4},\frac{1}{3}<y<2)$$

Result:

$f(x,y)=6x^2y \quad 0<x<1, 0<y<1$

Find: **P $(0<x<\frac{3}{4},\frac{1}{3}<y<2)=?$** **Out range**

$$f(x,y)=\int_0^{\frac{3}{4}}\int_{\frac{1}{3}}^{2}6x^2y\,dx\,dy=\int_0^{\frac{3}{4}}\int_{\frac{1}{3}}^{1}f(x,y)dx+\int_0^{\frac{3}{4}}\int_1^2 f(x,y)\,dxdy=$$

$$6\left(\frac{x^3}{3}I_0^{\frac{3}{4}}\right)*\frac{y^2}{2}I_{\frac{1}{3}}^{0}+0= \qquad \left(\left(\frac{3}{4}\right)^3-0\right)*\left(1^2-\frac{1}{3}^2\right)=\frac{27}{64}*\frac{8}{9}=\frac{3}{8}$$

3 – 2 Joint Cummulative Distribution F^n (j.c.d.f):

Let $(x_1,x_2,....,x_p)$ is defined to be a p-dimensional r.v's iff three exists function $f(x_1,x_2,....,x_p)$; such that ;

$$F(x_1,x_2,\dots,x_p)=p(X_1\le x_1,X_2\le x_2,\dots,X_p\le x_p)$$

$$=\sum_{-\infty}^{X_1}\sum_{-\infty}^{X_2}\dots\sum_{-\infty}^{X_p}p(x_1,x_2,\dots,x_p) \qquad if\ x_i\ are\ discrete\ r.v's$$

$$i=1,2,\dots,p$$

$$=\int_{-\infty}^{x_p}\int_{-\infty}^{x_{p-1}}\dots\int_{-\infty}^{x_1}f(x_1,x_2,....,x_p)d_{x1},d_{x2},\dots,d_{xp} \qquad if\ x_i\ are\,continuous\ r.v's$$

for all $(x_1,x_2,....,x_p)$, $f(x_1,x_2,....,x_p)$ is defined to be a joint p.d.f of $x_1,x_2,....,x_p$, and $F(x_1,x_2,....,x_p)$ is called a joint cumulative distribution f^n .(j.c.d.f)

- -

Def^n.: IF $x_1,x_2,....,x_p$ are jointly continuous r.v's ; then the knowlege of $F(x_1,x_2,....,x_p)$ is equivalent to the knowlege of $f(x_1,x_2,....,x_p)$.

$$f(x_1, x_2, \dots, x_p) = \frac{\partial^p F(x_1, x_2, \dots, x_p)}{\partial x_1, x_2, \dots, x_p}$$

Special case :- let x&y are two r.v's ; then the jointly continuous dist^n. f^n. F(x,y) is defined as ;

$$F(x,y) = pr(X \le x, Y \le y) = \begin{cases} \sum_{-\infty}^{x} \sum_{-\infty}^{y} p(x,y) if x \& y are dis. \\ \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y) dx dy if x \& y are con. \end{cases}$$

and

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x y d y} if x \& y are cont.$$

<u>* The properties of the (j.c.d.f).</u>

1- $0 \le F(x_1, x_2) \le 1)$

2- $F(x_1, x_2)$ is continuous function to the right hand it's non-decreasing f^n.

if $a_1 \le a_2, b_1 \le b_2$ where a_1, a_2, b_1, b_2 are numbers then:

$F(a_1, b_1) \le F(a_2, b_1) \le F(a_2, b_2)$

3- $F(-\infty, x_2) = \lim_{x \to -\infty} F(x_1, x_2) = 0$, $F(x_1, -\infty) = \lim_{x \to -\infty} F(x_1, x_2) = 0$

$F(\infty, \infty) = \lim_{\substack{x_1 \to \infty \\ x_2 \to \infty}} F(x_1, x_2) = 1$, $F(-\infty, -\infty) = \lim_{\substack{x_1 \to -\infty \\ x_2 \to -\infty}} F(x_1, x_2) = 0$

4- $\lim_{x_1 \to \infty} F(x_1, x_2) = F(x_2)$ *and* , $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

5- let a, b, c, d are real constant , if $(a < b) \& (c < d)$ and let $F(x_1, x_2)$ be the j.c.d.f of $x_1 \& x_2$ then; $pr(a \le x_1 \le b, c \le x_2 \le d)$

Example (5) et

$f(x,y,z) = e^{-(x+y+z)} 0 < x, y, z < \infty$

$= 0 \qquad o.w$

$Find: the j.c.d.f of x, y \& z.$

Result:

F(x,y,z)=$\int_0^x e^{-x}\boldsymbol{dx} * \int_0^y e^{-y}\boldsymbol{dy} * \int_0^z e^{-z}\boldsymbol{dz} = -e^{-x}I_0^x * -e^{-y}I_0^y * -e^{-z}\,I_0^z =$ $(1-e^{-x})(1-e^{-y})(1-e^{-z})$

$$F(x,y,z)=\begin{bmatrix} 0 & x\le 0; y\le 0; z\le 0 \\ (1-e^{-x})(1-e^{y})(1-e^{-z}) & 0<x,y,z<\infty \\ 1 & (x,y,z)\longrightarrow \infty \end{bmatrix}$$

Example (6) *:let*

$$f(x,y)=\begin{cases} \frac{1}{9}x(y-x) & 0<x<3 \ , \ 2<y<4 \\ 0 & o.w \end{cases}$$

Find : *the j.c.d.f of* $x \,\&\, y=(2)pr(0<x<1, 2<y<3).$

Result:

find:1) the j.c.d.f of x & y 2)Pr(0<x<1, 2≤ $y \le 3$)

1)

$$\boldsymbol{F(x,y)}\int_0^x\int_2^y \frac{1}{9}(xy-x^2)dydx$$

$$=\frac{1}{9}\int_2^y\left(\frac{x^2y}{2}-\frac{x^3}{3}I_0^x\right)dy=\frac{1}{9}\left(\frac{x^2y^2}{4}-\frac{x^3y}{3}I_2^y\right)$$

$$=\frac{1}{9}\left(\frac{x^2y^2}{4}-\frac{x^3y}{3}-x^2+\frac{2x^3}{3}\right)$$

$$F(x,y)=\begin{bmatrix} 0 & x\le 0; y\le 2 \\ \frac{1}{9}\left(\frac{x^2y^2}{4}-\frac{x^3y}{3}-x^2+\frac{2x^3}{3}\right) & 0<x<3; 2<y<4 \\ 1 & x\ge 3; y\ge 4 \end{bmatrix}$$

2)

Pr(0<x<1,2≤ $y \le 3) = F(1,3)-F(0,2)=\frac{1}{9}\left(\frac{9}{4}-1-1+\frac{2}{3}\right)-0=\frac{11}{108}$

3 – 3 The Marginal probability Distribution Fn.:

Defn.: let x & y are two r.v's with (j.p.d.f) , then $f(x)\&f(y)$ are called the marginal p.d.f of x & y respectively, which can be defined as follows:-

$$f(x) = \begin{cases} \sum_{\forall y} f(x,y) if x\&y are disc. r.v's \\ \int_{Ry} f(x,y)dy if x\&y are cont. r.v's \end{cases} \quad \text{if m.p.d.f of x}$$

$$f(y) = \begin{cases} \sum_{\forall x} f(x,y) if x\&y are disc. r.v's \\ \int_{Rx} f(x,y)dx if x\&y are cont. r.v's \end{cases} \quad \text{if m.p.d.f of y}$$

Example (7)

$\boxed{Example(9)}$: $consider the bivariate function$

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & o.w \end{cases}$$

$Find$:

1) $The marginal p.d.f of x.$

2) $The marginal p.d.f of y.$

3) $The joint \quad c.d.f of x\&y.$

4) $Marginal c.d.f of x.$

5) $pr(0<x<\frac{1}{2}; 0<y<\frac{1}{4}).$

6) $show that f(x,y) is a j.p.d.f of x\&y.$

Result:

1)the marginal pdf of x?

f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2} - 0 = x + \frac{1}{2}$

2)the marginal pdf of y?

F(y)= $\int_0^1 (x+y)dx = \frac{x^2}{2} + xy I_0^1 = \frac{1}{2} + y - 0 = y + \frac{1}{2}$

3) The joint c.d.f of x &y?

F(x,y)=$\int_0^x \int_0^y (x+y)dxdy = \int_0^x (xy + \frac{y^2}{2})dx = \frac{x^2y}{2} + \frac{xy^2}{2}$

$$F(x,y)=\begin{bmatrix} 0 & x \le 0; y \le 0 \\ \frac{x^2y}{2}+\frac{xy^2}{2} & 0 < x < 1; 0 < y < 1 \\ 1 & x \ge 1; y \ge 1 \end{bmatrix}$$

4)marginal cdf of x?

$$F(x)=\int_0^x \left(x+\frac{1}{2}\right)dx = \frac{x^2}{2}+\frac{x}{2}I_0^x = \frac{x^2}{2}+\frac{x}{2}-0 = \frac{x^2}{2}+\frac{x}{2}$$

$$F(x)=\begin{bmatrix} 0 & x \le 0 \\ \frac{x^2}{2}+\frac{x}{2} & 0 < x < 1 \\ 1 & x \ge 1 \end{bmatrix}$$

5)Pr(0<x<$\frac{1}{2}$; $0 < y < \frac{1}{4}$)?

$$F(\tfrac{1}{2},\tfrac{1}{4}) - F(0,0) = \left(\frac{1}{2}\right)^2 * \frac{1}{2} * \frac{1}{4} + \left(\frac{1}{4}\right)^2 * \frac{1}{2} * \frac{1}{2} = \frac{3}{64}$$

6)show that f(x,y) is j.p.d.f of x &y?

$$=\int_0^1\int_0^1 (x+y)dxdy = \int_0^1 (xy+\frac{y^2}{2})I_0^1 dx = \int_0^1 x+\frac{1}{2} = \frac{x^2}{2}+\frac{x}{2}I_0^1 = \frac{1}{2}+\frac{1}{2} = 1$$

Example (8) : *Find the marginal p.d.f of x & y from:*

x \ y	0	1	2	sum
0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{2}{8}$
sum	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	1

Result:

y \ x	**0**	**1**	**2**	**Sum y=f(y)**
0	**0**	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	**0**	$\frac{2}{8}$

Sum x= f(y)	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	**1**

$$f(x)=\begin{bmatrix} \frac{2}{8} & x=0 \\ \frac{4}{8} & x=1 \\ \frac{2}{8} & x=2 \\ 0 & o.w \end{bmatrix} \qquad f(y)=\begin{bmatrix} \frac{2}{8} & y=0 \\ \frac{4}{8} & y=1 \\ \frac{2}{8} & y=2 \\ 0 & o.w \end{bmatrix}$$

Example (9) *let*

$$f(x,y)=\begin{cases} \frac{x+2y}{18} x=1,2 \\ y=1,2 \\ 0 o.w \end{cases}$$

$Find: 1-f(x,y) 2-f(x)$

$3-f(y) 4-F(x)$

$5-F(y)$

Result:

1)f(x,y)?

f(x,y)= $\frac{x+2y}{18}$

2)f(x)?

f(x)=$\sum_1^2 \frac{x+2y}{18}=\frac{x+2(1)}{18}+\frac{x+2(2)}{18}=\frac{2x+6}{18}=\frac{x+3}{9}$

$$f(x)=\begin{bmatrix} \frac{x+3}{9} & x=1,2 \\ 0 & o.w \end{bmatrix}$$

3)f(y)?

f(y)= $\sum_1^2 \frac{x+2y}{18}=\frac{1+2(y)}{18}+\frac{2+2(y)}{18}=\frac{3+4y}{18}$

$$f(y)=\begin{bmatrix} \frac{3+4y}{18} & y=1,2 \\ 0 & o.w \end{bmatrix}$$

4)F(x)?

F(x)=$\sum_1^1 \frac{x+3}{9}=\frac{4}{9}$ F(x)=$\begin{bmatrix} 0 & x<1 \\ \frac{4}{9} & 1<x<2 \\ 1 & 2\le x \end{bmatrix}$

5)F(y)?

$F(y)=\sum_1^1 \frac{4y+3}{18} = \frac{7}{18}$ $F(y)=\begin{bmatrix} 0 & y<1 \\ \frac{7}{18} & 1<y<2 \\ 1 & y \geq 2 \end{bmatrix}$

3 – 4 Expectation Joint Mathematical:

Def^n.: Let $x_1, x_2, \ldots, x_p$ be a p-dimensional r.v's with j.p.d.f $f(x_1, x_2, \ldots, x_p)$, then the expected value of a p-dimensional function $g(x_1, x_2, \ldots, x_p)$ denoted by $E[g(x_1, x_2, \ldots, x_p)]$ is given by :

$$E[g(x_1,\ldots,x_p)] = \begin{cases} \sum_{x_1}\sum_{x_2}\cdots\sum_{x_p} g(x_1,\ldots,x_p)f(x_1,x_2,\ldots,x_p) if\, x_1,x_2,\ldots,x_p are disc. r.v's \\ \int_{Rx_1}\int_{Rx_2}\cdots\int_{Rx_p} g(x_1,\ldots,x_p)f(x_1,x_2,\ldots,x_p) if\, x_1,x_2,\ldots,x_p are cont. r.v's \end{cases}$$

Def^n.: let x & y are two r.v's with (j.p.d.f) , then $f(x,y)$, then the expected value of a function of two variables $g(x,y)$ is defined as follows:-

$$E[g(x,y)] = \begin{cases} \sum_y\sum_x g(x,y)f(x,y) if\, x,y are disc. r.v's \\ \int_{Ry}\int_{Rx} g(x,y)f(x,y)dxdy if\, x,y are cont. r.v's \end{cases}$$

Example (10) : *let*

$$f(x,y) = \begin{cases} \frac{x+y}{21} x = 1,2,3 \\ y = 1,2 \\ 0 o.w \end{cases}$$

$$Find: Ex, Ey, Exy, E(2x+3y), E(4x-2y)$$

Result:

$f(x,y)=\frac{x+y}{21}$ $x = 1,2,3 \,;\, y = 1,2$

f(x)=$\sum_1^2 \frac{x+y}{21} = \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21}$

f(y)= $\sum_1^3 \frac{x+y}{21} = \frac{y+1}{21} + \frac{y+2}{21} + \frac{3+y}{21} = \frac{2+y}{7}$

1)E(x)?

E(x)=$\sum_1^3 x * \frac{2x+3}{21} = 1 * \frac{2(1)+3}{21} + 2 * \frac{2(2)+3}{21} + 3 * \frac{2(3)+3}{21} = \frac{46}{21}$

2)E(y)?

E(y)=$\sum_1^2 y * \frac{2+y}{7} = 1 * \frac{2+1}{21} + 2 * \frac{2+2}{7} = \frac{11}{7}$

3)E(xy)?

E(xy)=$\sum_1^3 \sum_1^2 xy \frac{x+y}{21} = 1 * 1\frac{1+1}{21} + 1 * 2\frac{1+2}{21} + 2 * 1\frac{2+1}{21} + 2 * 2\frac{2+2}{21} + 3 *$
$1\frac{3+1}{21} + 3 * 2\frac{3+2}{21} = \frac{72}{21}$

4)E(2x+3y)?

2E(x)+3E(y)=2*$\frac{46}{21} + 3 * \frac{11}{7} = \frac{191}{21}$

5)E(4x-2y)?

Solved

4E(x)-2E(y)=4*$\frac{46}{21} - 2 * \frac{11}{7} = \frac{118}{21}$

Example (11) $tthej.p.d.fofx\&yisdefinedas;$

$$f(x,y) = \begin{cases} x + y0 < x < 1,0 < y < 1 \\ 0o.w \end{cases}$$

$$Find: 1)Ex2)Ey3)Exy4)E(2x+4y)5)Exy^2$$

Result:

f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$

f(y)=$\int_0^1 (x+y)dx = xy + \frac{x^2}{2} I_0^1 = y + \frac{1}{2}$

1)E(x)?

E(x)=$\int_0^1 x\left(x + \frac{1}{2}\right) dx = \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \frac{x^3}{3} + \frac{x^2}{2*2} I_0^1 = \frac{7}{12}$

2)E(y)=?

E(y)= $\int_0^1 y\left(y + \frac{1}{2}\right) dy = \int_0^1 \left(y^2 + \frac{y}{2}\right) dy = \frac{y^3}{3} + \frac{y^2}{2*2} I_0^1 = \frac{7}{12}$

3)E(x,y)?

E(x,y)=$\int_0^1 \int_0^1 xy(x+y)dydx = \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3} I_0^1 = \int_0^1 \frac{x^2}{2} + \frac{x}{3} = \frac{x^3}{6} + \frac{x^2}{6} I_0^1 = \frac{2}{6}$

4)E(2x+4y)?

2E(x)+4E(y)=2*$\frac{7}{12} + 4 * \frac{7}{12} = \frac{42}{12} = \frac{21}{6}$

5)E$(xy^2) = \int_0^1 \int_0^1 xy^2(x+y)dydx = \int_0^1 \frac{x^2y^3}{3} + \frac{xy^4}{4} I_0^1 = \int_0^1 \frac{x^2}{3} + \frac{x}{4} = \frac{x^3}{9} + \frac{x^2}{8} I_0^1 = \frac{17}{72}$

3 – 5 Covariance & Correlation Coefficient:

Def^n.: Let x & y be two r.v's with the marginal p.d.f of x & y, $f(x)\&f(y)$, and the j.p.d.f of $f(x,y)$ then the Correlation Coefficient between x & y denoted by ρ_{xy} is defined to be:-

$$correlation(x,y)=\frac{cov\,ariance(x,y)}{\sqrt{var\,iancex}.\sqrt{var\,iancey}}$$

$$\rho_{x,y}=\frac{cov(x,y)}{\sqrt{v(x)}.\sqrt{v(y)}}$$

$$\rho_{x,y}=\frac{E(x-M_x)(y-M_y)}{\sqrt{E(x-M_x)^2}.\sqrt{E(y-M_y)^2}}$$

$$\rho_{x,y}=\frac{Exy-M_x.M_y}{\sqrt{Ex^2-(Ex)^2}.\sqrt{Ey^2-(Ey)^2}}$$

$$\left[\rho_{x,y}=\frac{Exy-Ex.Ey}{\sqrt{Ex^2-(Ex)^2}.\sqrt{Ey^2-(Ey)^2}}\right]$$

$where: \sigma_x: -isthes\tan d\,arddeviationofx.$

$\sigma_y: -isthes\tan d\,arddeviationofy$

Example (12)

$\boxed{Example(20)}: let$

$$p(x,y)=\frac{x+y}{32}x=1,2y=1,2,3,4$$

$=0o.w$

$Findthecorrelationcoefficientbetweenx\&y.$

Result:

P(x,y)=$\frac{x+y}{32}$ $x=1,2\,;y=1,2,3,4$

X \ Y	1	2	3	4	Sum x=f(x)
1	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{4}{32}$	$\frac{5}{32}$	$\frac{14}{32}$
2	$\frac{3}{32}$	$\frac{4}{32}$	$\frac{5}{32}$	$\frac{6}{32}$	$\frac{18}{32}$
Sum y=f(y)	$\frac{5}{32}$	$\frac{7}{32}$	$\frac{9}{32}$	$\frac{11}{32}$	1

Find correlation between of x &y .

E(x)=$\sum_1^2 xp(x)=1p(1)+2p(2)=1*\frac{14}{32}+2*\frac{18}{32}=\frac{25}{16}$

$$E(x^2) = \sum_1^2 x^2 p(x) = 1^2 p(1) + 2^2 p(2) = 1 * * \frac{14}{32} + 4 * \frac{18}{32} = \frac{43}{16}$$

$$E(y) = \sum_1^4 y p(y) = 1p(1) + 2P(2) + 3p(3) + 4p(4) = \frac{5}{32} + 2 * \frac{7}{32} + 3 * \frac{9}{32} + 4 * \frac{11}{32} = \frac{90}{32}$$

$$E(y^2) = \sum_1^4 y^2 p(y) = 1^2 p(1) + 2^2 P(2) + 3^2 p(3) + 4^2 p(4) = \frac{5}{32} + 4 * \frac{7}{32} + 9 * \frac{9}{32} + 16 * \frac{11}{32} = \frac{145}{16}$$

$$E(xy) = \sum_1^4 \sum_1^2 xyp(x,y) = 1 * 1p(1,1) + 1 * 2p(1,2) + 1 * 3p(1,3) + 1 * 4p(1,4) + 2 * 1P(2,1) + 2 * 2P(2,2) + 2 * 3p(2,3) + 2 * 4p(2,4) = \frac{2}{32} + 2 * \frac{3}{32} + 3 * \frac{4}{32} + 4 * \frac{5}{32} + 2 * \frac{3}{32} + 4 * \frac{4}{32} + 6 * \frac{5}{32} + 8 * \frac{6}{32} = \frac{140}{32} = \frac{35}{8}$$

$$P_{xy} = \frac{Exy - Ex * Ey}{\sqrt{Ex^2 - (E(x))^2} * \sqrt{Ey^2 - (E(y))^2}} = \frac{\frac{35}{8} - \frac{25}{16} * \frac{45}{16}}{\sqrt{\frac{42}{16} - \left(\frac{25}{16}\right)^2} * \sqrt{\frac{145}{16} - \left(\frac{45}{16}\right)^2}} = -0.037$$

Example (13) t $Find: \rho_{xy}$

y \ x	1	2	3	sum
1	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{9}{15}$
2	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{6}{15}$
sum	$\frac{3}{15}$	$\frac{5}{15}$	$\frac{7}{15}$	1

Result

Y \ X	1	2	3	Sum y
1	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{9}{15}$
2	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{6}{15}$
Sum x	$\frac{3}{15}$	$\frac{5}{15}$	$\frac{7}{15}$	1

$$E(x) = \sum_1^3 xP(x) = \frac{3}{15} + 2 * \frac{5}{15} + 3 * \frac{7}{15} = \frac{34}{15}$$

$$E(x^2) = \sum_1^3 x^2 p(x) = \frac{3}{15} + 2^2 * \frac{5}{15} + 3^2 * \frac{7}{15} = \frac{86}{15}$$

$$E(y)=\sum_1^2 yp(y) = \frac{9}{15} + 2 * \frac{6}{15} = \frac{21}{15} = \frac{7}{5}$$

$$E(y^2) = \sum_1^2 y^2 p(y) = \frac{9}{15} + 2^2 * \frac{6}{15} = \frac{33}{15}$$

$$E(xy)=\sum_1^2 \sum_1^3 xyp(x.y) = \frac{2}{15} + 1 * 2 * \frac{1}{15} + 2 * 1 * \frac{4}{15} + 2 * 2 * \frac{1}{15} + 3 * 1 * \frac{3}{15} + 3 * 2 * \frac{4}{15} = \frac{49}{15}$$

$$P_{xy}=\frac{Exy-Ex*Ey}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}} = \frac{\frac{49}{15}-\frac{34}{15}*\frac{7}{5}}{\sqrt{\frac{86}{15}-\left(\frac{34}{15}\right)^2}*\sqrt{\frac{33}{15}-\left(\frac{7}{5}\right)^2}} = 0.2468$$

Example (14) t

$$f(x,y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & o.w \end{cases}$$

$$Find: \rho_{xy}$$

Result:

f(x,y)=x+y 0<x<1 ; 0<y<1

$$f(x)=\int_0^1 (x + y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$$

$$f(y)=\int_0^1 (x + y)dx = xy + \frac{x^2}{2} I_0^1 = y + \frac{1}{2}$$

$$E(x)=\int_0^1 x\left(x + \frac{1}{2}\right) dx = \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \frac{x^3}{3} + \frac{x^2}{2*2} I_0^1 = \frac{7}{12}$$

$$E(y)= \int_0^1 y\left(y + \frac{1}{2}\right) dy = \int_0^1 \left(y^2 + \frac{y}{2}\right) dy = \frac{y^3}{3} + \frac{y^2}{2*2} I_0^1 = \frac{7}{12}$$

$$E(xy)=\int_0^1 \int_0^1 xy(x + y)dydx = \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3} I_0^1 = \int_0^1 \frac{x^2}{2} + \frac{x}{3} = \frac{x^3}{6} + \frac{x^2}{6} I_0^1 = \frac{2}{6}$$

$$E(x^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx = \frac{x^4}{4} + \frac{x^3}{2*3} I_0^1 = \frac{10}{24}$$

$$E(y^2)= \int_0^1 y^2\left(y + \frac{1}{2}\right) dy = \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy = \frac{y^4}{4} + \frac{y^3}{2*3} I_0^1 = \frac{10}{24}$$

$$P_{xy}=\frac{Exy-Ex*Ey}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}} = \frac{\frac{2}{6}-\frac{7}{12}*\frac{7}{12}}{\sqrt{\frac{10}{24}-\left(\frac{7}{12}\right)^2}*\sqrt{\frac{10}{24}-\left(\frac{7}{12}\right)^2}} = -0.09$$

Example (15) t

$$f(x,y) = 2 \quad 0 < x < y, 0 < y < 1$$
$$= 0 \quad o.w$$

$$Find: \rho_{xy}$$

Result:

f(x,y)=2   0<x<y ; 0<y<1

f(x)=$\int_x^1 2dy = 2yI_x^1 = 2 - 2x$

f(y)=$\int_0^y 2dx = 2xI_0^y = 2y$

E(x)=$\int_0^1 x(2-2x)dx = \int_0^1 (2x - 2x^2)dx = x^2 - \frac{2x^3}{3} I_0^1 = \frac{1}{3}$

E(y)= $\int_0^1 y(2y)dy = \int_0^1 2y^2 dy = \frac{2y^3}{3} I_0^1 = \frac{2}{3}$

E(xy)=$\int_0^1 \int_0^y xy(2)dxdy = \int_0^1 x^2 y I_0^y = \int_0^1 y^2 y dy = \int_0^1 y^3 dy = \frac{y^4}{4} I_0^1 = \frac{1}{4}$

E(x^2)=$\int_0^1 x^2(2-2x)dx = \int_0^1 (2x^2 - 2x^3)dx = \frac{2x^3}{3} - \frac{2x^4}{4} I_0^1 = \frac{1}{6}$

E(y^2)= $\int_0^1 y^2(2y)dy = \int_0^1 2y^3 dy = \frac{2y^4}{4} I_0^1 = \frac{2}{4} = \frac{1}{2}$

P_{xy}=$\frac{Exy - Ex*Ey}{\sqrt{Ex^2 - (E(x))^2} * \sqrt{Ey^2 - (E(y))^2}} = \frac{\frac{1}{4} - \frac{1}{3} * \frac{2}{3}}{\sqrt{\frac{1}{6} - \left(\frac{1}{3}\right)^2} * \sqrt{\frac{1}{2} - \left(\frac{2}{3}\right)^2}} = 0.5$

3 – 6 Joint Moment Generating Function [j.m.g.f]:

Def^n.: Let $x_1, x_2, \ldots, x_p$ be a p-dimensional r.v's with j.p.d.f $f(x_1, x_2, \ldots, x_p)$, and $t_1, t_2, \ldots, t_p$ be other variables , IF (h_i) is positive number where $(-h_i < t_1 < h_i)$, if the expectation exist for all real values of t_i then the joint moment generating function of $x_1, x_2, \ldots, x_p$ is defined as follows :$M_{x_1,x_2,\ldots,x_p}(t_1, t_2, \ldots, t_p) = Ee^{t_1x_1 + t_2x_2 + \cdots + t_px_p} = Ee^{\sum_{i=1}^p t_ix_i}$

$$= \begin{cases} \sum_{x_1}\sum_{x_2}\cdots\sum_{x_p} e^{\sum_{i=1}^p t_ix_i} f(x_1, \ldots, x_p) if x_1, \ldots, x_p are disc. r.v's \\ \int_{Rx_1}\int_{Rx_2}\cdots\int_{Rx_p} e^{\sum_{i=1}^p t_ix_i} f(x_1, \ldots, x_p) dx_1, \ldots, dx_p if x_1, \ldots, x_p are cont. r.v's \end{cases}$$

then;

$Mx_1, \ldots, x_p(t_1 = 0, t_2 = 0, \ldots, t_p = 0) = Ee^0 = 1$

and;

$Mx_1, \ldots, x_p(t_1 = 0, t_2 = 0, \ldots, t_p = 0) \neq 0 t_{i+1} = 0, \ldots, t_p = 0 = Ee^{t_ix_i} = Mx_i(t)$

Def^n.: when p=2 , let x & y are two r.v's ,with j.p.d.f of $f(x,y)$, and if the expectation exists for all values of $(-h_1 < t_1 < h_1)$, $(-h_2 < t_2 < h_2)$, where $(h_1 \quad , h_2 \quad > 0)$, then the joint m.g.f of x & y is defined by:-

$$Mx,y(t_1,t_2) = Ee^{t_1x+t_2y} = \begin{cases} \sum_{\forall x}\sum_{\forall y} e^{t_1x+t_2y} f(x,y) if x,,y are disc.r.v's \\ \int_{Ry}\int_{Rx} e^{t_1x+t_2y} f(x,y)dxdy if x_1,\dots,x_p are cont.r.v's \end{cases}$$

then;

$$1) Mxy(t_1=0) = Ee^{t_1x} = Mx(t_1) = Ee^{t_1x}$$

$$Mxy(0=t_2) = Ee^{t_2y} = My(t_2) = Ee^{t_2y}$$

$$2) \quad E(xy) = \frac{\partial^2 Mxy(t_1,t_2)}{\partial t_1 \ \partial t_2}$$

$$E(x,y) = \begin{cases} \sum_x\sum_y xye^{t_1x+t_2y} f(x,y) \\ \int_{Ry}\int_{Rx} xye^{t_1x+t_2y} f(x,y)dxdy \end{cases}$$

In general;

$$E\,x^m\,y^k = \begin{cases} \sum_{\forall x}\sum_{\forall y} x^m y^k e^{t_1x^m+t_2y^k} f(x,y) \\ \int_{Ry}\int_{Rx} x^m y^k e^{t_1x^m+t_2y^k} f(x,y)dxdy \end{cases}$$

$$3) \quad Ex = \frac{\partial Mxy(t_1,t_2)}{\partial t_1}\Big|_{t_1=t_2=0} \quad , \quad Ex^2 = \frac{\partial^2 Mxy(t_1,t_2)}{\partial t_1^2}\Big|_{t_1=t_2=0}$$

$$4) \quad Ey = \frac{\partial Mxy(t_1,t_2)}{\partial t_2}\Big|_{t_1=t_2=0} \quad , \quad Ey^2 = \frac{\partial^2 Mxy(t_1,t_2)}{\partial t_2^2}\Big|_{t_1=t_2=0}$$

$$5) \quad \sigma_x^2 = Ex^2 - (Ex)^2$$

$$6) \quad \sigma_{xy} = \text{cov}(x,y) = Exy - Ex.Ey$$

Example (16) t

$$f(x,y,z) = \begin{cases} e^{-x-y-z} & \begin{matrix} 0 < x < \infty \\ 0 < y < \infty \\ 0 < z < \infty \end{matrix} \\ 0 & o.w \end{cases}$$

be the j.p.d.f of r.v's(x,y,z), then;

$Find: Mx,y,z(t_1,t_2,t_3)?$

Result:

f(x,y,z)= $\begin{bmatrix} e^{-x-y-z} & 0<x,y,z<\infty \\ 0 & o.w \end{bmatrix}$
be the j.p.d.f of r.vs (x,y,z) then

find : $M_{X,Y,Z}(t_1,t_2,t_3)$?
Solved

$M_{X,Y,Z}(t_1,t_2,t_3)=E(e^{-t_1x-t_2y-t_3z})=\int_0^\infty \int_0^\infty \int_0^\infty e^{-t_1x-t_2y-t_3z} * e^{-x-y-z}dxdydz = \int_0^\infty e^{-x(1-t_1)}dx * \int_0^\infty e^{-y(1-t_2)}dy * \int_0^\infty e^{-z(1-t_3)}dz = \frac{-e^{-x(1-t_1)}}{1-t_1} I_0^\infty * \frac{-e^{-y(1-t_2)}}{1-t_2} I_0^\infty * \frac{-e^{-z(1-t_3)}}{1-t_3} I_0^\infty =$
$=\frac{1}{(1-t_1)} * \frac{1}{(1-t_2)} * \frac{1}{(1-t_3)} I_{t_{1,2,3=0}} = 1$

Example (17) $\boxed{Example(31)}$: $let x \& y have the j.p.d.f;$

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0<x<\infty, 0<y<\infty \\ 0 & o.w \end{cases}$$

1) $Find$ the $j.m.g.f$ of $x\&y$.

2) $Find the$ $cov.$ $coeff.$ $between x\&y$.

Result:

f(x,y)= e^{-x-y} $0<x<\infty\,;0<y<\infty$
1)find the j.m.g.f of x & y
$M_{x,y}=E(e^{-t_1x-t_2y}) = \int_0^\infty \int_0^\infty e^{-t_1x-t_2y} * e^{-x-y}dxdy = \int_0^\infty e^{-x(1-t_1)}dx * \int_0^\infty e^{-y(1-t_2)}dy = \frac{-e^{-x(1-t_1)}}{1-t_1} I_0^\infty * \frac{-e^{-y(1-t_2)}}{1-t_2} I_0^\infty = \frac{1}{(1-t_1)} * \frac{1}{(1-t_2)} I_{t_{1,2=0}} = 1$
2)find cov. Coeff between x & y
$E(x)=\frac{\partial M x,y(t_1,t_2)}{\partial t1} = (1-t_1)^{-1}(1-t_2)^{-1}dt_1 = (1-t_1)^{-2}(1-t_2)^{-1} I_{t_{1,2=0}} = 1$
$E(y)=\frac{\partial M x,y(t_1,t_2)}{\partial t2} = (1-t_1)^{-1}(1-t_2)^{-1}dt_2 = (1-t_1)^{-1}(1-t_2)^{-2} I_{t_{1,2=0}} = 1$
$E(xy)=\frac{\partial M x,y(t_1,t_2)}{\partial t2t1} = (1-t_1)^{-1}(1-t_2)^{-1}dt_2 = (1-t_1)^{-2}(1-t_2)^{-2} I_{t_{1,2=0}} = 1$

$E(x^2) = \frac{\partial Mx,y(t_1,t_2)}{\partial^2 t1} = (1-t_1)^{-1}(1-t_2)^{-1}dt_1 = (1-t_1)^{-2}(1-t_2)^{-1} = 2(1-t_1)^{-3}(1-t_2)^{-1}I_{t_{1,2=0}} = 2$

$E(y^2) = \frac{\partial Mx,y(t_1,t_2)}{\partial^2 t1} = (1-t_1)^{-1}(1-t_2)^{-1}dt_1 = (1-t_1)^{-1}(1-t_2)^{-2} = (1-t_1)^{-1} * 2(1-t_2)^{-3}I_{t_{1,2=0}} = 2$

$P_{xy}=\frac{Exy-Ex*Ey}{\sqrt{Ex^2-\left(E(x)\right)^2}*\sqrt{Ey^2-\left(E(y)\right)^2}} = \frac{1-1*1}{\sqrt{2-(1)^2}*\sqrt{2-(1)^2}} = 0$

3 – 7 Stochastic Independence:

Def^n.: Let $x_1, x_2, ..., x_p$ be a p-dimensional r.v's with joint p.d.f $f(x_1, x_2, ..., x_p)$, and let $f(x_1), f(x_2), \dots, f(x_p)$ are the marginal p.d.f of $x_1, x_2, ..., x_p$, then the r.v's $x_1, x_2, ..., x_p$ are said to be stochastically independent ; iff ;

$$1) R(x_1, x_2, \dots, x_p) = R(x_1).R(x_2).\dots.R(x_p)$$

$$2) f(x_1, x_2, \dots, x_p) = f(x_1).f(x_2).\dots.f(x_p) = \prod_{i=1}^{p} f(x_i)$$

Def^n.: when p=2 ; let x & y be two r.v's ,with j.p.d.f of $f(x,y)$, and let $f(x), f(y)$ be the marginal p.d.f of x & y respectively then x & y are stochastically independent , iff ;

$$1) R(x,y) = R(x).R(y)$$

$$i.e: R[(x,y) = -\infty < x, y < \infty] = R[(x: -\infty < x < \infty].R[(y: -\infty < y < \infty]$$

$$2) f(x,y) = f(x).f(y) where: Rxy = Range of x \& y.$$

$$Rx = Range of x.$$

$$Ry = Range of y.$$

$$where f(x), f(y) are non-negative f^n.$$

$$i.e: f(x), f(y) > 0$$

Example (18) $Let x \& y are two r.v's with j.p.d.f$

$$f(x,y) = \begin{cases} 8\,xy & 0 < x < 1 \\ & 0 < y < 1 \\ 0 & o.w \end{cases}$$

$$show\ that\ x \& y\ are\ stoch.indep.?$$

Result:

f(x,y)=8xy 0<x<1 ; 0<y<1

show that x&y are stoch . indep?

f(x)=$\int_0^1 8xydy = \frac{8xy^2}{2} I_0^1 = 4x$

f(y)= $\int_0^1 8xydx = \frac{8yx^2}{2} I_0^1 = 4y$

E(x)=$\int_0^1 x * 4xdx = \int_0^1 4x^2 dx = \frac{4x^3}{3} I_0^1 = \frac{4}{3}$

E(y)= $\int_0^1 y * 4ydy = \int_0^1 4y^2 dy = \frac{4y^3}{3} I_0^1 = \frac{4}{3}$

E(xy)= $\int_0^1 \int_0^1 xy * 8xydxdy = \int_0^1 \int_0^1 8x^2y^2dxdy = \int_0^1 y^2dy(\frac{8x^3}{3} I_0^1) =$
$\int_0^1 \frac{8}{3}y^2dy = \frac{8}{3} * \frac{y^3}{3} I_0^1 = \frac{8}{9}$
E(xy)=E(x)*E(y)

$$\frac{8}{9} \neq \frac{4}{3} * \frac{4}{3} \quad thus\ do\ not\ stoch\ indep$$

Theorm 1 :

IF x&y are stochastically independent r.v·s

$$p(a < x < b, c < y < d) = p(a < x < b).p(c < y < d)$$

For evry $a < b, c < d$, where $a, b, canddare constant.

Example (19) $tx\&ybetwor.v{\cdot}swith$

$$f(x,y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & o.w \end{cases}$$

$Is\ x\&y\ stoch.indep.or\ \ not\ by\ u\,sin\,g\ theorm(1)$

Result:

f(x,y)=x+y 0<x<1 ; 0<y<1
is x & y stoch indep . or not by using theorem 1
f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$
f(y)=$\int_0^1 (x+y)dx = xy + \frac{x^2}{2} I_0^1 = y + \frac{1}{2}$
let a=0 b=$\frac{1}{2}$, $c = 0 \quad d = \frac{1}{3}$
P_r(a<x<b,c<y<d)=P_r(a<x<b)*P_r(c<y<d)
$=\int_0^{\frac{1}{2}} \int_0^{\frac{1}{3}} (x+y)dxdy = \int_0^{\frac{1}{2}} (x+\frac{1}{2})dx * \int_0^{\frac{1}{3}} (y+\frac{1}{2})dy$
$=\int_0^{\frac{1}{2}} (xy + \frac{y^2}{2}) I_0^{\frac{1}{3}} dx = \frac{x^2}{2} + \frac{x}{2} I_0^{\frac{1}{2}} * \frac{y^2}{2} + \frac{y}{2} I_0^{\frac{1}{3}}$
$=\int_0^{\frac{1}{2}} (\frac{x}{3} + \frac{1}{18})dx = \frac{3}{8} * \frac{2}{9}$
$=\frac{x^2}{3*2} + \frac{x}{18} I_0^{\frac{1}{2}} = \frac{3}{8} * \frac{2}{9}$

$$\frac{15}{216} \neq \frac{3}{8} * \frac{2}{9} \quad do\ not\ stoch\ indep$$

Theorm 2 :

IF x&y are two independent r.v·s with j.p.d.f F(x,y) , and F(x)&F(y)

are c.d.f of x&y respectively m then; x & y sto. Indep. , iff ;

$$F(x,y)=F(x).F(y)$$

Theorm 3 :

IF x&y are two independent r.v's with j.p.d.f f(x,y) , and let f(x)&f(y) are two marginal p.d.f and let u(x)&u(y) be two function of x&y then; x & y sto. Indep. , iff ;

$$E[u(x).u(y)]= E[u(x)].E[u(y)]$$

Example (20) *let the j.p.d.f of x & y be;*

$$f(x,y)=\begin{cases}\frac{1}{4} & (x,y)=(0,0)\ ,\ (1,1)\ ,\ (1,-1)\ ,\ (2,0)\\ 0 & o.w\end{cases}$$

1) *Are x & y independent* 2) *Calculate* ρ_{xy} .

Result:

1)are x &y independent

Y \ X	0	1	2	Sum y
-1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{2}{4}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
Sum x	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

E(x)=$\sum_0^2 xp(x) = 0*\frac{1}{4}+\frac{2}{4}+2*\frac{1}{4}=1$

E(y)= $\sum_{-1}^1 yp(y) = -1*\frac{1}{4}+0*\frac{2}{4}+\frac{1}{4}=0$

E(xy)=$\sum_0^2\sum_{-1}^1 xyp(x,y) = 0+1*-1*\frac{1}{4}+1*1*\frac{1}{4}+0=0$

E(xy)=E(x)*E(y)

0=1*0 thus are independent

2) Calculate P_{xy}?

E(x^2)=$\sum_0^2 x^2p(x) = 0^2*\frac{1}{4}+\frac{2}{4}+2^2*\frac{1}{4}=\frac{6}{4}=\frac{3}{2}$

E(y^2)= $\sum_{-1}^1 y^2p(y) = -1^2*\frac{1}{4}+0*\frac{2}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

$P_{xy}=\frac{Exy-Ex*Ey}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}}=\frac{0-1*0}{\sqrt{\frac{3}{2}-(1)^2}*\sqrt{\frac{1}{2}-(0)^2}}=0$

Example (21) *Find* $pr(0<x<\frac{1}{3}, 0<y<\frac{1}{3})$; *if the r.v's* $x\,\&\,y$ *have the j.p.d.f*

$$f(x,y)=\begin{cases}4x(1-y) & 0<x<1\ ,\ 0<y<1\\ 0 & o.w\end{cases}$$

show that $x\,\&\,y$ *are stoch. indep.*

Result:

f(x,y)=4x(1-y) 0<x<1; 0<y<1

$f(x)=\int_0^1(4x-4xy)dy=4xy-\frac{4xy^2}{2}I_0^1=4x-2x=2x$

$f(y)=\int_0^1(4x-4xy)dx=\frac{4x^2}{2}-\frac{4yx^2}{2}I_0^1=2-2y=2-2y$

$P_r(0<x<\frac{1}{3};0<y<\frac{1}{3})=p_r\left(0<x<\frac{1}{3}\right)*p_r(0<y<\frac{1}{3})$

$=\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}(4x-4xy)dxdy=\int_0^{\frac{1}{3}}2xdx*\int_0^{\frac{1}{3}}(2-2y)dy$

$=\int_0^{\frac{1}{3}}\left(2x^2-\frac{4yx^2}{2}I_0^{\frac{1}{3}}\right)dy=x^2\,I_0^{\frac{1}{3}}*2y-y^2I_0^{\frac{1}{3}}$

$=\int_0^{\frac{1}{3}}\frac{2}{9}+\frac{2y}{9}dy=\frac{1}{9}*\frac{5}{9}$

$=\frac{2y}{9}+\frac{2y^2}{9*2}I_0^{\frac{1}{3}}=\frac{5}{9}*\frac{1}{9}$ ⇨ $\frac{5}{81}=\frac{5}{9}*\frac{1}{9}$

$\frac{5}{81}=\frac{5}{9}*\frac{1}{9}$ *yes indep stoch*

Example (22) $tx\&ybetwor.v's withj.p.d.f$

$$p(x,y)=\begin{cases}4xy0<x,y<1\\ 0o.w\end{cases}$$

$Isx\&yarestoch.indep.\&findpr(0<x<\frac{1}{2},0.3<y<0.8)$

Result:

P(x,y)=4xy 0<x<1 ;0<y<1

Find $P_r(0<x<\frac{1}{2}\ ,0.3<y<0.8)$ =?

$f(x)=\int_0^1 4xydy=\frac{4xy^2}{2}I_0^1=2x$

f(y)= $\int_0^1 4xydx = \frac{4yx^2}{2} I_0^1 = 2y$

$P_r(0<x<\frac{1}{2}, 0.3 < y < 0.8) = P\left(0 < x < \frac{1}{2}\right) * P(0.3 < y < 0.8)$

$=\int_0^{\frac{1}{2}} \int_{0.3}^{0.8} 4xydydx = \int_0^{\frac{1}{2}} 2xdx * \int_{0.3}^{0.8} 2ydy$

$=\int_{0.3}^{0.8} \frac{4x^2y}{2} I_0^{0.5} dy = x^2 I_0^{0.5} * y^2 I_{0.3}^{0.8}$

$=\int_{0.3}^{0.8} \frac{y}{2} dy = 0.25 * 0.55$

$=\frac{y^2}{4} I_{0.3}^{0.8} = 0.25 * 0.55$

=0.1375=0.1375 yes are stoch indep

Example (23)

$$f(x,y) = \begin{cases} e^{-(x+y)} & x,y \geq 0 \\ 0 & o.w \end{cases}$$

show that x&y are stoch. indep. u sin g by theory 4

$$Mx,y(t_1,t_2) = Mx,y(t_1,0).Mx,y(0,t_2)$$

$$Mx,y=E(e^{t_1x+t_2y})$$

Result:

$f(x,y)=e^{-x-y} \quad x,y \geq 0$

$Mx,y=E(e^{t_1x+t_2y}) = \int_0^\infty \int_0^\infty e^{t_1x+t_2y} * e^{-x-y}dxdy = \int_0^\infty e^{-x(1-t_1)}dx *$
$\int_0^\infty e^{-y(1-t_2)}dy = \frac{-e^{-x(1-t_1)}}{1-t_1} I_0^\infty * \frac{-e^{-y(1-t_2)}}{1-t_2} I_0^\infty = \frac{1}{(1-t_1)} * \frac{1}{(1-t_2)} I_{t_{1,2=0}}$

$Mx(t_1)= \frac{1}{(1-t_1)} * \frac{1}{(1-t_2)} I_{t_2=0} = (1-t_1)^{-1}$

$My(t_2)= \frac{1}{(1-t_1)} * \frac{1}{(1-t_2)} I_{t_1=0} = (1-t_2)^{-1}$

Mxy(t1t2)=Mx(t1)*My(t2)

$(1-t1)^{-1}(1-t2)^{-1}=(1-t1)^{-1}(1-t2)^{-1}$ yes are stoch indep

Theorm 4 :

IF x&y be two r.v's with j.p.d.f f(x,y) , and let f(x)&f(y) be two marginal p.d.f x&y respectively , let $Mx,y(t_1,t_2)$ denoted the j.m.g.f $dist^n$, then x&y are stochastically Independent , iff ;

$$Mx,y(t_1,t_2) = Mx,y(t_1,0).Mx,y(0,t_2)$$

Example (24) *let* x & y *be two r.v's with* $\cong N(M_1,\sigma_1^2)$ & $\cong N(M_2,\sigma_2^2)$ *respectively , and let* x & y *be stoch. indep. show that* $\rho_{xy} = 0$.

Result:

$N(M2,\sigma_2^2) respectivley, let\ x\&y\ be\ stoch\ indep\ show\ that\ P_{xy} = 0.$

Cov(x,y)=E(x-M1)*(y-M2)=E(xy-M1y-M2X+M1M2)

Exy-M1Ey-M2Ex+M1M=Exy-M1M2-M1M2+M1M2=Exy-M1M2

Hence : x & y are indep. Then Exy=Ex*Ey thus cov(x,y)=Ex*Ey-M1M2

Cov(x,y)=Ex*Ey-M1*M2

=M1*M2-M1*M2=0

$$P_{xy}=\frac{cov(x,y)}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}} = \frac{0}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}} = 0$$

Example (25) $'x\&y are two independent r.v's$

$$prove\ that\ E(xy) = Ex.Ey$$

Result

E(xy)=E(x)*E(y)

Let x & y are discrete r.vs

Thus Exy=$\sum_{\forall y}\sum_{\forall x} XiYi : P(xi,yi)$

Because x & y are indep . then

P(Xi,Yi)=P(Xi)*P(Yi)

=EX*Ey

Note

1)if x & y indep, then fxy=0

2)If Exy=0 then not necessary to here X & Y indep.

Example (26)

$$f(x) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & o.w \end{cases}$$

$show\ that\ x\&y\ are\ stoch.indep.by\ u\ sin\ g\ theorm4$

Result

f(x,y)=x+y 0<x<1 ; 0<y<1

f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$

f(y)=$\int_0^1 (x+y)dx = xy + \frac{x^2}{2} I_0^1 = y + \frac{1}{2}$

E(x)=$\int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^2+\frac{x}{2}\right)dx = \frac{x^3}{3} + \frac{x^2}{2*2} I_0^1 = \frac{7}{12}$

E(y)= $\int_0^1 y\left(y+\frac{1}{2}\right)dy = \int_0^1 \left(y^2+\frac{y}{2}\right)dy = \frac{y^3}{3} + \frac{y^2}{2*2} I_0^1 = \frac{7}{12}$

E(xy)=$\int_0^1 \int_0^1 xy(x+y)dydx = \int_0^1 (\frac{x^2y^2}{2} + \frac{xy^3}{3}) I_0^1 dx = \int_0^1 (\frac{x^2}{2} + \frac{x}{3}) = \frac{x^3}{6} + \frac{x^2}{6} I_0^1 = \frac{2}{6}$

E(xy)=Ex*Ey

$$\frac{2}{6} \neq \frac{7}{12} * \frac{7}{12}$$

Example (27) $owthatther.v'sx\&ywiththej.p.d.f$

$$f(x) = \begin{cases} 12xy(1-y) & 0<x<1, 0<y<1 \\ 0 & o.w \end{cases}$$

$are\ stochastically\ indep.$

Result

f(x,y)=12xy-12xy^2 $0<x<1; 0<y<1$

f(x)=$\int_0^1 f(x,y)dy = \int_0^1 (12xy - 12xy^2)dy = 6xy^2 - 4xy^3 I_0^1 = 2x$

f(y)= $\int_0^1 f(x,y)dx = \int_0^1 (12xy - 12xy^2)dx = 6xy^2 - 6X^2y^2 I_0^1 = 6y - 6y^2$

F(x)=$\int_0^x 2xdx = x^2 I_0^x = x^2$

F(y)= $\int_0^y (6y - 6y^2)dy = 3y^2 - 2y^3 I_0^y = 3y^2 - 2y^3$

F(x,y)=$\int_0^y \int_0^x (12xy - 12xy^2)dxdy = \int_0^y (6x^2y - 6x^2y^2) I_0^x dy = \int_0^y (6x^2y - 6x^2y^2)dy = 3x^2y^2 - 2x^2y^3 I_0^y = 3x^2y^2 - 2x^2y^3$

F(x,y)=F(x)*F(y)

$$3x^2y^2 - x^2y^3 = 2x^2(3y^2 - 2y^3)\ are\ stoch\ indep$$

Conditional probability

Review

Probability

Stage (2)

$p(A/B) = \frac{P(A\cap B)}{P(B)}; P(B) > 0$

$$p(B/A) = \frac{P(A\cap B)}{P(A)}; P($$

Then

$$P(A\cap B) = P(B).P($$

$$P(A\cap B) = P(A).P(B/A)$$

3-8 Conditional Distribution Function

IF x& y are two r. v 's, with j. p. d. f f (x, y); then the conditional distribution function of x given that Y=y is defined as ;

$f(x/Y=y) = \frac{f(x,y)}{f(y)}, f(y) > 0$

and the con. Dist fun. Of Y given that X=x defined as ;

$$f(y/X=x) = \frac{f(x,y)}{f(x)}, f(x) > 0$$

Where f(x) & f(y) are the marginal p. d. f of x& y respectively if x& are two r. v 's , then con. Probability of ;

$p(a \le x \le b/Y=y) = \sum_{x=a}^{b} p(x/Y=y)\ if x\&y are dist.$

$$\int_a^b f(x/Y=y)dx if x\&y are cont.$$

Remark :

If x& y are two indep r. v 's , then

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{f(x).f(y)}{f(y)} = f(x)$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{f(y).f(x)}{f(x)} = f(y)$$

Properties of the conditional p. d. f

F(x/y) is a p. d. f

- $0 \leq f(x/y) \leq 1$
- $\sum_{\forall x} f(x/y) = 1 \, for discr. v's$

$$\int_{\Re x} f(x/y)dx = 1 for con. r. v's$$

proof:

$$\int_{Rx} f(x/y)dx = 1$$

$$= \int_{Rx} \frac{f(x,y)}{f(y)}dx = \frac{1}{f(y)}\int_{Rx} f(x,y)dx = \frac{1}{f(y)}.f(y) = 1$$

Also

$$= \sum_{\forall x} f(x/y) = 1$$

$$\sum_{\forall x} \frac{f(x,y)}{f(y)} = \frac{1}{f(y)}\sum_{\forall x} f(x,y) = \frac{1}{f(y)}.f(y) = 1$$

Example (28) :t x& y be two r.v 's with j. p. d. f

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & o.w \end{cases}$$

Find:

$$1. p(x < \frac{1}{2}) 2. pr(y < \frac{1}{3}) 3. pr(x < \frac{1}{2}/y = \frac{3}{4})$$

$$4. pr(y < \frac{1}{3}/x = \frac{1}{6}) 5. pr(x < \frac{1}{2}, y < \frac{3}{4}) 6. pr(x = \frac{1}{2}/y = \frac{1}{3})$$

Result

f(x,y)=2 0<x<y<1

f(x)=$\int_x^1 2dy = 2yI_x^1 = 2 - 2x$

f(y)=$\int_0^y 2dx = 2xI_0^y = 2y$

1)$P_r(X<\frac{1}{2})$ =?

$P_r(X<\frac{1}{2}) = \int_0^{\frac{1}{2}} f(x)dx = \int_0^{\frac{1}{2}}(2-2x)dx = 2x - X^2 I_0^{\frac{1}{2}} = 1 - \frac{1}{4} = \frac{3}{4}$

2) $P_r(Y<\frac{1}{3}) = \int_0^{\frac{1}{3}} f(y)dy = \int_0^{\frac{1}{3}}(2y)dy = y^2 I_0^{\frac{1}{3}} = \frac{1}{9}$

3)$P_r(X<\frac{1}{2}/y=\frac{3}{4}) = \frac{f(x,y)}{f(y=\frac{3}{4})} = \frac{2}{2*\frac{3}{4}} = \int_0^{\frac{1}{2}} \frac{4}{3} dx = \frac{4x}{3} I_0^{\frac{1}{2}} = \frac{2}{3}$

4) $P_r(y<\frac{1}{3}/x=\frac{1}{6}) = \frac{f(x,y)}{f(x=\frac{1}{6})} = \frac{2}{2-2*\frac{1}{6}} = \int_0^{\frac{1}{3}} \frac{6}{5} dx = \frac{6x}{5} I_0^{\frac{1}{3}} = \frac{2}{5}$

5)) $P_r(x<\frac{1}{2}/y<\frac{3}{4})$ =?

$f(x,y)=\int_0^{\frac{3}{4}} \int_0^{y} 2dxdy = \int_0^{\frac{3}{4}} 2xI_0^{y} dy = \int_0^{\frac{3}{4}} 2ydy = y^2 I_0^{\frac{3}{4}} = \frac{9}{16}$

6)$P(x=\frac{1}{2}/y=\frac{1}{2}) = 0$ *because contionus*

Example (29) :t

$$f(x,y) = \frac{x+2y}{18} \qquad y = 1,2x = 1,2$$

Find :

1. cond. P. d. f of x given y=1 [f(x)/y=1]
2. cond. P. d. f of y given x=2 [f(y)/x=2]
3. if x &y are indep r.v 's

Result

$f(x,y)=\frac{x+2y}{18} \quad y = 1,2: x = 1,2$

1)cond.p.d.f x given y=1{f(x/y=1)}?

$f(y)=\sum_1^2 \frac{x+2y}{18} = \frac{3+4y}{18}$

$f(x)= \sum_1^2 \frac{x+2y}{18} = \frac{2x+6}{18} = \frac{x+3}{9}$

thus $f(x/y=1)=\frac{f(x,y=1)}{f(y=1)} = \frac{\frac{x+2(1)}{18}}{\frac{3+4(1)}{18}} = \frac{x+2}{7}$

2)cond.p.d.f y given x=2{f(y/x=2)}?

$f(y/x=2)=\frac{f(x,y=2)}{f(x=2)} = \frac{\frac{2+2(y)}{18}}{\frac{2(2)+6}{18}} = \frac{2+2y}{10}$

3) if x & y are indep r.vs?

Y \ X	1	2	Sum y
1	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{7}{18}$
2	$\frac{5}{18}$	$\frac{6}{18}$	$\frac{11}{18}$

Sum x	$\frac{8}{18}$	$\frac{10}{18}$	1

E(x)=$\sum_1^2 xp(x) = 1*\frac{8}{18}+2*\frac{10}{18}=\frac{28}{18}$

E(y)= $\sum_1^2 yp(y) = 1*\frac{7}{18}+2*\frac{11}{18}=\frac{29}{18}$

E(xy)=$\sum_1^2\sum_1^2 xyp(x,y) = \frac{3}{18}+1*2*\frac{5}{18}+2*1*\frac{4}{18}+2*2*\frac{6}{18}=\frac{45}{18}$

$$\frac{45}{18}\neq\frac{28}{18}*\frac{29}{18}\ are\ not\ stoch\ indep$$

Example (30) :t x&y be two r. v 's with follow in j. p. d. f

y \ x	0	1	f(y)
0	0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$
2	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	$\frac{1}{8}$	0	$\frac{1}{8}$
f(x)	$\frac{4}{8}$	$\frac{4}{8}$	1

find

1. con. P.d.f of y given x=1 p(y/x=1)
2. con. P.d.f of x given y=2 p(x/y=1)
3. if x &y are indep

Result

Example 49) let x & y are two r.vs with j.p.d.f following

Y \ X	0	1	Sum y
0	0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$
2	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	$\frac{1}{8}$	0	$\frac{1}{8}$
Sum x	$\frac{4}{8}$	$\frac{4}{8}$	1

1)con.p.d.f of y given x=1

f(y/x=1)=$\frac{f(x=1,y=0)}{f(x=1)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$ & $\frac{f(x=1,y=1)}{f(x=1)} = \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{2}{4}$ & $\frac{f(x=1,y=2)}{f(x=1)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$ & $\frac{f(x=1,y=3)}{f(x=1)} = \frac{0}{\frac{4}{8}} = 0$

f(y/x=1)=$\begin{bmatrix} \frac{1}{4} & (1,0) \\ \frac{2}{4} & (1,1) \\ \frac{1}{4} & (1,2) \\ 0 & (1,3) \\ 0 & o.w \end{bmatrix}$

2)con.p.d.f of x given y=2?

f(x/y=2)=$\frac{f(x,y=2)}{f(y=2)}$ & $\frac{f(x=0,y=2)}{f(y=2)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}$ & $\frac{f(x=1,y=2)}{f(y=2)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$

f(x/y=2)=$\begin{bmatrix} \frac{2}{3} & (0,2) \\ \frac{1}{3} & (1,2) \\ 0 & o.w \end{bmatrix}$

3)if x & y stoch indep?

E(x)=$\sum_0^1 xp(x) = 0 + \frac{4}{8} = \frac{4}{8}$

E(y)= $\sum_0^3 yp(y) = 0 + \frac{3}{8} + 2 * \frac{3}{8} + 3 * \frac{1}{8} = \frac{12}{8}$

E(xy)=$\sum_0^1 \sum_0^3 xyp(x,y) = 0 + 1 * 1 * \frac{2}{8} + 1 * 2 * \frac{1}{8} + 1 * 3 * 0 = \frac{4}{8}$

$$\frac{4}{8} \neq \frac{4}{8} * \frac{12}{8} \quad are\ not\ stoch\ indep$$

Example (31) $\boxed{Example(50)}$:let x& y having the following j.$p.d.f$

$$f(x,y) = 21x^2y^3 0 < x < y < 1$$

$$find; f(x), f(y), f(x/Y = y), f(Y/X = x)$$

Result

f(x,y)=$21x^2y^3 \quad 0 < x < y < 1$

1)f(x)=$\int_x^1 21x^2y^3dy = \frac{21x^2y^4}{4} I_x^1 = \frac{21x^2}{4} - \frac{21x^6}{4}$

$f(x) \begin{bmatrix} \frac{21x^2}{4} - \frac{21x^6}{4} & 0 < x < 1 \\ 0 & o.w \end{bmatrix}$

2)f(y)= $\int_0^y 21x^2y^3dx = \frac{21x^3y^3}{3} I_0^y = 7y^6$

$$f(y)\int_0^1 21x^2y^3dx = 21y^3\frac{x^3}{3}I_0^1 = 7y^3$$

3)f(x/Y=y)?

f(x/Y=y)=f(x/y)=$\frac{f(x,y)}{f(y)} = \frac{21x^2y^3}{7y^3} = 3x^2$

4)f(y/X=x)?

f(y/X=x)=f(y/x)=$\frac{f(x,y)}{f(x)} = \frac{21x^2y^3}{\frac{21x^2}{4}-\frac{21x^6}{4}} = \frac{4y^3}{1-x^4}$

3-9 Conditional Cumulative Distribution Function [cond . c. d. f]

Let x&y are two r.v's with cond p.d.f of x given that Y=y ,then the con .c.d.f of x given thatY =y is defined;

$$F(X/Y=y) = \begin{cases}\sum_{-\infty}^{x} p(X/Y=y) for disc.v.r's\\ \int_{-\infty}^{x} f(X/Y=y)dx for cont.r.v's\end{cases}$$

And similarty :

$$F(Y/X=x) = \begin{cases}\sum_{-\infty}^{y} p(Y/X=x) for disc.v.r's\\ \int_{-\infty}^{y} f(Y/X=x)dy for cont.r.v's\end{cases}$$

Then if x& y are conti m nuous r. v 's then;

$f(X/Y=y) = \frac{d}{dx}F(X/Y=y)$

$$f(Y/X=x) = \frac{d}{dy}F(Y/X=x)$$

Example (32)

$\boxed{Example(55)} Let$

$$f(x,y) = \begin{cases}x+y & 0<x,y<1\\ 0 & o.w\end{cases}$$

Find F(Y/X)?

Result

f(x,y)=x+y 0<x<1;0<y<1

f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$

f(y/x)=$\frac{x+y}{x+\frac{1}{2}}$

find F(y, x)?

F(x/y)=$\int_0^y f(y/x)\, dy = \int_0^y \frac{x+y}{x+\frac{1}{2}} dy = \frac{1}{x+y}\left(xy + \frac{y^2}{2} I_0^y\right) = \frac{xy+\frac{y^2}{2}}{x+\frac{1}{2}}$

Example (33) :t

$$F(x, y) = \frac{x+y}{21} x = 1,2,3 y = 1,2$$

$= 0o.w$

Find ; F(Y/X)

Result

f(x,y)=$\frac{x+y}{21}$ $x = 1,2,3\ ; y = 1,2$

find F(y/x)

j.p.d

Y \ X	1	2	3	Sum y
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
Sum x	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	1

f(y/x)

Y \ X	1	2	3
1	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$
2	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{5}{9}$

F(y/x)

Y \ X	1	2	3
y<1	0	0	0
1≤ y <2	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$
$y \geq 2$	1	1	1

3-10 Conditional Expectation and Variance

- **Conditional Expectation**

Let x& y be two r. v 's with con. P. d. f f(x/y) , then let g(x,y) be a fun. Of two r. v 's , then the cond. Expectation of g(x,y) given that Y=y is defined to be ;

$$E[g(x,y)/Y=y]=\begin{cases}\sum_{\forall x} g(x,y)f(x/y) disc.\\ \int_{Rx} g(x,y)f(x/y)dx cont.\end{cases}$$

$$E[g(x,y)/X=x]=\begin{cases}\sum_{\forall y} g(x,y)f(y/x) disc.\\ \int_{Ry} g(x,y)f(y/x)dy cont.\end{cases}$$

As special case of g(x,y)=x, we define

E(X/Y=y), which is called the con. Mean of x given that Y=y is defined as;

$$E(X/Y=y)=\begin{cases}\sum_{\forall x} xf(X/Y=y) disc\\ \int_{Rx} xf(X/Y=y) con.\end{cases}$$

And if g(x,y)=Y ; then;

$$E(Y/X=x)=\begin{cases}\sum_{\forall y} yf(Y/X=x) disc\\ \int_{Ry} yf(Y/X=x) con.\end{cases}$$

and if $g(x,y)=x^2$; then

$E[X^2/Y=y]$ is called the squares mean of given that Y=y

$$E[X^2/Y=y]=\begin{cases}\sum_{\forall x} X^2 f(X/Y=y) for disc.\\ \int_{Rx} X^2 f(X/Y=y)dx for cont.\end{cases}$$

In general case $E[X^r/Y=y]$ is called con. Moment of order (r) about original of variable x given that Y=y .

$$E[X^r/Y=y]=\begin{cases}\sum_{\forall x} X^r f(X/Y=y) for disc.\\ \int_{Rx} X^r f(X/Y=y)dx for cont.\end{cases}$$

- **Conditional variance**

$V(X/Y=y)=E[X^2/Y=y]-[E(X/Y=y)]^2$

$V(Y/X=x)=E[Y^2/X=x]-[E(Y/X=x)]^2$

Remark :

$$E[X/Y] \neq E[\frac{x}{y}]$$

$$where E[X/Y] = \int Xf(x/y)\,dx$$

$$but E(\frac{x}{y}) = \int\int \frac{x}{y} f(x,y)\,dxdy$$

Example (34) t f(x,y)=x+y 0<x<1 0<y<1

Find 1.E(y/x) 2.v(y/x)

Result

f(x,y)=x+y 0<x<1 ; 0<y<1

f(x)=$\int_0^1 (x+y)dy = xy + \frac{y^2}{2} I_0^1 = x + \frac{1}{2}$

f(y)=$\int_0^1 (x+y)dx = xy + \frac{x^2}{2} I_0^1 = y + \frac{1}{2}$

f(y/x)=$\frac{x+y}{x+\frac{1}{2}}$

1)E(y/x)?

E(y/x)=$\int_{Ry} yf(y/x)dy = \int_0^1 y * \frac{x+y}{x+\frac{1}{2}} dy = \frac{1}{x+\frac{1}{2}}\left(\frac{xy^2}{2} + \frac{y^3}{3} I_0^1\right) = \frac{\frac{3x+2}{3}}{2x+1} = \frac{3x+2}{6x+3}$

2)v(y/x)?

V(y/x)=E(y^2/x)-(E(y/x))^2

E(y^2/x)=$\int_0^1 y^2 * \frac{x+y}{x+\frac{1}{2}} dy = \frac{1}{x+\frac{1}{2}}\left(\frac{xy^3}{3} + \frac{y^4}{4} I_0^1\right) = \frac{\frac{4x+3}{6}}{2x+1} = \frac{4x+3}{12x+6}$

V(y/x)=$\frac{\frac{4x+3}{6}}{2x+1} - \left(\frac{\frac{3x+2}{3}}{2x+1}\right)^2 = \frac{4x+3}{12x+6} - \frac{9x^2+12x+4}{(6x+3)^2} =$

$\frac{24x^2+30x+9-(18x^2+24x+8)}{2(6x+3)^2} = \frac{6x^2+6x+1}{2(6x+3)^2}$

$\boxed{Example(58)}$:let

$$f(x,y) = \frac{x+2y}{18}(x,y) = (1,1),(1,2)(2,1),(2,2)$$

$$= 0 o.w$$

Find E[Y/X]. V[Y/X]

Result

f(x,y)=$\frac{x+2y}{18}$ $(x,y) = (1,1)(1,2)(2,1)(2,2)$

Y \ X	1	2	Sum y
1	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{7}{18}$
2	$\frac{5}{18}$	$\frac{6}{18}$	$\frac{11}{18}$
Sum X	$\frac{8}{18}$	$\frac{10}{18}$	1

f(y/x)

Y \ X	1	2
1	$\frac{3}{8}$	$\frac{4}{10}$
2	$\frac{5}{8}$	$\frac{6}{10}$

1)E(y/x)?

E(y/x=1)=$\sum_1^2 yf(y/x=1) = 1*\frac{3}{8}+2*\frac{5}{8}=\frac{13}{8}$

E(y/x=2)= $\sum_1^2 yf(y/x=2) = 1*\frac{4}{10}+2*\frac{6}{10}=\frac{16}{10}$

E(y/x)= $\frac{13}{8}+\frac{16}{10}=\frac{129}{40}=3.225$

2)V(y/x)=v(y/x=1)+v(y/x=2)

E(y^2/x)=$\sum_1^2 y^2 f(y/x=1) = \frac{3}{8}+2^2*\frac{5}{8}=\frac{23}{8}$

E(y^2/x=2)= $\sum_1^2 y^2 f(y/x=2) = 1*\frac{4}{10}+2^2*\frac{6}{10}=\frac{28}{10}$

V(y/x=1)=E(y^2/x=1)-(E(y/x=1)^2=$\frac{23}{8}-\left(\frac{13}{8}\right)^2=\frac{15}{64}$

V(y/x=2)=E(y^2/x=2)-(E(y/x=2)^2=$\frac{28}{10}-\left(\frac{16}{10}\right)^2=\frac{24}{100}$

V(y/x)=v(y/x=1)+v(y/x=2)= $\frac{15}{64}+\frac{24}{100}=\frac{759}{1600}$

Example (35) x & y have the joint p.d.f f(x,y)

(x,y)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
F(x,y)	1/18	3/18	4/18	3/18	6/18	1/18

Find

1. f(x) 2.f(y) 3.Ex 4.Ey 5.Pxy 6.F(x,y)
2. 7.E(x/y) 8.E(y/x) 9.F(x/y) 10. F(y/x) 11.V(x/y) 12.sto. Indep. Or not ?

Result

X,Y	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
f(x,y)	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{1}{18}$

1)f(x)?

Y \ X	0	1	2	Sum Y
0	$\frac{1}{18}$	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{11}{18}$
1	$\frac{3}{18}$	$\frac{3}{18}$	$\frac{1}{18}$	$\frac{7}{18}$
Sum x	$\frac{4}{18}$	$\frac{7}{18}$	$\frac{7}{18}$	1

$$f(x)=\begin{bmatrix} \frac{4}{18} & x=0 \\ \frac{7}{18} & x=1 \\ \frac{7}{18} & x=2 \\ 0 & o.w \end{bmatrix}$$

2)f(y)

$$f(y)=\begin{bmatrix} \frac{11}{18} & y=0 \\ \frac{7}{18} & y=1 \\ 0 & 0.w \end{bmatrix}$$

3)E(x)=?

$$E(x)=\sum_0^2 xp(x) = 0+\frac{7}{18}+2*\frac{7}{18}=\frac{21}{18}=\frac{7}{6}$$

4)E(y)?

$$E(y)=\sum_0^1 yp(y) = 0+\frac{11}{18}+1*\frac{7}{18}=\frac{7}{18}$$

5)Pxy?

$$E(xy)=\sum_0^1\sum_0^2 xyp(x,y) = 0+0+1*1*\frac{3}{18}+0+2*1*\frac{1}{18}=\frac{5}{18}$$

$$E(y^2)=\sum_0^1 y^2p(y) = 0+\frac{11}{18}+1^2*\frac{7}{18}=\frac{7}{18}$$

$$E(x^2)=\sum_0^2 x^2p(x) = 0+\frac{7}{18}+2^2*\frac{7}{18}=\frac{35}{18}$$

$$P_{xy}=\frac{Exy-Ex*Ey}{\sqrt{Ex^2-(E(x))^2}*\sqrt{Ey^2-(E(y))^2}}=\frac{\frac{5}{18}-\frac{21}{18}*\frac{7}{18}}{\sqrt{\frac{35}{18}-\left(\frac{21}{18}\right)^2}*\sqrt{\frac{7}{18}-\left(\frac{7}{18}\right)^2}}=-0.472$$

6)F(x,y)?

Y \ X	x< 0	$0\leq x<1$	$1\leq x<2$	x≥ 2
y< 0	0	0	0	0
$0\leq y<1$	0	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{11}{18}$
y≥ 1	0	$\frac{4}{18}$	$\frac{11}{18}$	1

7)E(x/y)?

F(x/y)

Y	0	1	2
0	$\frac{1}{11}$	$\frac{4}{11}$	$\frac{6}{11}$
1	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$

E(x/y)=E(x/y=0)+E(x/y=1)

E(x/y=0)=$\sum_0^2 xf(x/$y=0)=0+$1*\frac{4}{11}+2*\frac{6}{11}=\frac{16}{11}$

E(x/y=1)=$\sum_0^2 xf(x/$y=1)=0+$1*\frac{3}{7}+2*\frac{1}{7}=\frac{5}{7}$

E(x/y)=$\frac{16}{11}+\frac{5}{7}=\frac{167}{77}$

8)E(y/x)=E(y/x=0)+E(y/x=1)+E(y/x=2)

f(y/x)

Y	0	1	2
0	$\frac{1}{4}$	$\frac{4}{7}$	$\frac{6}{7}$
1	$\frac{3}{4}$	$\frac{3}{7}$	$\frac{1}{7}$

E(y/x=0)=$\sum_0^1 yf\ (y/$x=0)=0+$\frac{3}{4}*1=\frac{3}{4}$

E(y/x=1)=$\sum_0^1 yf\ (y/$x=1)=0+$\frac{3}{7}*1=\frac{3}{7}$

E(y/x=2)=$\sum_0^1 yf\ (y/$x=2)=0+$\frac{1}{7}*1=\frac{1}{7}$

E(y/x)=$\frac{3}{4}+\frac{3}{7}+\frac{1}{7}=\frac{37}{28}$

9)F(x/y)?

Y \ X	x< 0	$0\le x<1$	$1\le x<2$	x≥ 2
0	0	$\frac{1}{11}$	$\frac{5}{11}$	1
1	0	$\frac{3}{4}$	$\frac{6}{7}$	$\frac{7}{7}=1$

10)F(y/x)

Y \ X	0	1	2
y< 0	0	0	0
$0\le y<1$	$\frac{1}{4}$	$\frac{4}{7}$	$\frac{6}{7}$
y≥ 1	1	$\frac{7}{7}=1$	$\frac{7}{7}=1$

11)v(x/y)?

E(x/y=0)=$\sum_0^2 xf(x/y=0)$=0+$1*\frac{4}{11}+2*\frac{6}{11}=\frac{16}{11}$

E(x/y=1)=$\sum_0^2 xf(x/y=1)$=0+$1*\frac{3}{7}+2*\frac{1}{7}=\frac{5}{7}$

V(x/y)=v(x/y=0)+v(x/y=1)

V(x/y=0)=E(x^2/y=0)-(E(x/y=0)^2=$\frac{28}{11}-\left(\frac{16}{11}\right)^2=\frac{52}{121}$

E(x^2/y=0)=$\sum_0^2 x^2 f$(x/y)=0+$\frac{4}{11}+2^2*\frac{6}{11}=\frac{28}{11}$

V(x/y=1)=E(x^2/y=1)-(E(x/y=1)^2= $1-\left(\frac{5}{7}\right)^2=\frac{24}{49}$

E(x^2/y=1)=$\sum_0^2 xf(x^2/y=1)$=0+$1*\frac{3}{7}+2^2*\frac{1}{7}=\frac{7}{7}=1$

V(x/y)=v(x/y=0)+v(x/y=1)= $\frac{52}{121}+\frac{24}{49}=\frac{5452}{5929}=0.9195$

12)stoch indep?

E(xy)=E(x)*E(y)

$$\frac{5}{18}\neq\frac{21}{18}*\frac{7}{18}\ \textit{are not stoch independent}$$

Exercise of Chapter Three

Exer. (1)

$$f(x_1,x_2) = \begin{cases} 4x_1x_2 0 < x_1 < 1 \\ 0 < x_2 < 1 \\ 0 o.w \end{cases}$$

$be the j.p.d.f of x_1 \& x_2.$

$Find:$

$1) p(0 < x_1 < \frac{1}{2}, \frac{1}{4} < x_2 < 1)$

$2) p(x_1 = x_2)$

$3) p(x_1 < x_2)$

$4) p(x_1 \leq x_2)$

Exer. (2)

$et the pro. set fun. p(A) of two r.v's x \& y be.$

$$p(A) = \sum_A \sum f(x,y), where f(x,y) = \frac{1}{52}, (x,y) \in A$$

$A = \{(x,y); (x,y) = (0,1),(0,2),\dots,(0,13),(1,1),\dots,(1,13),\dots,(3,1),\dots,(3,13)\}$

$compute p(A) = p[(x,y) \in A]$

$a) when A = \{(x,y); (x,y) = (0,4),(1,3),(2,2)\}$

$b) when A = \{(x,y); x + y = 4, (x,y) \in A\}$

Exer. (3)

IF $x \& y$ having the $j.p.d.f$ as;

$$f(x,y) = \begin{cases} \dfrac{xy}{96} & 0 < x < 4 \ , \ 1 < y < 5 \\ 0 & o.w \end{cases}$$

Find: $f(y)$, $f(x)$

Exer. (4) : let the j.p.d.f of x & y be:

$$f(x,y)=\begin{cases} e^{-x-y} & 0<x<\infty \;,\; 0<y<\infty \\ 0 & o.w \end{cases}$$

Find: 1) $E(x)$, 2) $E(y)$ 3) Exy

Exer. (5) Example(18): let

$$f(x,y)=\begin{cases} 2x0<x<1,0<y<1 \\ 0o.w \end{cases}$$

Compute: $E(x+y)E(x+y)^2-[E(x+y)]^2$

Exer. (6)

$f(x,y)=20<x<y,0<y<1$

$=0o.w$

Find: 1)Ex2)Ey3)Exy

Exer. (7) : letthej.p.d.fofx&y

$$f(x,y)=\begin{cases} \frac{1}{4}(x,y)=(0,0),(1,1),(1,-1),(2,0) \\ 0o.w \end{cases}$$

calculate $cov(x,y)$andρ_{xy}.

Exer. (8) : letx&yhavethej.p.d.fdiscrete

(x,y)	$f(x,y)$
(0,0)	(1,6)
(1,0)	(2,6)
(1,1)	(2,6)
(2,1)	(1,6)

$0 o.w$

$Find or calculate the correlation of coefficient between x \& y$

Exer. (9) $'xample(26)$: $let x \& y are two r.v's have the j.p.d.f$

$$f(x,y) = \begin{cases} \frac{1}{3} (x,y) = (0,0),(1,1),(2,2) \\ 0 o.w \end{cases}$$

$compute the correlation of coefficient between x \& y.$

Exer. (10)

$$f(x,y) = \frac{1}{3} (x,y) = (0,0),(1,1),(2,0)$$

$$= 0 o.w$$

$Find: \rho_{xy}.$

Exer. (11)

$$f(x,y) = \frac{1}{3} (x,y) = (0,2),(1,1),(2,0)$$

$$= 0 o.w$$

$Find: \rho_{xy}.$

Exer. (12) :let $x \& y$ have the following $j.p.d.f$;

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & o.w \end{cases}$$

Find : Ex , Ey , ρ_{xy}

Exer. (13) $x \& y be two r.v's with j.p.d.f$

$$p(x,y) = \begin{cases} \frac{1}{16} x, y = 1,2,3,4 \\ 0 o.w \end{cases}$$

$show that x \& y are stoch. indep.$

Exer. (15)

$$f(x,y) = 2e^{-x-y} 0 < x < y, 0 < y < \infty$$

$$= 0 o.w$$

showthatx&yarer.v'sindependentornot.

Exer. (16)

$$f(x_1,x_2,x_3) = \begin{cases} \frac{1}{4} (x_1,x_2,x_3) = (0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1) \\ 0 o.w \end{cases}$$

$$let f_{ij}(x_i,x_j) = \frac{1}{4}(x_i,x_j) = [(0,0),(1,0),(0,1),(1,1)]$$

$$f_i(x_i) = \frac{1}{2} x_i = 0,1 i \neq j$$

showthatther.v'sareindep.

Exer. (17)

Let x & y be two r.v's , haning j.p.d.f

$$p(x,y) = \begin{cases} \frac{1}{18} & (0,0) \ , \ (2,1) \\ \frac{3}{18} & (0,1) \ , \ (1,1) \\ \frac{4}{18} & (1,0) \\ \frac{6}{18} & (2,0) \\ 0 & o.w \end{cases}$$

1) *Are x & y are indep.*
2) *Find* ρ_{xy} .

Exer. (18)

Example(46): *Let*

$$f(x_1/x_2) = C_1 \frac{x_1}{x_2^2} 0 < x_1 < x_2 < 1$$

$$= 0 o.w$$

And $f(x_2) = C_2 x_2^4 0 < x_2 < 1$

1. the value of $C_1 \& C_2$
2. $f(x_1,x_2)$
3. $p(\frac{1}{4} < x_1 < \frac{1}{2} / x_2 = \frac{5}{8})$
4. $pr(\frac{1}{4} < x_1 < \frac{1}{2})$

$pr(\frac{1}{4} < x_2 < \frac{1}{2})$

Exer. (19)

$\boxed{Example(51)}$:let x& y betwo $r.v'swith$

$$f(x/y) = \frac{2x+4y}{1+4y} 0 < x < 1$$

$$0 < y < 1$$

$and f(y) = \frac{1}{2}(1+4y) 0 < y < 1$

Find

1. j. p. d.f of x& y
2. show that f(x/y) is

Exer. (20)

):let x& y having the following j. $p.d.f$

f(x,y)=x+y   0<x<1   0<y<1

1. marginal p.d.f of x.
2. marginal p.d.f of y
3. conditional p.d.f of (y/x)
4. $f(y/x = \frac{1}{2})$
5. x& y are stoc .indep?

Exer. (21)

:t x& y having the following j. $p.m.f$

$$p(x,y) = \begin{cases} {}^{1}/_{12}\ (1,2)(3,2) \\ {}^{2}/_{12}(2,2)(3,3)(1,3) \\ {}^{4}/_{12}\ (2,4) \end{cases}$$ Find

1. J. c. d. f
2. p(x=2/y=3)
3. p(y/x=2)

Exer. (22)

$\boxed{Example(54)}$:let x& y having the following j. $p.d.f$

$$f(x,y) = \frac{e^{-2}}{x!\,(y-x)!} x = 0,1,\ldots,y y = 0,1,\ldots..$$

$$= oo.w$$

Find :

1. $them.g.f Mxy(t_1,t_2) of the j.dist.$
2. compute the mean & variance , and the corr. Coff. x&y
3. determine the conditional p.d.f of (x/y).

Exer. (23) f(x,y)=2 0<x<y<1

=0 o.w

Find 1.E(x/y)

2.E(y/x)

3.v(x/y)

4.v(y/x)

Exer. (24) $t the j.p.d.f of x \& y be:$

$$p(x,y) = \begin{cases} \frac{xy^2}{30} x = 1,2,3 \\ y = 1,2 \\ 0 o.w \end{cases}$$

$Find: f(x), f(y)?$

Exer. (25) $t x \& y are two r.v's with:$

$$f(x_1,x_2) = \begin{cases} e^{-(x+y)} 0 < x < \infty, 0 < y < \infty \\ 0 o.w \end{cases}$$

$Find:$

1)$The m\arg inal p.d.f of x.$

2)$The m\arg inal p.d.f of y.$

3)$The m\arg inal c.d.f of x.$

4)$The m\arg inal c.d.f of y.$

5)$show that f(x,y) is a j.p.d.f of x \& y.$

Name:-..: ناو

Code & Group (A) (B) (C)

Department of Statistics

Monthly Exam
Chapter (III)

ID Exam

Sub.: Mathematical Statistics
Date:- 14 – 4- 2011
Time90 minutes

Q1) ***40 Marks***

$$f(x;y)=\begin{cases} cx^2y^3 & 0<x<\frac{1}{2} \quad 0<y<1 \\ 0 & o.w \end{cases}$$

Find 1) c? 2) $V(x/y)$

Q2) ***40 Marks***

$$f(x;y)=\begin{cases} \frac{1}{15} & (1,3),(3,1),(2,3) \\ \frac{2}{15} & (2,2) \\ \frac{3}{15} & (1,1),(2,1) \\ \frac{4}{15} & (3,2) \\ 0 & o.w \end{cases}$$

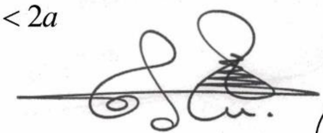
1. Calculate correlation of coefficient between x & y
2. X & y are stochastic independent or not?
3. Find the mean[E(y/x)]?
4. Find the c.d.f F(y/x)?

Q3) ***Let*** ***20 Marks***

$$f(x,y)=\frac{(2m+2)!}{m!m!}\left(\frac{1}{2a}\right)^{2m+2}x^m(2a-y)^m \qquad 0<x<y<2a$$

proof) $f(x;y)$ is the j.p.d.f of x and y.

Which means $\int_0^{2a}\int_0^{y} f(x;y)\,dx\,dy=1$



Dler Hussein Kadir
The examiner

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Best of Luck