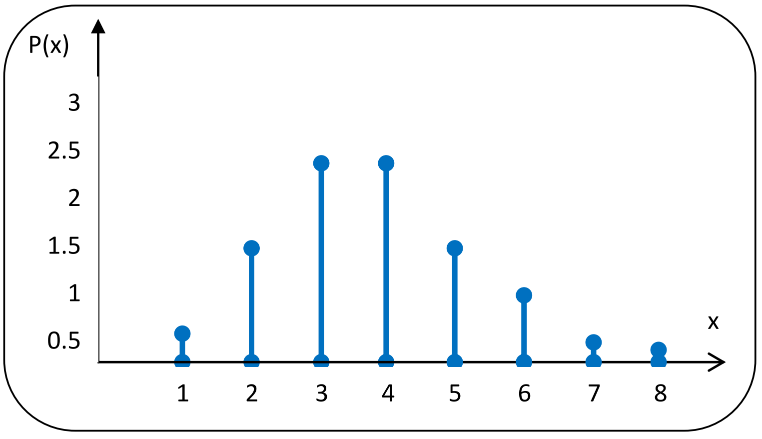
CHAPTER (4)

(Discrete Distribution)

Subjects
4 – 1 Discrete uniform Distributions.
4 – 2 Bernoulli Distribution.
4 – 3 Binomial Distribution.
4 – 4 The Poisson Distribution.
4 – 5 The Geometric Distribution.
4 – 6 Negative Binomial Distribution.
4 – 7 The Hyper Geometric Distribution

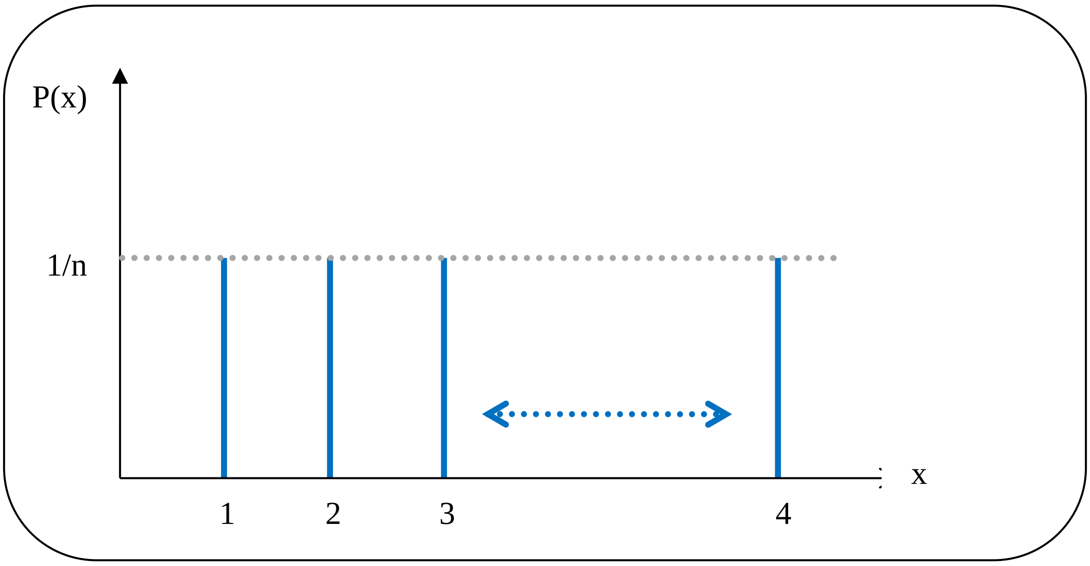


4 – 1 Discrete Uniform Distribution $X \sim D.u(n)$

A r.v is defined to have a discrete uniform Distribution. If p.m.f of x is given by:

$$P(x) = \begin{cases} \frac{1}{n} & x = 1,2,\ldots,n \\ 0 & o.w \end{cases}$$

Where the parameter (n) is positive integer :



Is p.m.f of uniform Dist.

Properties of uniform Distribution.

1) *If x has a uniform dist. Then the c.d.f of x F(x)*

$$F(x) = Pr(X \le x) = \sum_{x=1}^{x} P(x) = \sum_{x=1}^{x} \frac{1}{n} = \frac{x}{n}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{n} & 1 \le x < 2 \\ \frac{2}{n} & 2 \le x < 3 \\ \vdots \\ \frac{n}{n} = 1 & x \ge n \end{cases}$$

Important Note

All the distributions have the condition

$$\sum_{all\ x} p(x) = 1$$

2) *Mean uniform Dist.*

$E(x) = \frac{n+1}{2}$

Note: $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

$E(x) = \sum_1^n xp(x) = \sum_1^n x\frac{1}{n} = \frac{1}{n}\sum_1^n x = \frac{1}{n} * \frac{n(n+1)}{2} = \frac{n+1}{2}$

3) Variance of x.

$V(x) = \frac{n^2-1}{12}$

Note: $\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$

$E(x^2) = \sum_1^n x^2p(x) = \sum_1^n x^2\frac{1}{n} = \frac{1}{n}\sum_1^n x^2 = \frac{1}{n} * \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$

$V(x) = E(x^2)-(E(x))^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{n^2+2n+1}{4} = \frac{2n^2+n+2n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2-1}{12}$

4) The moment generating function (m.g.f) of uniform Dist.

$M_x(t) = \frac{1}{n} * \frac{e^t(1-e^{tn})}{1-e^t}$

Note: $\sum r^x = \frac{r(1-r^n)}{1-r}$

$M_xt = \sum e^{tx}\frac{1}{n} = \frac{1}{n}\sum(e^t)^x = \frac{1}{n} * \frac{e^t(1-e^{tn})}{1-e^t}$

Example(1):- *Let x~ D.U (6)*

Find $E(x), V(x), M_x(t)$

Solution

Thus $p(x) = \frac{1}{6}$ n=6

1) E(x) =?

$E(x) = \frac{n+1}{2} = \frac{6+1}{2} = \frac{7}{2}$

2) V(x)?

$V(x) = \frac{n^2-1}{12} = \frac{6^2-1}{12} = \frac{35}{12}$

3) Mx(t)?

$M_xt = \frac{1}{n} * \frac{e^t(1-e^{tn})}{1-e^t} = \frac{1}{6} * \frac{e^t(1-e^{6t})}{1-e^t}$

4 – 2 Bernoulli Distribution $X \sim br(p)$

In probability theory and Statistics, the Bernoulli distribution, named after Swiss Scientist Jacob Bernoulli: is a discrete probability distribution, which takes value 1 with success probability q = 1- p. So if X is a random variable with this distribution, we have

where parameter p is between Zero and On

$$P(x=1)=1-\Pr(x=0)=1-q=p$$

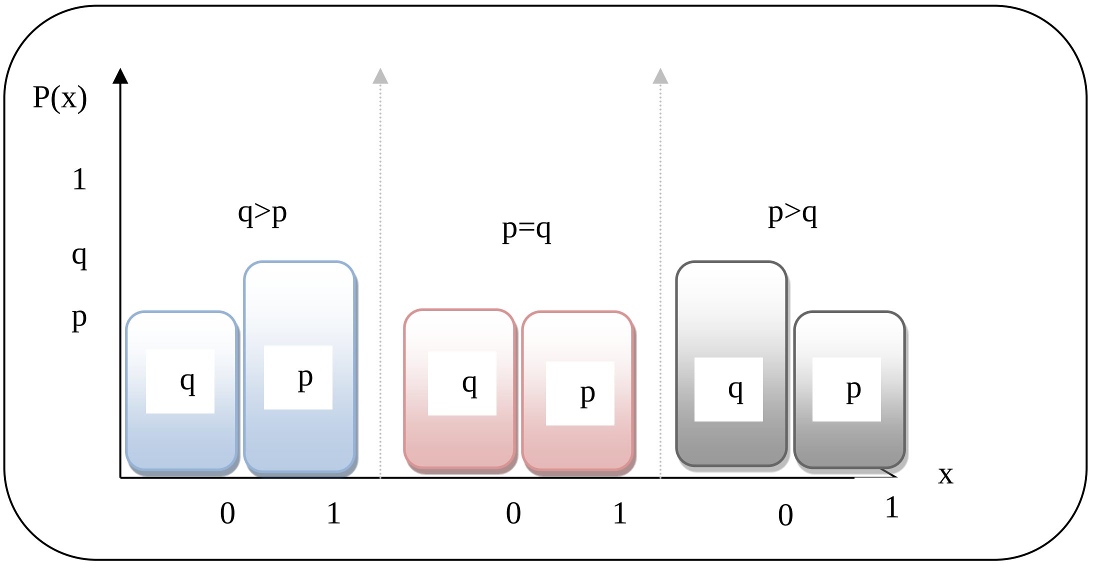
This can also be expressed as

$$p(x;p)=p^{x}(1-p)^{1-x}=p^{x}q^{1-x}$$
$$0 \quad otherwise$$

The p.m.f of this distribution is

$$p(x;p)=\begin{cases} p & if\ x=1 \\ 1-p & if\ x=0 \\ 0 & otherwise \end{cases}$$

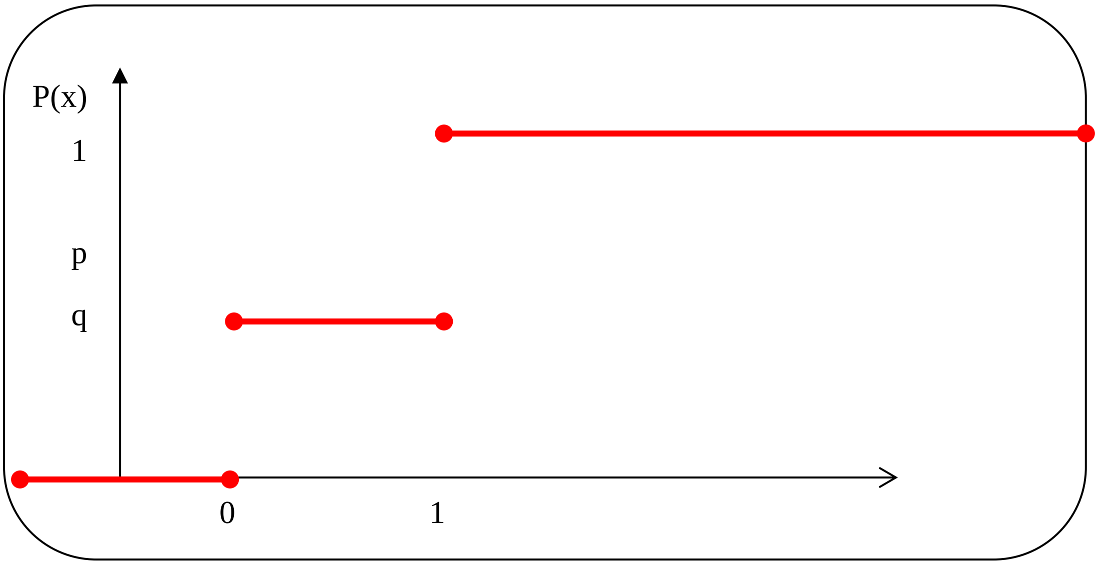
$0 < P < 1, P \in$
$X \sim br(p)$



*** Properties of Bernoulli Distribution.**

1) $c.d.f$

$$F(x) = \begin{cases} 0 for x < 0 \\ q for 0 \le x < 1 \\ p + q = 1 for 1 \le x \end{cases}$$



2) Mean Bernoulli Dist.

$\boxed{E(X) = p}$ $proof(4).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

Proof :-

$$E(x) = p$$

$$E(x) = \sum_{x=0}^{1} x p(x)$$

$$= \sum_{x=0}^{1} x\, p^x (1-p)^{1-x}$$

$$= \sum_{x=0}^{1} x\, p^x q^{1-x}$$

$$= (0)\ p^0 q^{1-0} + (1)\ p^1 q^{1-1}$$

$$= (0) + p$$

$$\therefore \quad E(x) = p$$

3)Variance of

$\boxed{V(X) = pq}$ $proof(5).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

Proof :-

$$var(x) = p\,q$$
$$var(x) = Ex^2 - (Ex)^2$$
$$Ex = p$$
$$Ex^2 = \sum_{x=0}^{1} x^2\, p(x)$$
$$= \sum_{x=0}^{1} x^2\, p^x\, q^{1-x}$$
$$= (0)^2\, p^0\, q^{1-0} + (1)^2\, p^1\, q^{1-1}$$
$$= 0 + p$$
$$= p$$
$$\therefore var(x) = p - p^2$$
$$= p(1-p)$$
$$= pq$$
$$var(x) = p\,q$$

4) The moment generating function (m.g.f) of uniform Dist.

$\boxed{M_x(t) = q + pe^t}$ $proof(6).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.w

5) Mode (Mo)

$$x = \begin{cases} 0 & q > p \\ 0,1 & if\ q = p \\ 1 & if\ q < p \end{cases}$$

6) α_1 "$Skewnes$"

$$\boxed{\alpha_1 = \frac{q-p}{\sqrt{pq}}}$$

$proof(7).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.w

$\boxed{\alpha_1 = \frac{E(x-\mu)^3}{\sigma^3{}_x}\ Skewnes}$

7) α_2 "Kurtosis"

$$\alpha_2 = \frac{6p^2 - 6p + 1}{p(1-p)}$$ $proof(8).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.w

$$\alpha_2 = \frac{E(x-\mu)^4}{\sigma^4{}_x} Kurtosis$$

$Note$:

$\mu = \mu_1 = E(x)$

$\mu_2 = E(x^2)$

$\mu_3 = E(x^3)$

$\mu_4 = E(x^4)$

$8 - Addition$

• *If* $x_1, x_2, \ldots, x_n$ *are independent identically distributed (i.i.d) random variables.all Bernoulli distributed with success probability P. then*

$$Y = \sum_{x=1}^{n} Z_x \sim Binomial(n, p)$$

Example (2) $-Let\ x \sim br(\frac{1}{3});$

Example (2)

Find ;

1) $E(x)$ 2) $V(x)$ 3) $M_x(t)$

Solution

P=$\frac{1}{3}$, $q = \frac{2}{3}$

1) E(x) = $p = \frac{1}{3}$

2) V(x) = pq=$\frac{1}{3} * \frac{2}{3} = \frac{2}{9}$

3) $M_x t = q + e^t p = \frac{2}{3} + \frac{1}{3} e^t$

Example (3) $-Let\ x \sim br(\frac{1}{2});$

Find 1) $p.m.f$ 2) $F(x)$ 3) $Mode$

4) $M_x(t)$ 5) $Kurtosis$ 6) $Skewnes$

Solution

P=$\frac{1}{2}$, $q = \frac{1}{2}$

1) pmf?

$$P(x) = \begin{bmatrix} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1-x} & x = 0,1 \\ 0 & o.w \end{bmatrix}$$

2)cdf?

$$F(x) = \begin{bmatrix} 0 & x < 0 \\ \sum \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1-x} & 0 \le x < 1 \\ 1 & x \ge 1 \end{bmatrix} = \begin{bmatrix} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{2} + \frac{1}{2} = 1 & 1 \le x \end{bmatrix}$$

3) Mode x=0 and x=1 because p=q

4) $M_x t = \frac{1}{2} + \frac{1}{2} e^t$

5) Kurtosis

$$= \alpha_2 kurtosis = \frac{6p^2 - 6p + 1}{pq} = \frac{6\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 1}{\frac{1}{2} * \frac{1}{2}} = \frac{\frac{6}{4} - \frac{6}{2} + 1}{\frac{1}{4}} = \frac{\frac{-2}{4}}{\frac{1}{4}} = -2$$

6) Skweness $\propto_1 = \frac{q-p}{\sqrt{pq}} = \frac{\frac{1}{2} - \frac{1}{1}}{\sqrt{\frac{1}{2} * \frac{1}{2}}} = 0$

$$\alpha_1 skweness = \frac{q-p}{\sqrt{pq}} = \frac{\frac{1}{2} - \frac{1}{2}}{\sqrt{\frac{1}{2} * \frac{1}{2}}} = 0$$

Example (4) – Let $M_x(t) = \frac{1}{2} + \frac{1}{2} e^t$ or $= \frac{1 + e^t}{2}$;

Find

1) $p.m.f$ 2) $F(x)$ 3) $Mode$

4) $V(x)$ 5) $Kurtosis$ 6) $Skewnes$

Solution

1) $$P(x) = \begin{bmatrix} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1-x} & x = 0,1 \\ 0 & o.w \end{bmatrix}$$

2)cdf?

$$F(x) = \begin{bmatrix} 0 & x < 0 \\ \sum \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1-x} & 0 \le x < 1 \\ 1 & x \ge 1 \end{bmatrix}$$

3) Mode x=0 and x=1 because p=q

4) V(x)?

$V(x) = pq = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$

5) Kurtosis

$=\alpha_2 kurtosis = \frac{6p^2-6p+1}{pq} = \frac{6\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)+1}{\frac{1}{2}*\frac{1}{2}} = \frac{\frac{6}{4}-\frac{6}{2}+1}{\frac{1}{4}} = \frac{\frac{-2}{4}}{\frac{1}{4}} = -2$

6) Skweness

$$\alpha_1 skweness = \frac{q-p}{\sqrt{pq}} = \frac{\frac{1}{2}-\frac{1}{2}}{\sqrt{\frac{1}{2}*\frac{1}{2}}} = 0$$

4 – 3 Binomial Distribution $X \sim b(n,p)$

In (n) trails, let the pr mm no. of an event occurring in each trials, be equal to (p), and let all trials be independent. Then the density of the r.v. x (The number of occurrences in (n) trials are:

In other words : The total number of success in (n) indep. Bernoulli trial is a r.v x having a binomial dist. With probability . mass function . Is given by :

In other words : A r.v. X is said to have a binomial dist. If the p.d.f is given by :

$$p(x;n,p) = P(X = x) = \begin{cases} C_x^n p^x q^{n-x} & x = 0,1,\dots,n \\ 0 & o.w \end{cases}$$

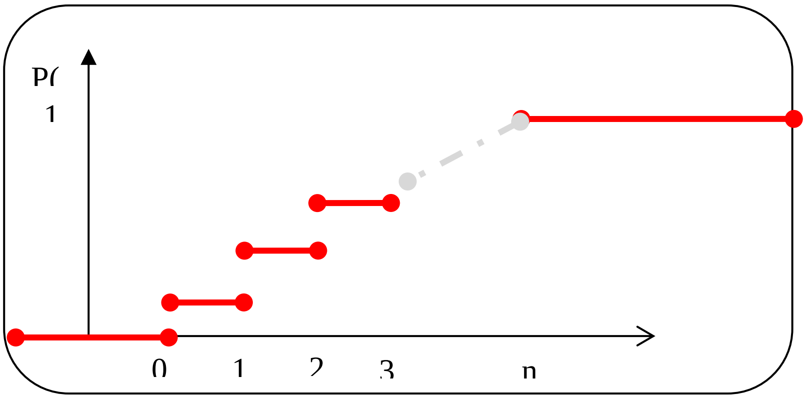
Where (n) and (p) positive parameter, such that (0 ≤ P ≤ 1).

n = No. of items of success in (n) trials.
n – x = No. of items of failure in (n) trials.
P = No. of success.
q = No. of failure.

* *The properties of the Binomial dist.*

1) $c.d.f$

$$F(x) = Pr(X \leq x) = \left\{ \begin{array}{ll} 0 & x < 0 \\ q^n & 0 \leq x < 1 \\ npq^{n-1} & 1 \leq x < 2 \\ npq^{n-1} + c_2^n p^2 q^{n-2} & 2 \leq x < 3 \\ .. & \\ .. & \\ 1 & n \leq x \end{array} \right\} \sum_{u=0}^{x} C_u^n p^u q^{n-u} \quad x = 0,1,2,\dots,n$$



2) Mean Binomial Dist.

$\boxed{E(X) = np}\ proof(9).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

Proof :-

$$E(x) = np$$

$$E(x) = \sum_{x=0}^{n} xp(x) \Rightarrow \sum_{x=0}^{n} x\ C_x^n\ p^x\ q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{1} x\ \frac{n!}{x!(n-x)!}\ p^x\ q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{1} x\ \frac{n(n-1)!}{x(x-1)!(n-x)!}\ p^{x-1+1}\ q^{n-x}$$

$$\Rightarrow np\sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!(n-x)!}\ p^{x-1}\ q^{n-x}$$

let

$$m = n-1 \quad \Rightarrow \quad n = m+1$$

$$y = x-1 \quad \Rightarrow \quad x = y+1 \quad \Rightarrow \quad y+1 = 0 \quad \therefore\ y = -1$$

$$m - y = n-1-(x-1)$$

$$= n-1-x+1$$

$$m - y = n - x$$

$$\Rightarrow np\sum_{y=-1}^{m+1} \frac{m!}{y!(m-y)!}\ p^y\ q^{m-y} \qquad y = 0,1,2,\ldots,m$$

o.w

$$p(y=-1) = 0$$

$$\text{o.w}\ (m+1)$$

$$\Rightarrow np\sum_{y=0}^{m} C_y^m\ p^y\ q^{m-y} \qquad \text{by low (11)} \quad \sum_{x=0}^{n} C_x^n\ p^x\ q^{n-x} = (p+q)^n$$

$$\Rightarrow np\ (p+q)^m$$

$$\Rightarrow np\ (1)^m$$

$$\therefore \quad E(x) = np$$

3)Variance of x

$\boxed{V(X) = npq}\ proof(10).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

Proof :

$$\text{var}(x) = npq$$

$$\text{var}(x) = Ex^2 - (Ex)^2$$

$$Ex = np$$

$$Ex^2 = \sum_{x=0}^{n} x^2\, C_x^n\, p^x\, q^{n-x} \Rightarrow \sum_{x=0}^{n} x^2 - x + x\; C_x^n\, p^x\, q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{n} x^2 - x\; C_x^n\, p^x\, q^{n-x} + \sum_{x=0}^{n} x\; C_x^n\, p^x\, q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{n} x(x-1)\, \frac{n!}{x!(n-x)!}\, p^x\, q^{n-x} + np$$

$$\Rightarrow \sum_{x=0}^{n} x(x-1)\, \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!}\, p^{x-2+2}\, q^{n-x} + np$$

$$\Rightarrow n(n-1)\,p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)!(n-x)!}\, p^{x-2}\, q^{n-x} + np$$

let

$$m = n-2 \Rightarrow n = m+2$$

$$y = x-2 \Rightarrow x = y+2 \Rightarrow y+2 = 0 \quad \therefore\; y = -2$$

$$m - y = n - x$$

$$\Rightarrow n^2p^2 - np^2 \sum_{y=-2}^{m+2} \frac{m!}{y!(m-y)!}\, p^y\, q^{m-y} + np$$

$$\Rightarrow n^2p^2 - np^2 \sum_{y=0}^{m} C_y^m\, p^y\, q^{m-y} + np$$

$$\Rightarrow n^2p^2 - np^2\,(p+q)^m + np$$

$$\Rightarrow n^2p^2 - np^2 + np$$

$$\text{var}(x) = n^2p^2 - np^2 + np - (np)^2$$

$$= -np^2 + np$$

$$\text{var}(x) = np(1-p)$$

$$\text{var}(x) = npq$$

4) The moment generating function (m.g.f) of uniform Dist.

$$M_x(t) = (p + qe^t)^n$$

$proof(11).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.W

$5) Addition$

$If X \sim b(n,p)$

$The n y = \sum_{i=1}^{n} x_i \; has a dist. is binomial [y \sim b(\sum n, p)]$

$proof(12).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.W

^^^

Example (5) $If has a M_x(t) = (\frac{2}{3} + \frac{1}{3}e^t)^5$

$Find the p.d.f of X \& Mean \& Varince$

Solution

1) p(x)= $C_x^5 \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x} \quad x = 0,1,2,3,4,5$

=0 o.w

2)E(x)np=5* $\frac{1}{3} = \frac{5}{3}$

3)v(x)=npq= $\frac{2}{3} * \frac{1}{3} * 5 = \frac{10}{9}$

Example (6) $et X \sim b(n,p) show that$

$$1) E(\frac{x}{n}) = p \quad 2) E(\frac{x}{n} - p)^2 = \frac{p^4}{n}$$

Solution

1)E $(\frac{x}{n}) = p$

$= \frac{E(x)}{n} = \frac{np}{n} = p$

2)E $(\frac{x}{n} - p)^2 = \frac{pq}{n}$

$= \frac{E(x-np)^2}{n^2} = \frac{E(x^2) - 2npE(x) + n^2p^2}{n^2} = \frac{v(x)(E(x^2) - 2np*np + n^2p^2)}{n^2} = \frac{npq}{n^2} = \frac{pq}{n}$

Example (7) $et y be the number of success in (n) indep.$

$exp e riment having pro. of success$

$, p = \frac{2}{3}, if n = 3 find: Pr(2 \le y); if n = 5 find: Pr(3 \le y)$

Solution

P=$\frac{2}{3}$, $if\ n = 3\ find\ p(2 \le y): if\ n = 5\ find\ \ p(3 \le y)$

P(y) = $C_y^3 \left(\frac{2}{3}\right)^y \left(\frac{1}{3}\right)^{3-y}$ $\quad y = 0,1,2,3$

=0 o.w

P(y≥ 2) $= P(2) + p(3) = C_2^3 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{3-2} + C_3^3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^{3-3} = \frac{20}{27}$

P(y)= $C_y^5 \left(\frac{2}{3}\right)^y \left(\frac{1}{3}\right)^{5-y}$ $\quad y = 0,1,2,3,4,5$

=0 o.w

P(y≥ 3) $= p(3) + p(4) + p(5) = C_3^5 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^{5-3} + C_4^5 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{5-4} +$ $C_5^5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^{5-5} = 0.79$

Example (8) $fy \sim bin(n, \frac{1}{3}), and let\ Pr(y \ge 1) \ge 0.80$
$find(n)?.$

Solution

$P(y)= C_y^n \left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{n-y}$ $\quad x = 0,1, \ldots, n$

$=0$ $\quad o.w$

$P(y\ge 1) = 1 - p(y < 1) = 1 - p(0) = 1 - C_0^n \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{n-0} =$
$0.8 \quad thus\ C_0^n \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{n-0} = 0.2 \Rightarrow n = \frac{ln\frac{2}{3}}{ln0.2} = 3.96 \approx 4$

Example (9) $fX \sim bin(2, p), and y \sim b(4, p)$
$if\ Pr(x \ge 1) = \frac{5}{9} find\ Pr(y \ge 1).$

Solution

P(x)= $C_x^2 (p)^x (q)^{2-x}$

$p(x \ge 1) = 1 - p(x < 1) \Rightarrow = \frac{5}{9} = 1 - \{C_0^2 (p)^0 (q)^{2-0}\} = \Rightarrow \frac{5}{9} = p(1 - q^2)\ p(x \ge 1) = q^2$

$= 1 - \frac{5}{q} \quad q = \sqrt{\frac{4}{q}} \Rightarrow q = \frac{2}{3} \Rightarrow \quad \therefore\ p = \frac{1}{3}$

P(y)= $C_y^4 \left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{4-y}$

Thus p(y≥ 1) $= 1 - p(0) = 1 - C_0^4 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{2-0} = \frac{65}{81}$

4 – 4 The Poisson Distribution $X \sim P(\lambda)$

Is the limiting form of the binomial dist., when $n \to \infty$ and $P \to 0$. So that (np) is finite quantity such as (λ) in general if $n \geq 50$ and $\lambda \leq 5$ it can be taken to be a case of Poisson dist. such event are know as are events .

Applications :

1 – Number of deaths from a disease such as heart. attack

2 - The number of defective material per packing manufactured.

3 – arrivals at a service counter.

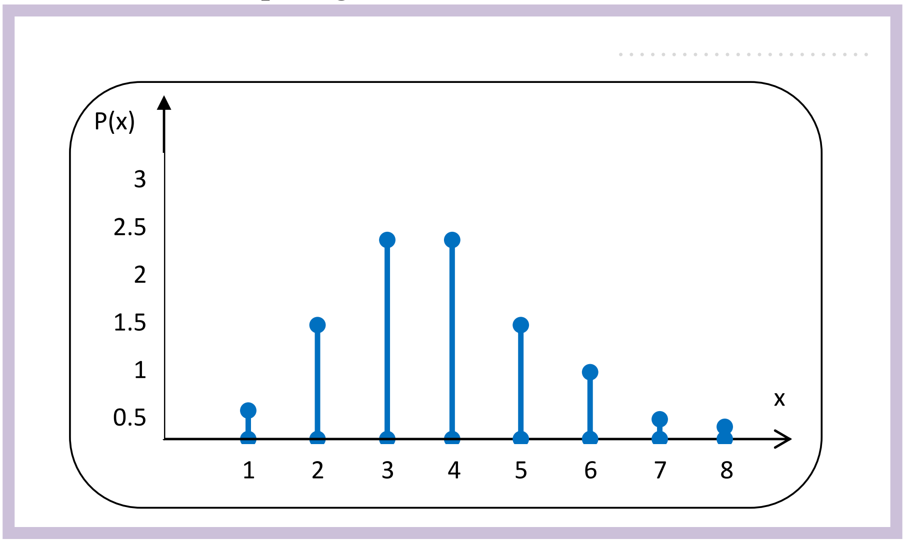
4 – The number of rail road accidents in same unit of time.

Definition : A r.v. x is defined to have a Poisson dist. if the p.m.f of X given by:

$$Pr(X;\lambda) = Pr(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,\ldots \\ 0 o.w \end{cases}$$

Where the parameter ($\lambda > 0$) is a constant integer of fun. of any positive real number R^+ [λ=np]

***Remark** : Poisson : n large & p small*
Binomial : n small & p large



* Properties of a Poisson Dist. $X \sim P(\lambda)$

1) $The\ c.d.f\ of\ X \sim P(\lambda)$

$$F(x) = Pr(X \le x) = \begin{cases} 0 & x < 0 \\ \sum_{u=0}^{x} \frac{e^{\lambda}\lambda^{u}}{u!} & x = 0,1,2,\dots,\infty\ or\ 0 \le x \le \infty \\ 1 & x \to \infty \end{cases}$$

2) *Mean Poisson Dist.*

$\boxed{E(X) = \lambda}$ $proof(13).(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots)$

Proof :-

$$\mathrm{E}(x) = \lambda$$

$$\mathrm{E}(x) = \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^{x-1+1}}{x(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}\lambda^{1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \qquad \text{by low}(12) \quad \sum_{x=0}^{\infty} \frac{r^{x}}{x!} = e^{r}$$

$$= e^{-\lambda}\, \lambda\, e^{\lambda}$$

$$\therefore \quad \mathrm{E}(x) = \lambda$$

3) *Variance of x*

$\boxed{V(X) = \lambda}$ $proof(14).(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots)$

Proof :-

$$\mathrm{var}(x) = \lambda$$

$$\mathrm{var}(x) = Ex^{2} - (Ex)^{2}$$

$$Ex = \lambda$$

$$Ex^{2} = \sum_{x=0}^{\infty} x^{2} p(x)$$

$$= \sum_{x=0}^{\infty} x^2 - x + x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} x^2 - x \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} x(\text{x-1}) \frac{e^{-\lambda}\lambda^{x-2+2}}{x(x-1)(x-2)!} + \lambda$$

$$\Rightarrow e^{-\lambda}\lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$\Rightarrow e^{-\lambda}\ \lambda^2\ e^{\lambda} + \lambda$$

$$\Rightarrow \lambda^2 + \lambda$$

$$\therefore \text{var}(x) = \lambda^2 + \lambda - \lambda^2$$

$$\therefore \text{var}(x) = \lambda$$

4) The moment generating function (m.g.f) of uniform Dist.

$$M_x(t) = e^{\lambda(e^t-1)}$$

$proof\,(15).\,(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ H.W

$5) The additive property or addition$

$let\, x_1, x_1, \ldots, x_n\, be\, a\, poisson\, dist.\, with\, \lambda_1, \lambda_1, \ldots, \lambda_n\, and\, x's\, are\, indep.\, Then$

$$y = x_1 + x_2 + \ldots + x_n\, has\, a\, poisson\, dist.\, P_O(\sum_{i=1}^{n} \lambda_i)$$

Example (10) $etX \sim P(\lambda), and\, M_x(t) = e^{4(e^t-1)}$

$$Find: Pr(M - 2\sigma \le x \le M + 2\sigma)?$$

Solution

E(x)= $\lambda = 4$ and v(x)= $\lambda = 4$

P(4-2*2≤x≤4+2*2)=P(0<x<8)=

p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6)+p(7)+p(8)

Example (11) $et\, Pr(x = 1) = 2\, Pr(x = 2) X \sim P(\lambda) Find\, Pr(x = 4)$

Solution

P(x=1) = $\frac{e^{-\lambda} * \lambda^1}{1!} = \lambda e^{-\lambda}$,P(x=2)= $\frac{e^{-\lambda} * \lambda^2}{2!} = \frac{\lambda^2 e^{-\lambda}}{2}$

But p(x=1) = 2P(x=2)

$$\lambda e^{-\lambda} = 2 * \frac{\lambda^2 e^{-\lambda}}{2} \Rightarrow = \lambda - \lambda^2 = 0 \Rightarrow \lambda(1-\lambda) = 0 \quad \lambda = 0 \text{ or } \lambda = 1$$

Thus p(x=4) = $\frac{e^{-\lambda} * \lambda^4}{4!} = \frac{\lambda^4}{4!} * e^{-\lambda} = \frac{\lambda^4}{4!} * e^{-1} = 0.015$

Example (12) $\sim poi(\lambda) and let\, Pr(X \ge 1) \ge 0.99$

$$Find: the value of \lambda and write p.d.f of X.$$

Solution

P$(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0) = 0.99$ thus P(x=0)=1-0.99=0.01

$\frac{e^{-\lambda}4^0}{0!} = 0.01\ thus\ e^{-\lambda} = 0.01 \quad lne^{-\lambda} = ln0.01 \Rightarrow -\lambda = -4.6 \qquad \lambda = 4.6$

$$p(x) = \begin{bmatrix} \frac{e^{-4.6}(4.6)^x}{x!} & x = 0,1,\dots,\infty \\ 0 & O.W \end{bmatrix}$$

Example (13) $let x_1 be a r.v. with P(4)$

$x_2 be another r.v. with P(3) and x_1, x_2 be Stoc. indep.$

$let y = x_1 + x_2$

$find 1) The p.m.f of y. 2) The mean and variance of y. 3) The m.g.f of y.$ Solution

1)find pmf of y?

$$x_1 \sim P(4) \quad x_2 \sim P(3)$$

$Y \sim po(\lambda) = \therefore y \sim P(4+3) \therefore y \sim P(7)$

$$p(4) = \begin{bmatrix} \frac{e^{-7}7^y}{y!} & y = 1,2,\dots,\infty \\ 0 & O.W \end{bmatrix}$$

2)E(y)= $\lambda = 7 \quad and \quad V(y) = \lambda = 7$

3)the m.g.f of y?

$M_x t = e^{\lambda(e^t-1)} = e^{7(e^t-1)}$

4 – 5 Geometric Dist. X~ G(p)

Independent Bernoulli trial are performed until the first success appears, define X be the No. of trials needed to get the first success, then a r.v. x is defined to have Geometric Dist. If the p.m.f of x given by :

$$P(x) = P(x,p) = \begin{cases} pq^x & x = 0,1,2,\dots \\ 0 & o.w \end{cases}$$

Where the parameter (p) satisfies (0 ≤ P ≤ 1)

X : No. of failed trials before getting the success trial.

$$p(X = 1) = S = p$$
$$p(X = 2) = fS = qp$$
$$p(X = 3) = ffS = q^2p$$
$$p(X = 4) = fffS = q^3p$$

.
.
.

$$p(X = x) = ff \dots fSX \sim G(p)$$

* Properties of Geometric Dist.

1) $The c.d.f of X \sim G(p)$

$$F(x) = Pr(X \leq x) = \begin{cases} 0 & x < 0 \\ \sum_{u=0}^{x} pq^x = 1 - q^{x+1} & 0 \leq x < \infty \\ 1 & x \to \infty \end{cases}$$

2) Mean Poisson Dist.

$$\boxed{E(X) = \frac{q}{p}} proof(16).(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots)$$

Proof :-16

$$E(x) = \frac{q}{p}$$

$$E(x) = \sum_{x=0}^{\infty} xp(x) \quad \Rightarrow \quad \sum_{x=0}^{\infty} xpq^x$$

$$\Rightarrow \text{ p} \sum_{x=0}^{\infty} xq^{x-1+1}$$

$$\Rightarrow \text{ pq} \sum_{x=0}^{\infty} xq^{x-1}$$

$$\Rightarrow \text{ pq} \sum_{x=0}^{\infty} \frac{\partial}{\partial q} q^x \qquad \frac{\partial}{\partial q} q^x = \text{xq}^{\text{x-1}}$$

$$\Rightarrow \text{ pq} \frac{\partial}{\partial q} \sum_{x=0}^{\infty} q^x \qquad by \text{ low}(2) \quad \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$$

$$\Rightarrow \text{ pq} \frac{\partial}{\partial q} \frac{1}{1-q} \qquad \Rightarrow \text{ pq} \frac{1}{(1-q)^2}$$

$$\Rightarrow \frac{\text{pq}}{p^2}$$

$$\therefore \text{ Ex} = \frac{q}{p}$$

3)Variance of x

$\boxed{V(X)=\frac{q}{p^2}}$ $proof(17).(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots)$

Proof :-17

$$\text{var}(x)=\frac{q}{p^2}$$

$$\text{var}(x)=Ex^2-(Ex)^2 \qquad Ex=\frac{q}{p}$$

$$Ex^2=\sum_{x=0}^{\infty}x^2p(x)$$

$$\Rightarrow\sum_{x=0}^{\infty}x^2-x+x\ p(x) \qquad \Rightarrow \sum_{x=0}^{\infty}x^2-x\ p(x)+\sum_{x=0}^{\infty}x\ p(x)$$

$$\Rightarrow\sum_{x=0}^{\infty}x(x-1)pq^x+E(x) \qquad \Rightarrow \text{p}\sum_{x=0}^{\infty}x(x-1)q^x+\frac{q}{p}$$

$$\Rightarrow p\sum_{x=0}^{\infty}x(x-1)q^{x-2+2}+\frac{q}{p}$$

$$\Rightarrow \text{pq}^2\sum_{x=0}^{\infty}x(x-1)q^{x-2}+\frac{q}{p} \qquad \frac{\partial}{\partial q}q^x=xq^{x-1}$$

$$\Rightarrow \text{pq}^2\sum_{x=0}^{\infty}\frac{\partial^2}{\partial^2 q}q^x+\frac{q}{p} \qquad \frac{\partial^2}{\partial^2 q}q^x=x(\text{x-1})q^{x-2}$$

$$\Rightarrow \text{pq}^2\frac{\partial^2}{\partial^2 q}\sum_{x=0}^{\infty}q^x+\frac{q}{p}$$

$$\Rightarrow \text{pq}^2\frac{\partial^2}{\partial^2 q}\ \frac{1}{1-q}+\frac{q}{p}$$

$$\Rightarrow \text{pq}^2\frac{\partial}{\partial q}\ \frac{1}{(1-q)^2}+\frac{q}{p}$$

$$\Rightarrow \text{pq}^2\frac{\partial}{\partial q}\ (1-q)^{-2}+\frac{q}{p}$$

$$\Rightarrow \text{pq}^2(-2)\ (1-q)^{-3}.(-1)+\frac{q}{p}$$

$$\Rightarrow\frac{2\text{pq}^2}{(1-q)^3}+\frac{q}{p} \qquad \Rightarrow \frac{2\text{pq}^2}{p^3}+\frac{q}{p}$$

$$\therefore\ \text{Ex}^2=\frac{2\text{q}^2}{p^2}+\frac{q}{p}$$

$$\therefore\ \text{var}(x)=\frac{2\text{q}^2}{p^2}+\frac{q}{p}-\frac{q^2}{p^2}\Rightarrow\frac{2\text{q}^2+pq-q^2}{p^2}$$

$$\Rightarrow \frac{q^2+pq}{p^2} \qquad \Rightarrow \frac{q(q+p)}{p^2} \qquad \Rightarrow \therefore\ \text{var}(x)=\frac{q}{p^2}$$

4) The moment generating function (m.g.f) of uniform Dist.

$M_x(t) = \frac{p}{1-e^t q}$ $proof(18).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$ HW

Example (14) $et X \sim G(\frac{1}{4})$

$Find: 1) p.m.f of x. 2) E(x). 3) M_x(t).$

Solution

1)pmf of x?

$P(x;p) = \begin{cases} p\,q^x & x = 0,1,2,\ldots. \\ 0 & o.w \end{cases}$

$P(x)= \begin{bmatrix} \frac{1}{4}\left(\frac{3}{4}\right)^x & x = 0,1\ldots. \\ 0 & o.w \end{bmatrix}$

2)E(x)?

$E(x)=\frac{q}{p} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$

2)v(x)?

$V(x)= \frac{q}{p^2} = \frac{\frac{3}{4}}{\left(\frac{1}{4}\right)^2} = 12$

$3) Mx_{(t)} = \frac{p}{1-qe^t} = \frac{\frac{1}{4}}{1-\frac{3}{4}e^t}$

Example (15) $et X \sim G(p) and\ Pr(X \geq 1) = \frac{5}{6} find P(x)?$

Solution

$x \sim G(p) \quad P(x \geq 1) = \frac{5}{6}\ find\ p(x)?$

$= P(x \geq 1) = 1 - P(x < 1) = 1 - (p(0)) = \frac{5}{6}$

$1-(q^0 p)=\frac{5}{6} \Rightarrow p = \frac{1}{6}$

Thus

$Mx_{(t)} = \frac{p}{1 - qe^t} = \frac{\frac{1}{6}}{1 - \frac{5}{6}e^t}$

$p(x) = \begin{bmatrix} \frac{1}{6} * \frac{5}{6} & x = 0,1,2,\ldots.,\infty \\ 0 & O.W \end{bmatrix}$

4 – 6 Negative Binomial Distribution N.b(r,p)

Independent Bernoulli trials are performed until (r) success, appear, Define the r.v. x is the number of failure trials before getting the r-th success trial, then a r.v. x is defined to have (N.B) if the p.d.f of x given by:

$$P(x) = \begin{cases} C_x^{x+r-1} p^r q^x, x = 0,1,2,\dots \\ 0 o.w \end{cases}$$

Where the parameters (r & p) satisfy r = 1,2,… 0 < p < 1

x: No. of failure trials before getting the r-th success .

r: No. of successes case .

$$If x = 0 \to Pr(x = 0) = \{sss\dots s\}$$
$$x = 1 \to Pr(x = 1) = \{fsss\dots s\}$$
$$x = 2 \to Pr(x = 2) = \{ffsss\dots s\}$$

* Properties of N.B. dist.

Let X be a r.v. have (N.B). Then

1) *The c.d.f of N . B dist .:*

;

$$F(x) = \Pr(X = x) = \begin{cases} 0 & x < 0 \\ p^r \sum C_x^{x+r-1} q^x & x = 0,1,\dots \\ 1 & x \to \infty \end{cases}$$

2) *The mean of X :*

$$E(x) = \frac{rq}{p} \qquad proof\ (19).(\dots\dots\dots\dots\dots\dots\dots\dots\dots)$$

Proof :-

$$\mathrm{E}(x) = \frac{rq}{p}$$

$$\mathrm{E}(x) = \sum_{x=0}^{\infty} xp(x) \quad \Rightarrow \quad \sum_{x=0}^{\infty} xC_x^{x+r-1} p^r q^x$$

$$But \quad \Rightarrow \quad [C_x^{x+r-1} = (-1)^x C_x^{-r}]$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x\,(-1)^x C_x^{-r} q^x$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x\, C_x^{-r}(-q)^x \Rightarrow p^r \sum_{x=0}^{\infty} x \frac{-r!}{x!\,(-r-x)!}(-q)^x$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x \frac{-r(-r-1)!}{x(x-1)!\,(-r-x)!}(-q)^{x-1+1}$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} \frac{-r(-r-1)!}{(x-1)!\,(-r-x)!}(-q)^{x-1}(-q)^1$$

$$\Rightarrow rqp^r \sum_{x=0}^{\infty} \frac{(-r-1)!}{(x-1)!\,(-r-x)!}(-q)^{x-1}(1)^{-r-x}$$

$$\Rightarrow rqp^r \sum_{x=0}^{\infty} \frac{(-r-1)!}{(x-1)!\,(-r-x)!}(-q)^{x-1}(1)^{-r-x}$$

let m = -r-1 y = x-1 m-y = -r-1-(x-1)

x = y + 1 = -r-x

y + 1 = 0 ∴ y = -1

$$\Rightarrow rqp^r \sum_{y=-1}^{\infty} \frac{m!}{y!\,(m-y)!}(-q)^y(1)^{m-y}$$

$$\Rightarrow rqp^r \sum_{y=0}^{\infty} C_y^m(-q)^y(1)^{m-y} \qquad \Rightarrow rqp^r(1-q)^m$$

$$\Rightarrow rqp^r(p)^{-r-1} \qquad \Rightarrow \quad rqp^r p^{-r} p^{-1}$$

$$\therefore \quad Ex = \frac{rq}{p}$$

3)*The variance of X :*

- $V(x) = \dfrac{rq}{p^2}$ *proof* (20).(.......................................)

Proof :-

$$\text{var}(x) = \frac{rq}{p^2}$$

$$\text{var}(x) = \frac{rq}{p^2}$$

$$\text{var}(x) = Ex^2 - (Ex)^2 \qquad Ex = \frac{rq}{p}$$

$$Ex^2 = \sum_{x=0}^{\infty} x^2 p(x)$$

$$\Rightarrow \sum_{x=0}^{\infty} x^2 - x + x\ \ p(x) \qquad \Rightarrow \sum_{x=0}^{\infty} x^2 - x\, p(x) + \sum_{x=0}^{\infty} x\ \ p(x)$$

$$\Rightarrow \sum_{x=0}^{\infty} x(x-1) C_x^{x+r-1} p^r q^x + \ Ex \qquad \text{But } [C_x^{x+r-1} = (-1)^x C_x^{-r}]$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x(x-1)(-1)^x C_x^{-r} q^x + \frac{rq}{p}$$

$$\Rightarrow \quad p^r \sum_{x=0}^{\infty} x(x-1) C_x^{-r} (-q)^x + \frac{rq}{p}$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x(x-1) \frac{-r!}{x!\,(-r-x)!} (-q)^x + \frac{rq}{p}$$

$$\Rightarrow p^r \sum_{x=0}^{\infty} x(x-1) \frac{-r(-r-1)(-r-2)!}{x(x-1)(x-2)!\,(-r-x)!} (-q)^{x-2+2} + \frac{rq}{p}$$

$$\Rightarrow p^r q^2 r(r+1) \sum_{x=0}^{\infty} \frac{(-r-2)!}{(x-2)!\,(-r-x)!} (-q)^{x-2} + \frac{rq}{p}$$

let m = -r-2 y = x-2 m-y = -r-2-x + 2 m-y = -r-x

$$\Rightarrow p^r q^2 (r^2+r) \sum_{y=-2}^{\infty} \frac{m!}{y!\,(m-y)!} (-q)^y (1)^{m-y} + \frac{rq}{p}$$

o.w p(-1 and -2)

$$\Rightarrow p^r q^2 (r^2+r) \sum_{y=0}^{\infty} C_y^m (-q)^y (1)^{m-y} + \frac{rq}{p}$$

$$\Rightarrow r^2 p^r q^2 + r p^r q^2 \,(1-q)^m + \frac{rq}{p} \qquad \Rightarrow \qquad (r^2 p^r q^2 + r p^r q^2)\,(p)^{-r-2} + \frac{rq}{p}$$

$$\Rightarrow \quad (r^2 p^r q^2 + r p^r q^2)\, p^{-r-2} + \frac{rq}{p}$$

$$\Rightarrow \quad r^2\ p^{r-r-2}\ q^2 + r\ p^{r-r-2}\, q^2 + \frac{rq}{p}$$

$$\Rightarrow \quad r^2\ p^{-2}\ q^2 + r\ p^{-2}\, q^2 + \frac{rq}{p}$$

$$\therefore \ Ex^2 = r^2 \left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + \frac{rq}{p}$$

$$\therefore \ \mathrm{var}(x) = r^2 \left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + \frac{rq}{p} - \left(\frac{rq}{p}\right)^2$$

$$\Rightarrow \quad r\,\frac{q^2}{p^2} + r\,\frac{q}{p} \qquad \Rightarrow \qquad r\,\frac{q}{p}\left[\frac{q}{p}+1\right]$$

$$\Rightarrow \quad r\,\frac{q}{p}\left[\frac{q+p}{p}\right] \qquad \Rightarrow \qquad r\,\frac{q}{p}\left[\frac{1}{p}\right] \qquad \Rightarrow \ \therefore \ \mathrm{var}(x) = \frac{rq}{p^2}$$

4) *The m.g.f of X*

$$M_x(t) = \left(\frac{p}{1-qe^t}\right)^r$$ *proof* (21).(.......................................) *HW*

6) *Additive Property*

Let $x_1, x_2, \ldots, x_n$ *be a r.v. and indep. such that*

$$\sum X_i \sim N.B\left(\sum r_i, p\right), \qquad i = 1, 2, \ldots, n$$

proof (22).(.......................................) *HW*

Remark *: If in negative Binomial. Dist. r = 1, then the N.B. density specializes to the geometric*

$$P(x) = C_x^{x+r-1}\ p^r\ q^x \qquad x = 0,1,\ldots$$

$$where \quad r = 1$$

$$P(x) = C_x^x\ p\ q^x = pq^x$$

Example (16) $etX \sim N.B(6,0.4), then find:$

$1) p(x).$
$2) Mean \& Var of x.$
$3) The m.g.f of x.$
$4)\ Pr(\ x \geq 1).$

Solution

1)P(x)?

$$P(x) = \begin{cases} C_x^{x+r-1}\ p^r\ q^x & x = 0,1,2,\ldots. \\ 0 & o.w \end{cases}$$

$$P(x) = \begin{cases} C_x^{x+5}\ (0.4)^6\ (0.6)^x & x = 0,1,2,\ldots. \\ 0 & o.w \end{cases}$$

2)mean and var of x?

$$E(x) = \frac{rq}{p} = \frac{6*(0.6)}{0.4} = 9$$

$$V(x) = \frac{rq}{p^2} = \frac{6*(0.6)}{(0.4)^2} = 22.5$$

$$3) Mxt = \left(\frac{p}{1-qe^t}\right)^r = \left(\frac{0.4}{1-0.6e^t}\right)^6$$

$$4) p(x \geq 1) = 1 - p(x < 1) = 1 - p(x = 0) = 1 - \left[C_0^5\ (0.4)^6 (0.6)^0\right] = 0.996$$

Example (17) $etX_1 \sim N.B(7,0.5) and X_2 \sim N.B(5,0.5) and$
$y = x_1 + x_2 find: 1) The dist. of y, and write the p.d.f of y.$
$2) Mean \&\ var\ o\ fy.$

Solution

$X_1 NB(7,0.5), x_2 \sim NB(5,0.5) \quad y = x_1 + x_2$

1)Dist y pdf?

Solve/ $y \sim NB(12,0.5)$

$$P(y)=\begin{cases} C_y^{y+r-1}\ p^r\ q^x & x=0,1,2,.. \\ 0 & o.w \end{cases}$$

$$P(y)=\begin{cases} C_y^{y+11}\ 0.5^{12}\ 0.5^x & y=0,1,2,.. \\ 0 & o.w \end{cases}$$

2)E(y) & V(y)?

$$E(x)=\frac{rq}{p}=\frac{12*(0.5)}{0.5}=12$$

$$V(x)=\frac{rq}{p^2}=\frac{12*(0.5)}{(0.5)^2}=24$$

4 – 7 The Hyper Geometric Dist. X~H.G(N,k,n)

Suppose that (n) objects are to be dram at random, one at time from a collection of (N) objects (k) of one kind and (N – k) of another kind. The one kind of objects will be thought of as a ((success)) and coded (1) : the other kind is coded (0); then a r.v. x is defined to have a hyper geometric dist. if the p.m.f of x given by :-

$$P(X)=P(X;N,k,n)=\begin{cases} \frac{C_x^k C_{n-x}^{N-k}}{C_n^N} & x=0,1,\dots,n \\ 0 & o.w \end{cases}$$

Where N,k,n are parameter such that N ≥ n, N ≥ k and N, k, n are all positive integer X ~ H.G(N, k, n).

*** Remark :**

X=a, a+1, …, b
a = max(0,n – (N – k))
b = min(n, k)

$$\text{If} \quad k<X \rightarrow C_x^k=0$$
$$N-k<n \rightarrow 0$$

* Properties of H.G Dist.

$$1)F(x)=Pr(X\le x)=\begin{cases} 0 & x<0 \\ \sum_{u=0}^{x}\frac{C_u^k C_{n-u}^{N-k}}{C_n^N} & x=0,1,\dots,n \\ 1 & x\ge n \end{cases}$$

2)The mean:

$E(x)=k(\frac{n}{N})$ proof (23). (.....................................)

3)The var(x)

$$V(x) = \frac{nk}{N}(1 - \frac{k}{N})(\frac{N-n}{N-1}) proof(24).(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

4) $Them. g. f of X not exist.$

Proof :- 23

$$E(x) = \frac{nk}{N}$$

$$E(x) = \sum_{x=0}^{n} x p(x) \Rightarrow \sum_{x=0}^{n} x \frac{C_x^k \; C_{n-x}^{N-k}}{C_n^N}$$

$$\Rightarrow \frac{1}{C_n^N} \sum_{x=0}^{n} x \frac{k!}{x!(k-x)!} C_{n-x}^{N-k}$$

$$\Rightarrow \frac{1}{\frac{N!}{n!(N-n)!}} \sum_{x=0}^{n} x \frac{k!}{x(x-1)!\,(k-x)!} C_{n-x}^{N-k}$$

$$\Rightarrow \frac{1}{\frac{N(N-1)!}{n(n-1)!\,(N-n)!}} \sum_{x=0}^{n} \frac{k(k-1)!}{(x-1)!\,(k-x)!} C_{n-x}^{N-k}$$

$$\Rightarrow \frac{k}{\frac{N}{n}} \frac{1}{C_{n-1}^{N-1}} \sum_{x=0}^{n} C_{x-1}^{k-1} C_{n-x}^{N-k}$$

$$\Rightarrow \frac{nk}{N} \sum_{x=0}^{n} \frac{C_{x-1}^{k-1} C_{n-x}^{N-k}}{C_{n-1}^{N-1}}$$

let $n^* = n-1 \quad N^* = N-1 \quad k^* = k-1 \quad x^* = x-1$

$n = n^* + 1 \quad N = N^* + 1 \quad k = k^* + 1 \quad x = x^* + 1$

$x^* + 1 = 0 \quad \therefore \quad x^* = -1$

$$\Rightarrow \frac{nk}{N} \sum_{x^*=-1}^{n^*+1} \frac{C_{x^*}^{k^*} C_{n^*-x^*}^{N^*-k^*}}{C_{n^*}^{N^*}} \qquad x^* = 0,1,2,\ldots,n^*$$

$$\Rightarrow \frac{nk}{N} \sum_{x^*=0}^{n^*} \frac{C_{x^*}^{k^*} C_{n^*-x^*}^{N^*-k^*}}{C_{n^*}^{N^*}} \qquad \therefore \quad \sum_{x=0}^{n} p(x) = 1$$

$$\therefore \quad Ex = \frac{nk}{N}$$

Proof :- 24

$$\text{var}(x) = \frac{rk}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$$\text{var}(x) = Ex^2 - (Ex)^2 \qquad Ex = \frac{nk}{N}$$

$$Ex^2 = \sum_{x=0}^{n} x^2 p(x) \qquad \Rightarrow \sum_{x=0}^{n} x^2 - x + x \; p(x)$$

$$\Rightarrow \sum_{x=0}^{n} x^2 - x\, p(x) + \sum_{x=0}^{n} x \; p(x)$$

$$\Rightarrow \sum_{x=0}^{n} x(x-1)\frac{C_x^k C_{n-x}^{N-k}}{C_n^N} + \; Ex$$

$$\Rightarrow \frac{1}{C_n^N}\sum_{x=0}^{n} x(x-1)\frac{k!}{x!\,(k-x)!}C_{n-x}^{N-k} + \; \frac{nk}{N}$$

$$\Rightarrow \frac{1}{\frac{N!}{n!\,(N-n)!}}\sum_{x=0}^{n} x(x-1)\frac{k(k-1)(k-2)!}{x(x-1)(x-2)!\,(k-x)!}C_{n-x}^{N-k} + \; \frac{nk}{N}$$

$$\Rightarrow \frac{1}{\frac{N(N-1)(N-2)!}{n(n-1)(n-2)!\,(N-n)!}}\sum_{x=0}^{n} \frac{k(k-1)(k-2)!}{(x-2)!\,(k-x)!}C_{n-x}^{N-k} + \; \frac{nk}{N}$$

$$\Rightarrow \frac{k(k-1)}{\frac{N(N-1)}{n(n-1)}}\frac{1}{C_{n-2}^{N-2}}\sum_{x=0}^{n} C_{x-2}^{k-2}C_{n-x}^{N-k} + \; \frac{nk}{N}$$

$$\Rightarrow \frac{k(k-1)n(n-1)}{N(N-1)}\sum_{x=0}^{n} \frac{C_{x-2}^{k-2}C_{n-x}^{N-k}}{C_{n-2}^{N-2}} + \; \frac{nk}{N}$$

let $n^* = n-2$ $\quad N^* = N-2$ $\quad k^* = k-2$ $\quad x^* = x-2$

$n = n^* + 2$ $\quad N = N^* + 2$ $\quad k = k^* + 2$ $\quad x = x^* + 2$

$x^* + 2 = 0 \quad \therefore \quad x^* = -2$

$$\Rightarrow \frac{nk(k-1)(n-1)}{N(N-1)}\sum_{x^*=0}^{n^*} \frac{C_{x^*}^{k^*} C_{n^*-x^*}^{N^*-k^*}}{C_n^{N^*}} + \frac{nk}{N}$$

$$\Rightarrow \frac{nk(k-1)(n-1)}{N(N-1)} + \frac{nk}{N} \quad \Rightarrow \quad \therefore \quad \frac{nk(k-1)(n-1)+(N-1)nk}{N(N-1)}$$

$$\therefore \quad Ex^2 = \frac{nk(n-1)(k-1)+Nnk-nk}{N(N-1)}$$

$$var(x) = \frac{nk(n-1)(k-1)+Nnk-nk}{N(N-1)} - \left(\frac{nk}{N}\right)^2$$

$$var(x) = \frac{n^2k^2-nk^2-n^2k+nk+Nnk-nk}{N(N-1)} - \frac{n^2k^2}{N^2}$$

$$= \frac{Nn^2k^2-Nnk^2-Nn^2k+Nnk+N^2nk-Nnk-(N-1)n^2k^2}{N^2(N-1)}$$

$$= \frac{Nn^2k^2-Nnk^2-Nn^2k+Nnk+N^2nk-Nnk-Nn^2k^2+n^2k^2}{N^2(N-1)}$$

$$= \frac{-Nnk^2-Nn^2k+N^2nk+n^2k^2}{N.N(N-1)}$$

$$\Rightarrow \frac{nk[-Nk-Nn+N^2+nk]}{N.N(N-1)}$$

$$\Rightarrow \frac{nk[-Nk+N^2-Nn+nk]}{N.N(N-1)} \quad \Rightarrow \frac{nk[-N(k+N)+n(-N+k)]}{N.N(N-1)}$$

$$\Rightarrow \frac{nk[n(k-N)-N(k-N)]}{N.N(N-1)} \quad \Rightarrow \frac{nk[k-N][n-N]}{N.N(N-1)}$$

$$\Rightarrow \frac{nk[-(N-k)][-(N-n)]}{N.N(N-1)} \quad \Rightarrow \frac{nk(N-k)(N-n)}{N.N(N-1)}$$

$$\Rightarrow \frac{nk}{N}\frac{N-k}{N}\frac{N-n}{N-1}$$

$$\therefore \quad \text{var}(x) = \frac{nk}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Example (18) $a box contains (20) balls [12 red balls \& 8 white balls],$ $we selected 8 balls from this box at random and without replacement. (let x no. of a red balls)$ $find: 1) p.m.f of X. 2) Mean \& V(x). 3) Pr(x \geq 2).$

Solution

1)Pmf ?N=20

$$P(x;N,k,n) = \begin{cases} \frac{C_x^k \; C_{n-x}^{N-k}}{C_n^N} & x = 0,1,2,\dots,n \\ 0 & o.w \end{cases}$$

$$P(x;N,k,n) = \begin{cases} \frac{C_x^8 \; C_{8-x}^{12}}{C_8^{20}} & x = 0,1,2,\dots,8 \\ 0 & o.w \end{cases}$$

2) E(x) & v(x)?

$E(x)=\frac{nk}{N} = \frac{8*8}{20} = 3.2$

$$\text{var}(x) = \frac{nk}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = \frac{8*8}{20}\left(1-\frac{8}{20}\right)\left(\frac{20-8}{20-1}\right) = 1.21$$

$3)P(x\geq 2) = 1 - P(x<2) = 1-(P(0)+p(1)) = 1-\left(\frac{C_0^8 \; C_8^{12}}{C_8^{20}} + \frac{C_1^8 \; C_7^{12}}{C_8^{20}}\right) =$ 0.94577

Example (19) *box contains* (10) *lights bulbs* [3 *are defective* & 7 *are non defective*], *if draw* 3 *bulbs at random with out replacement*, *and let* x *represent*. *No of defective bulbs in drawn sample*. *find* 1)*The p.m.f of* X & *the range of* X. 2)*The pro*. *of each point in the range*. 3)*Mean* & *variance of* X.
4) $\Pr(1<x\leq 3)$, $\Pr(0\leq x<2)$, $\Pr(x=4)$, $\Pr(2\leq x<3)$.

Solution

1) The p.m.f of x & range of x?

N=10 k=3 N-k=7 n=3

$$P(x;N,k,n) = \begin{cases} \frac{C_x^k \; C_{n-x}^{N-k}}{C_n^N} & x = 0,1,2,\ldots,n \\ 0 & o.w \end{cases}$$

$$P(x;N,k,n) = \begin{cases} \frac{C_x^3 \; C_{3-x}^{7}}{C_3^{10}} & x = 0,1,2,3 \\ 0 & o.w \end{cases}$$

2) The probability each point?

P(x=0)= $\frac{C_0^3 \; C_3^7}{C_3^{10}} = \frac{35}{120}$

P(x=1)= $\frac{C_1^3 \; C_2^7}{C_3^{10}} = \frac{63}{120}$

P(x=2)= $\frac{C_2^3 \; C_1^7}{C_3^{10}} = \frac{21}{120}$

P(x=3)= $\frac{C_3^3 \; C_0^7}{C_3^{10}} = \frac{1}{120}$

3) Mean and Var. of x?

E(x) = E(x)=$\frac{nk}{N} = \frac{3*3}{10} = \frac{9}{10}$

$$\text{var}(x) = \frac{nk}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = \frac{3*3}{10}\left(1-\frac{3}{10}\right)\left(\frac{10-3}{10-1}\right) = \frac{49}{100}$$

4)P$P_r(1<x \leq 3) = P(x=2) + p(x=3) = \frac{C_2^3 \; C_1^7}{C_3^{10}} + \frac{C_3^3 \; C_0^7}{C_3^{10}} = \frac{22}{120}$

$P_r(0 \leq x < 2) = P(x=0) + p(x=1) = \frac{C_0^3 \; C_3^7}{C_3^{10}} + \frac{C_1^3 \; C_2^7}{C_3^{10}} = \frac{98}{120}$

P_r(x=4)= not have resulte because r bigger than n if used.

$P_r(2 \leq x < 3) = P(x=2) = \frac{C_2^3 \; C_1^7}{C_3^{10}} = \frac{21}{120}$

Exercise of Chapter Four

Exer. (1) ...t x~ D.U(7)

Find

1) $Mean \& Var(x)$

2) $\Pr(x > \mu_x)$

3) $\Pr(\mu - \sigma_x \le x \le \mu + \sigma_x)$

4) $Mean \& Variance\ of\ \ y = 2 + 3x$

Exer. (2) $:-Let\ X\ be\ a\ r.v\ with\ D.U(8)$

$Find: 1)\ p.d.f \qquad 2)\ c.d.f \qquad 3)\ Mean\ \& V(x)$

$4)\ \Pr(x \le 4) \qquad 5)\ \Pr(x \ge 3).$

Exer. (3) $-Let\ x \sim D.U(n)\ find\ the\ mean\ \& Var\ of$

$y = a + bx$ [$where\ a\ \&\ b\ are\ real\ cons\tan t$]

Exer. (4) $:-Let\ x \sim br(\frac{3}{4});$

$Find\ \ 1)\ \Pr(X < \alpha_1)? \qquad 2)\ p_r(\mu < x < \mu - \alpha_2)?$

Exer. (5) $If has a M_x(t) = (\frac{1}{3} + \frac{2}{3}e^t)^5$

Find;

$1) the p.d.f of X 2) Mean 3) V(x) 4)\ Pr(M - 2\sigma < X < M + 2\sigma)$

Exer. (6) $let X be a binomial dist X \sim b(7, \frac{1}{2})$

$Find: 1) p.d.f of X 2) Mean \& Variance$

$3) The M.g.f of X 4)\ Pr(0 \le X \le 1) 5)\ Pr(X = 5)$

Exer. (7) $let X \sim b(n, p), let y = n - x$

$show that:$

$1) y \sim b(n, q)$

$2) find\ cov(x, n - x)$

Exer. (8) $t x_1, x_2, x_3 be a random sample space 3 and is mutually indep.$

$have same dist. function. [c.d.f. F(x)] and,$

$Let y = middle value of x_1, x_2, x_3 let X_i \le y, i = 1,2,3$

$is the i - tn trial of success find p.d.f of y?$

Exer. (9) $let y \sim bin(n, \frac{1}{4}), and \ Pr(y \geq 1) \geq 0.70$

$find(n)?.$

Exer. (10) $let X be a r.v. has a Poisson dist. with \lambda = 3$

$Find: Pr(x = 2), Pr(x \leq 3), Pr(x \geq 5), Pr(4 \leq x \leq 8)$

Exer. (11) $let X be a r.v., and let p.m.f f(x) be positive one and$

$only one the nonnegative\ int\ e\ gers, given that:$

$f(x) = \frac{4}{x} f(x-1) x = 1,2,3,\ldots$

$find f(x).$

Exer. (12) $let X have a poisson dist. with M = 100 chybyhev's inquality$

$to deter\ min\ s\ is a lower bound. find: Pr(75 < x < 125)?.$

Exer. (13) $if x has poisson an\ Pr(X = 0) = \frac{1}{2}$

$what is E(x)?$

Exer. (15) $Let x \sim p(\lambda)$

x	0	1	2	3	4	5	6	7
$p(x)$	50	160	130	90	66	4	0	0

$Find 1) E(x) 2) V(x) 3) p.m.f 4) M_x(t)$

Exer. (16) $Let X \sim G(p) and\ Pr(X \geq 2) = 0.25$

$Find: 1) p.m.f of x.$
$2) E(x) \& V(x).$
$3) M_x(t).$

Exer. (17) :5 *card are drawn without replacement from an ordinary pack of* (52) *cards, if r.v .x represent the No. of red (diamond) in drawn sample k .*

Find :1)*The p.m.f of x .* 2)*The pro. that* (4) *cards are red .*

3)*At more two red (dia.) card .* 4)*Mean &Var of* (*x*).

Exer. (18) $oxcontains(20)balls[12redballs\&8whiteballs],$

$weselected8ballsfromthisboxatrandomandwithoutreplacement.$

$Findtheprobabilityof:$

$1)drawingthreeredballs.$

$2)atleasttworedballs.$

$3)atmosttworedballs.$

$4)lessthantworedballs.$

$5)Notmorethantworedballs.$

$6)morethanfourredballs.$

Name:-..: ناو

Code & Group (A) (B) (C)

Department of Statistics

Monthly Exam
Chapter (4)

ID Exam

Sub.: Mathematical Statistics
Date:- 5 – 5- 2011
Time: 90 minutes

Q₁) *Let* $x \sim br(1,p)$ *show that* $\alpha_1 = \frac{q-p}{\sqrt{pq}}$, $\alpha_1 = \text{"Skewnes"}$ — *25 Marks*

Q₂) *Let* $x \sim H.G(28, n, k)$

knowing that $E(x) = 1$ *and* $V(x) = \frac{2}{3}$

Find n, k ? — *25 Marks*

Q₃) *Let* $x \sim bin(n,p)$ and; — *25 Marks*

X_i	0	1	2	3
f_i	4	2	2	1

Find 1) $M_x(t)$? 2) $p_r(x>2)$ 3) $p_r(x<v(x))$

Q₄) : If — *25 Marks*

1) $M_{x1}(t) = 4(25 - 30e^t + 9e^{2t})^{-1}$

2) $M_{x2}(t) = \left(\frac{4}{1-2e^t+e^{2t}}\right)^{-2}$

3) $M_{x2}(t) = \left(\frac{2-2e^t}{e^t - e^{3t}}\right)^{-1}$

Find:

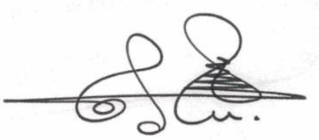
1) *p.m.f for each of them.*

2) $E(x_1)$. $E(x_2)$. $E(x_3)$.

2) $V(x_1)$. $V(x_2)$. $V(x_3)$.

100

Best of Luck



Dler Hussein Kadir
The examiner