# Chapter three

## 2.4  Solving linear programming problems: the simplex method

## 2.5   The Simplex method:-

The simplex method is an algebraic procedure. However, its basic concepts are geometric. Understanding these geometric concepts provides a strong intuitive feeling for how the simplex method operates and what makes it so efficient.

1. **Steps to the simplex method in tabular form:**
   **Step1- standard form:**
   **Step2-A-Normalize restrictions:** converts all the constraints to equality by adding slack, artificial variables, surplus as follow:

| Constraint type | Variable to be added |
|---|---|
| $\leq$ | +slack (s) |
| $=$ | +Artificial (R) |
| $\geq$ | -Surplus(s)+artificial(R) |

**Definition slack or surplus:** The term "slack" applies to less than or equal constraints, and the term "surplus" applies to greater than or equal constraint. If a constraint is binding, then the corresponding slack or surplus value will equal zero. When a less than or equal constraint is not binding, then there is some unutilized, or slack, resource. The slack value is the amount of the resource, as represented by the less than or equal constraint, that is not being used. When greater than or equal constraint is not binding, then the surplus is the extra amount over the constraint that is being produced or utilized. The units of the slack or surplus values are same as the units of the corresponding constraints.

**Definition Artificial variable:** Artificial variables are added to those constraints with equality (=) and greater than or equal to ($\geq$)sign. An artificial is added to the constraints to get an initial solution to an LP problem.

**B- Constructs the initial simplex tableau.**

**Step3- test for optimality:**

Case 1: Maximization problem

The current solution is optimal if every coefficient in the objective function row is non-negative.

Case 2: Minimization problem

The current basic function solution is optimal if every coefficient in the objective function row is non-positive.

**Step 4: Iteration**

**Step 1;** determine the **entering basic variable** by selecting the variable with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the z-row. Put a box around the column below this variable, and call it the "pivot column"

**Step 2:** Determine the **leaving basic variable by applying the minimum ratio test as following:**

**1.** Pick out each coefficient in the pivot column that is strictly ($>0$)

2. Divide each of these coefficients into the right hand side entry for the same row.

3. Identify the row that has the smallest of these ratios.

4. The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simples tableau. Put a box around this row and call it the "pivot row"

**Step 3**: solve for the new BF solution by using elementary row operations (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:

1. Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column).

2. For each other row that has a negative coefficient, add to this row the product of the absolute value of this coefficient and the new pivot row.

3. For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new "pivot row".

# Example:

$$z_{max} = 3x_1 + 5x_2$$

Subject                                                      to

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

**Solution:-1. Standard form**

$\mathbf{z}$**maximize**                                                              ,

$$z - 3x_1 - 5x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$

Subject to                                $x_1 + s_1 = 4$

$$2x_2 + s_2 = 12$$

$$3x_1 + 2x_2 + s_3 = 18$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

**2. Initial tableau:**

| Basic | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS | ratio |
|-------|-------|-------|-------|-------|-------|-----|-------|
| Z | -3 | -5 | 0 | 0 | 0 | 0 | |
| $s_1$ | 1 | 0 | 1 | 0 | 0 | 4 | None |
| $s_2$ | 0 | 2 | 0 | 1 | 0 | 12 | 6 |
| $s_3$ | 3 | 2 | 0 | 0 | 1 | 18 | 9 |

Table 1

| most negative | least ratio |
|---------------|-------------|
| -5 | 6 |

| Entering variable | leaving variable | |
|-------------------|------------------|--|
| pivot column | pivot row | pivot number |
| x2 | s2 | 2 |

## 3. Optimality test:

- By investing the z-row of the initial tableau, we find that there are some negative numbers. Therefore, the current solution is not optimal
- Step 1: Determine the entering variable by selecting the variable with the most negative in the z-row
- From the initial tableau, in the z-row, the coefficient of $x_1$ is -3 and the coefficient of $x_2$ is -5; therefore, the most negative is -5. Consequently, $x_2$ is the **entering variable.**
- $x_2$ is surrounded by a box and it is called the pivot column.
- Step 2: determining the leaving variable by using the minimum ratio test.

**4.** Solving for the new solution by using the eliminatory row operations as following:

**1-New pivot row= old pivot row ÷ pivot number**

**2-New row= old row+ the coefficient of this row in the pivot column*(new pivot row)**

Table 2

| most negative | least ratio |
|---------------|-------------|
| -3 | 2 |

| Entering variable | leaving variable | |
|-------------------|------------------|--|
| pivot colum | pivot row | pivot number |
| x2 | s2 | 3 |

**1-New pivot row= old pivot row ÷ pivot number**

**2-New row= old row+ the coefficient of this row in the pivot column*(new pivot row)**

| Basic | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHL | Ratio |
|-------|-------|-------|-------|-------|-------|-----|-------|
| Z | -3 | 0 | 0 | 2.5 | 0 | 30 | |
| $s_1$ | 1 | 0 | 1 | 0 | 0 | 4 | 4 |
| $x_2$ | 0 | 1 | 0 | 0.5 | 0 | 6 | None |
| $s_3$ | 3 | 0 | 0 | -1 | 1 | 6 | 2 |

| Basic | x1 | x2 | s1 | s2 | s3 | RHL |
|-------|-----|-----|-----|---------|---------|-----|
| Z | 0 | 0 | 0 | 1.5 | 1 | 36 |
| s1 | 0 | 0 | 1 | 0.33333 | -0.3333 | 2 |
| x2 | 0 | 1 | 0 | 0.5 | 1 | 6 |
| x1 | 1 | 0 | 0 | -0.3333 | 0.33333 | 2 |

Table 3

This solution is optimal; since there is no negative solution in the z-row: basic variables are $x_1 = 2$, $x_2 = 6$ and $s_1 = 2$; the non-basic variables are $s_1 = s_3 = 0$, z=36

$$z = 3 \times 2 + 5 \times 6$$

$$z = 6 + 30$$

$$z = 36$$

## 2.6 <u>Artificial variable</u>

➢ Artificial variable is added to the LHS of an equation of a ≥ and = (are called M-method) constraint in order to convert an equation (=).

➢ Artificial variable cannot take negative value just like other variables and are assigned non-negativity restrictions, i.e. $R_i \geq 0$.

➢ An artificial variable is fictitious and do not have any physical meaning.

➢ **Assign –M to an artificial variable in the objective function of a maximization** problem and **assign +M to an artificial variable in the objective function of a minimization problem,** where M is a big penalty or large coefficient.

**Steps of the big M-method:**

**Step1:** express the LP problem in the standard form by introducing slack variable.

**Step2:** add non-negative artificial variable (R) corresponding to constraints having "≥"and"="equations.

**Step3:** set up the initial solution.

**Step4:** 21.

### Example1:

$$Z_{max} = 4x_1 + 5x_2$$

Subject to

$$2x_1 + 5x_2 \leq 6$$

$$3x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

### Sol:

$$Z_{max} = 4x_1 + 5x_2 - MR1$$

$$2x_1 + 5x_2 + S_1 = 6$$

$$3x_1 + x_2 - S_2 + R_1 = 3$$

$$x_1, x_2 \geq 0$$

$$Z - 4x_1 - 5x_2 + 0S1 + 0S_2 + MR1 = 0$$

| BV | X₁ | X₂ | S₂ | S₁ | R₁ | RHS |
|---|---|---|---|---|---|---|
| Z | -4 | -5 | 0 | 0 | M | 0 |
| S2 | 2 | 5 | 0 | 1 | 0 | 6 |
| R1 | 3 | 1 | -1 | 0 | 1 | 3 |

| BV | X₁ | X₂ | S₂ | S₁ | R₁ | RHS |
|---|---|---|---|---|---|---|
| Z | -4-3M | -5-M | 0 | 0 | 0 | -3M |
| S2 | 2 | 5 | 0 | 1 | 0 | 6 |
| R1 | **3** | 1 | -1 | 0 | 1 | 3 |

| BV | X₁ | X₂ | S₂ | S₁ | R₁ | RHS |
|---|---|---|---|---|---|---|
| Z | 0 | $-\dfrac{11}{3}$ | $-\dfrac{4}{3}$ | 0 | $\dfrac{1}{3}(4+M)$ | 4 |
| S2 | 0 | 3 | 0 | 1 | $-\dfrac{2}{1}$ | 4 |
| X1 | **1** | $\dfrac{1}{3}$ | $-\dfrac{1}{3}$ | 0 | $\dfrac{1}{3}$ | 1 |

| BV | X₁ | X₂ | S₂ | S₁ | R₁ | RHS |
|---|---|---|---|---|---|---|
| Z | 0 | 0 | $-\dfrac{4}{3}$ | 0 | $\dfrac{1}{3}(4+M)$ | $\dfrac{288}{39}$ |
| X2 | 0 | 1 | $\dfrac{2}{13}$ | $\dfrac{3}{13}$ | $-\dfrac{2}{13}$ | $\dfrac{12}{13}$ |
| X1 | 1 | 0 | $-\dfrac{15}{39}$ | $-\dfrac{3}{39}$ | $\dfrac{15}{39}$ | $\dfrac{9}{13}$ |

| BV | X₁ | X₂ | S₂ | S₁ | R₁ | RHS |
|---|---|---|---|---|---|---|
| Z | 0 | 5 | 0 | 2 | *M* | 12 |
| X2 | 0 | $\dfrac{13}{2}$ | 1 | $\dfrac{3}{2}$ | 1 | 6 |
| X1 | 1 | $\dfrac{3}{2}$ | 0 | $\dfrac{1}{2}$ | 0 | 3 |

**Therefore:**

**Z=12,  X1=3 , X2=0**

**Example2:( H.W)**

$$Z_{max} = 3x_1 - x_2$$

Subject                                                                                    to

$$2x_1 + x_2 \leq 2$$

$$x_1 + 3x_2 \geq 3$$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution:

Ex : find the optimal solution for the following LPP.

Max z= 5x1+12x2+4x3

    X1+2x2+x3≤5

    2x1+x2+3x3=2

    X1≥0 x2≥0 x3≥0

Convert to the standard form:  Z=5x1+12x2+4x3-MR

    X1+2x2+x3+s1=5

    2x1-x2+3x3+R=2

    R=2-2X1+X2-3X3

Substitute in Z    Z = 5X1+12X2+4X3-M(2-2X1+X3-3X3)

=(5+2M) X1+(12-M)X2+(4+3M)X3-2M

$Z-(5+2M)X_1-(12-M)X_2-(4+3M)X_3=-2M$

$X_1+2X_2+X_3+S_1=5$

$2X_1-X_2+3X_3+R=2$

| Basic var. | X1 | X2 | X3 | S1 | R | SOLU |
|---|---|---|---|---|---|---|
| Z | -5-2M | -12+M | -4-3M | 0 | 0 | -2M |
| S1 | 1 | 2 | 1 | 1 | 0 | 5 |
| R | 2 | -1 | 3 | 0 | 1 | 2 |
| Z | -7/3 | -40/3 | 0 | 0 | 4/3+M | 8/3 |
| S1 | 1/3 | 7/3 | 0 | 1 | -1/3 | 13/3 |
| X3 | 2/3 | -1/3 | 1 | 0 | 1/3 | 2/3 |
| Z | -3/7 | 0 | 0 | 40/7 | -4/7+M | 192/7 |
| X2 | 1/7 | 1 | 0 | 3/7 | -1/7 | 13/7 |
| X3 | 5/7 | 0 | 1 | 1/7 | 2/7 | 9/7 |
| Z | 0 | 0 | 3/5 | 29/5 | -2/5+M | 141/5 |
| X2 | 0 | 1 | -1/5 | 2/5 | -1/5 | 3/5 |
| X1 | 1 | 0 | 7/5 | 1/5 | 2/5 | 9/5 |

## EX: Find the optimal solution for the following LPP.

## Minimization case of simplex method

For most part of finding solution for minimization problem using simplex method are handled in the same fashion as maximization problem. The three key exceptions are:

➢ The coefficients of artificial (M) have positive sign in the objective function.
➢ The selection of pivot column (entering variables) is based on the largest positive number in the ($\Delta j$) row.
➢ The solution is optimal when all values in the ($\Delta j$) row are non-positive.

The other alternative method of solving minimization problem is by converting it to maximization.i.e., Min Z =Max (-Z).

**Example**:
$$z_{min} = 12x_1 + 20x_2$$

*subject to*:
$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

Standard form:
$$Z_{Min} = 12x_1 + 20x_2 + 0S_1 + 0S_2 + MR_1 + MR_1$$

*subject to*:
$$6x_1 + 8x_2 - s_1 + R_1 = 100$$
$$7x_1 + 12x_2 - s_2 + R_2 = 120$$
$$x_1, x_2, s_1, s_2, R_1R_2, \geq 0$$

| BV | x1 | x2 | s1 | s2 | R1 | R2 | RHS |
|----|----|----|----|----|----|----|-----|
| Cj | 12 | 20 | 0 | 0 | M | M | 0 |
| R1 | 6 | 8 | -1 | 0 | 1 | 0 | 100 |
| R2 | 7 | 12 | 0 | -1 | 0 | 1 | 120 |

Table1

| BV | x1 | x2 | s1 | s2 | R1 | R2 | RHS |
|----|----|----|----|----|----|----|-----|
| Z | 13M- 12 | 20M-20 | (-M) | (-M) | 0 | 0 | 220M |
| R1 | (4/3) | 0 | -1 | (2/3) | 1 | 0 | 20 |
| X2 | (7/12) | 1 | 0 | (-1/12) | 0 | 1 | 10 |

Table2

| BV | x1 | x2 | s1 | s2 | R1 | R2 | RHS |
|----|----|----|----|----|----|----|-----|
| Z | 4/3M-1/3 | 0 | (-M) | 2/3M-5/3 | 0 | 0 | 220M |
| R1 | (4/3) | 0 | -1 | (2/3) | 1 | 0 | 20 |
| X2 | (7/12) | 1 | 0 | (-1/12) | 0 | 1 | 10 |

Table3

| BV | x1 | x2 | s1 | s2 | RHS |
|----|----|----|----|----|-----|
| Z | 0 | 0 | (-1/4) | -9 | 220 |
| x1 | 1 | 0 | (-3/4) | (1/2) | 15 |
| x2 | 0 | 1 | (7/16) | (-3/4) | (5/4) |

Example: (H.W)

min z = $2x_1 + 3x_2$

st    $0.5x_1 + 0.25x_2 \leq 4$

$x_1 + 3x_2 \geq 20$

$x_1 + x_2 = 10$

$x_1, x_2, > 0$

MIN z=$4x1+x2$

S.T

$3x1+x2=3$

$4x1+3x2 \geq 6$

$X1+2x2 \leq 4$

$X1 \geq 0, x2 \geq 0$

## Dual problem:

Associated with any linear programming problem is another linear problem, called the dual problem. Now we explain how to find the dual problem to the given linear programming problem, we discuss the economic interpretation of the dual problem and we discuss the relations that exist between an linear programming problem (called primal) and its dual problem. We consider the linear programming problem with normal form:

$$\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

$x_j \geq o(\text{j=1, 2,}\cdots\text{, n})$

The original problem is called the primal.


The dual problem is defined as follows:

$$\text{Min } z = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

S.to
$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \leq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \leq c_2$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} x_m \leq c_n$$

$y_i \geq o(\text{i=1, 2,}\cdots\text{, m})$

## Rules for construction of dual:

The general conclusion from the preceding example is that the variables and constraints in the primal and dual problems are defined by rules in the following table:

| **Maximization** | ↔ | **minimization** |
|---|---|---|
| Dual program | ↔ | primal program |
| Primal program | ↔ | dual program |

| Constraint | | variables |
|---|---|---|
| $\geq$ | $\leftrightarrow$ | $\leq 0$ |
| $\leq$ | $\leftrightarrow$ | $\geq 0$ |
| $=$ | $\leftrightarrow$ | free |
| Variables | | constraint |
| $\geq 0$ | $\leftrightarrow$ | $\geq$ |
| $\leq 0$ | $\leftrightarrow$ | $\leq$ |
| Free | $\leftrightarrow$ | $=$ |

## Example 1:

Primal problem:

$$z_{min} = 2x_1 + x_2$$

$$3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$[x_1 + 2x_2 \leq 3] \times -1$$

$$x_1, x_2 \geq 0$$

Primal problem:

$$z_{min} = 2x_1 + x_2$$

$$3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$-x_1 - 2x_2 \geq -3$$

$$x_1, x_2 \geq 0$$

Dual problem:

$$z_{max=}3y_1 + 6y_2 - 3y_3$$

$$3y_1 + 4y_2 - y_3 \leq 2$$

$$y_1 - 3y_2 - 2y_3 \leq 1$$

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$$y_1, y_2, y_3 \geq 0$$

## Example 2:

Primal problem:

$$z_{max} = 7x_1 + 5x_2$$

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + 3x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Dual problem:

$$z_{min} = 4y_1 + 6y_2 + y_3$$

$$y_1 + 2y_2 - y_3 \geq 7$$

$$2y_1 + 3y_2 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

## Example 3:- Primal problem:

$$z_{max} = 2x_1 + 4x_2 + 3x_3$$

$$2x_1 + 2x_2 + 3x_3 \leq 6$$

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

Dual problem:

$$z_{min} = 6y_1 + 4y_2$$

$$2y_1 + y_2 \geq 2$$

$$2y_1 + 2y_2 \geq 4$$

$$3y_1 + y_2 \geq 3$$

$$y_1, y_2 \geq 0$$

**Solution the primal problem by simplex method:**

$$z - 2x_1 - 4x_2 - 3x_3 + 0s_1 + 0s_2$$

$$2x_1 + 2x_2 + 3x_3 + s_1 = 6$$

$$x_1 + 2x_2 + x_3 + s_2 = 4$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

| basic | x1 | x2 | x3 | s1 | s2 | RHS | Ratio |
|-------|----|----|----|----|----|-----|-------|
| z | -2 | -4 | -3 | 0 | 0 | 0 | |
| s1 | 2 | 2 | 3 | 1 | 0 | 6 | 3 |
| s2 | 1 | 2 | 1 | 0 | 1 | 4 | 2 |

Table1

| entering var | leaving var | |
|--------------|-------------|--|
| piovt column | pivot row | pivot element |
| x2 | s2 | 2 |

| most negative | least ratio |
|---------------|-------------|
| -4 | 2 |

| Basic | x1 | x2 | x3 | s1 | s2 | RHS | Ratio |
|-------|----|----|----|----|----|-----|-------|
| Z | 0 | 0 | -1 | 0 | 2 | 8 | |
| s1 | 1 | 0 | 2 | 1 | -1 | 2 | 1 |
| x2 | 0.5 | 1 | 0.5 | 0 | 0.5 | 2 | 4 |

Table2

| most negative | least ratio |
|---------------|-------------|
| -1 | 1 |

| basic | x1 | x2 | x3 | s1 | s2 | RHS |
|-------|----|----|----|----|----|-----|
| Z | 0.5 | 0 | 0 | 0.5 | 1.5 | 9 |
| x1 | 0.5 | 0 | 1 | 0.5 | -0.5 | 1 |
| x2 | 0.25 | 1 | 0 | -0.25 | 0.75 | 1.5 |

Table3

| entering var | leaving var | |
|--------------|-------------|--|
| piovt column | pivot row | pivot element |
| x3 | s1 | 2 |

**Since** z=9, $x_1 = 1, x_2 = 1.5$

$$2 \times 0 + 4 \times 1.5 + 3 * 1$$

$$6 + 3 = 9$$

# Examples:

# 3   Chapter Four

## 3.1   <u>Introduction</u>

The transportation problem is one of special case of linear programming Problem Whose objective is to minimize the cost of transportation of goods to carry from One place to another place.

Such types of problems have more number of variables and constraints Which Makes the method complex to solve by the simple LPP methods. Therefore, to Make the calculation simple we require the transportation Model.

In transportation problems the goods are transported from a set of sources to The Set of destination to meet the requirements. The objective is to satisfy the Demand at the destination from the supply constraints at the minimum Transportation cost.

To achieve this, we must know the location, units to be supplied and the Demands of the units.

This model is useful for choosing optimum transportation routes which calls Us to Allocate the production of various plants to several warehouse.

- Transportation problem involves the movement of goods from:
  - Plants to warehouse.
  - Warehouses to wholesalers.
  - Wholesalers to retailers.
  - Retailers to customers.
- We have two different ways to solve this problem:
  - Initial basic feasible solution.
  - Optimality test.

### 3.1.1 Definition of some phrases:

**The transformation problem**: is a special type of linear programming problem, Where the objective is to minimize the cost of distributing from a number of Sources to a number of destinations.

**Origin:** A point of supply in a transportation problem.

**Destination:** A point of demand in transportation problem.

**Transportation table:** Tabular framework for solving the transportation problem.

<u>Mathematical Formulation of Transportation Problem</u>

Let there be m-warehouses which contain various amounts of a commodity to be Shifted to n-destinations. let **i$^{th}$** ware-house contain $a_i$ commodities, i=1,2,..,m While **j$^{th}$** destination requires

$b_j$ commodities , j=1,2,…,n . we assume that the Total available commodities may not be equal to the total demand. let $C_{ij}$ be the Shipping cost of a unit commodity from $i^{th}$ warehouse to $j^{th}$ destination. Let $t_{in}$ Be the number of units shipped from $i^{th}$ warehouse to $j^{th}$ destination .

The main aim of the problem is to determine a feasible shipping plan from various Warehouse to various destinations such that total transportation cost will be Minimum. It is assumed that the quantities such as $a_i$, $b_j$, and $C_{ij}$ are known. Such problem is known as "Transportation Problem ".

Transportation problem can be represented in the tabular from as below:

| To \ From | $D_1$ | $D_2$ | … | $D_n$ | Supply $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | $C_{11}$ $X_{11}$ | $C_{22}$ $X_{12}$ | … | $C_{1n}$ $X_{1n}$ | $a_1$ |
| $S_2$ | $C_{21}$ $X_{21}$ | $C_{22}$ $X_{22}$ | … | $C_{2n}$ $X_{2n}$ | $a_2$ |
| ⋮ | ⋮ | ⋮ | | ⋮ | ⋮ |
| $S_m$ | $C_{m1}$ $X_{m1}$ | $C_{m2}$ $X_{m2}$ | … | $C_{mn}$ $X_{mn}$ | $a_m$ |
| Demand $b_j$ | $b_1$ | $b_2$ | … | $b_n$ | $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ |

**(Table 1.1)**

We write transportation problem as the linear programming model:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} t_{ij} \qquad (1,1)$$

Subject to:

$$\sum_{j=1}^{n} t_{ij} \leq a_i \qquad\qquad i=1, 2,\dots,m \qquad\qquad (1.2)$$

$$\sum_{i=1}^{m} t_{ij} \geq b_j \qquad\qquad j=1,2,\dots,n \qquad\qquad (1.3)$$

$$t_{ij} \geq 0 \qquad\qquad \text{for all (i,j)} \qquad\qquad (1.4)$$

## 3.2   Types of transportation problem

There are two types of transportation problem:

- Balanced transportation problem.
- Unbalanced transportation problem.
  1. **Balanced transportation problem:** if the total supply equals total demand, Then the problem is said to be balanced transportation problem. Mathematically, it is represented by:

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

  2. **Unbalanced transportation problem:** if total supply is not equal to total Demand ( requirement ), then the problem is said to be unbalanced Transportation problem. Mathematically, it is represented by:

$$\sum_{i=1}^{m} S_i \neq \sum_{j=1}^{n} D_j$$

### 3.2.1   Basic methods of solving transportation problem

Initial feasible solution is one where demand and supply is being satisfied.

The obtained solution maybe optimal or may not be optimal.

### 3.2.2   There are different methods to find the feasible solution, the most commonly Use Methods are:
  1. North-west corner method.
  2. Least cost method.
  3. Vogel's approximation method (VAM).
  4. Row minima method.
  5. Column minima method.

### 3.2.3   Optimality test for the initial basic feasible solution
- Before checking for the optimality we need to make sure that the feasible Solution is acceptable or not.
- If it satisfied the following condition then it is an acceptable solution to Check for optimality.
- There should be m+n-1 allocation in the given matrix, where
  m= no. of rows
  n= no. of column in the transportation matrix

Optimality test is performed to check that there is no alternative solution Which is yield from the basic feasible solution.

There are two methods to test the initial feasible solution for optimality:

1. Modified distribution method (MODI).
2. Stepping stone method

## 3.3  Initial basic feasible solution

### *3.3.1.1   North west corner:*

This method is used the obtain the initial feasible solution by allocating the cells from the beginning of the table.

*i.e.* from the upper left hand corner of the table.

This is one of the method to find initial feasible solution, the steps as follows:

**Step 1.** We start allocating from the upper left corner which the north west corner of the given matrix.

**Step 2.** Then compare the capacity and demand whichever is lower allocate it to (1,1) cell.

**Step 3.** If the demand in the column is satisfied then move to the right of the Matrix i.e. in the next column, and if the capacity in the row satisfied the move Downward in the matrix i.e. to the next row of the matrix.

**Step 4.** And if both demand and capacity are satisfied then move to the diagonal Square i.e. moving towards next column and to the next row.

**Step 5.** Repeat the above step till all the demand and capacity is satisfied.

**Example 2.1:** Determine an initial feasible solution to the following Transportation problem using north-west corner rule**:**

|      | D1 | D2 | D3 | D4 | D5 | S  |
|------|----|----|----|----|----|----|
| S1   | 3  | 4  | 6  | 8  | 9  | 20 |
| S2   | 2  | 10 | 1  | 5  | 8  | 30 |
| S3   | 7  | 11 | 20 | 40 | 3  | 15 |
| S4   | 2  | 1  | 9  | 14 | 16 | 13 |
| D    | 40 | 6  | 8  | 18 | 6  |    |

**(Table 2.1)**

**Solution:**  First check the total supply and demand are equal:

Supply = 20+30+15+13 = 78                    Demand = 40+6+8+18+6 = 78

Since, supply = demand

Hence, the above problem is said to be a balanced problem.

|      | D1 | D2 | D3 | D4 | D5 | S |
|------|----|----|----|----|----|----|
| S1 | 3 | 4 | 6 | 8 | 9 | 20 |
|    | **20** |   |   |   |   |   |
| S2 | 2 | 10 | 1 | 5 | 8 | 30 |
|    | **20** | **6** | **4** |   |   |   |
| S3 | 7 | 11 | 20 | 40 | 3 | 15 |
|    |   |   | **4** | **11** |   |   |
| S4 | 2 | 1 | 9 | 14 | 16 | 13 |
|    |   |   |   | **7** | **6** |   |
| D | 40 | 6 | 8 | 18 | 6 |   |

**(Table 2.2)**

= (20*3) + (20*2) + (6*10) + (4*1) + (4*20) + (11*40) + (7*14) + ( 6 * 16 )

= 60 + 40 + 60 + 4 + 80 + 440 + 98 + 96 = **878**

### 3.3.1.2    Least COSt method

It is used to obtain the initial feasible solution by allocating to the lowest cost cell and then allocating to the second lowest cost cell and so on.

This is another method to find the initial feasible solution, the steps as follows:

**Step 1**. We start allocating to the lowest cell due to which row will be crossed off if Capacity is exhausted or column will be crossed off if requirement if fulfilled.

**Step 2.** Then allocate to the second lowest cell and repeat the step 1.

**Step 3.** In the case of tie, then select the cell where allocation of number of units Are there.

**Example 2.2:** Determine an initial feasible solution of the following by using

Least cost method:

| | D1 | D2 | D3 | |
|---|---|---|---|---|
| S1 | 2 | 5 | 4 | 4 |
| S2 | 8 | 7 | 5 | 7 |
| S3 | 3 | 1 | 7 | 6 |
| S4 | 5 | 8 | 9 | 13 |
| | 6 | 8 | 16 | |

**(Table 2.3)**

**Solution:** First, check the total supply and demand are equal:

Supply = 4+7+6+13 = 30      and      Demand = 6+8+16 = 30

Since, supply = demand then the above problem is said to be a balanced problem.

| | D1 | D2 | D3 | |
|---|---|---|---|---|
| S1 | (2) 2 5 **4** | 5 | 4 | 4 |
| S2 | 8 | 7 | (4) 5 **7** | 7 |
| S3 | 3 | (1) 1 **6** | 7 | 6 |
| S4 | (3) 5 **2** | (5) 8 **2** | (6) 9 **9** | 13 |
| | 6 | 8 | 16 | |

**(Table 2.4)**

$= (6*1) + (4*2) + (2*5) + (7*5) + (2*8) + (9*9)$

$= 8 + 35 + 6 + 10 + 16 + 81 = \mathbf{156}$

### 3.3.1.3　　Vogel approximation method (VAM)

This is also one of the method used to obtain the initial feasible solution by Allocating to the cell which are having high penalties. This method often obtain The optimal solution.

*This method gives the better optimal solution than the northwest corner method as it does not take into the consideration of cell cost.

Steps to solve the Vogel's approximation:

**Step 1**. Firstly, calculate the difference between the two minimum elements in Each row and column.

**Step 2**. Select that row or column which has the largest difference.

**Step 3.** Allocate to the minimum cost in the select row or column with capacity or Demand which have the lower value.

**Step 4.** Then cross out the row or column whose assignment has been completed.

**Step 5.** Again determine the row and column difference for all the rows and Column except the cross out row or column.

**Step 6.** Repeat the producer till all the assignment have been made.

**Example 2.3 :** Determine an initial basic feasible solution of the following by *VAM method*:

|     | D1 | D2 | D3 | S  |
|-----|----|----|----|----|
| S1  | 2  | 5  | 4  | 4  |
| S2  | 8  | 7  | 5  | 7  |
| S3  | 3  | 1  | 7  | 6  |
| S4  | 5  | 8  | 9  | 13 |
| D   | 6  | 8  | 16 |    |

**(Table 2.5)**

**Solution:** First, check the supply and demand are equal:

Supply $= 4 + 7 + 6 + 13 = 30$,   and       demand $= 6 + 8 + 16 = 30$

Since, supply $=$ demand so, the above problem is said to be a balanced problem.

|      | D1   |   | D2  |   | D3   |   | S  |     |     |     |   |   |   |   |   |   |
|------|------|---|-----|---|------|---|----|-----|-----|-----|---|---|---|---|---|---|
| S1   | (2)  | 2 | 5   |   | 4    |   | 4  |     |     |     |   |   |   |   |   |   |
|      | 4    |   | x   |   | X    |   |    | 2   | 2   | -   | - | - | - |   |   |   |
| S2   | 8    |   | 7   |   | (3)  | 5 | 7  |     |     |     |   |   |   |   |   |   |
|      | X    |   | x   |   | 7    |   |    | 2   | 2   | 2   | - | - | - |   |   |   |
| S3   | 3    |   | (1) | 1 | 7    |   | 6  |     |     |     |   |   |   |   |   |   |
|      | X    |   | 6   |   | X    |   |    | 2   | -   | -   | - | - | - |   |   |   |
| S4   | (6)  | 5 | (5) | 8 | (4)  | 9 | 13 |     |     |     |   |   |   |   |   |   |
|      | 2    |   | 2   |   | 9    |   |    | 3   | 3   | 3   | 3 | 3 | 5 |   |   |   |
| D    | 6    |   | 8   |   | 16   |   |    |     |     |     |   |   |   |   |   |   |
|      | 1    |   | 4   |   | 1    |   |    |     |     |     |   |   |   |   |   |   |
|      | 3    |   | 2   |   | 1    |   |    |     |     |     |   |   |   |   |   |   |
|      | 3    |   | 1   |   | 4    |   |    |     |     |     |   |   |   |   |   |   |
|      | 3    |   | 1   |   | 9    |   |    |     |     |     |   |   |   |   |   |   |
|      | 5    |   | 8   |   | -    |   |    |     |     |     |   |   |   |   |   |   |
|      | 5    |   | -   |   | -    |   |    |     |     |     |   |   |   |   |   |   |

**(Table 2.6 )**

$= (6*1) + (4*2) + (7*5) + (9*9) + (2*5) + (2*8)$

$= 6 + 8 + 35 + 81 + 10 + 16 = \textbf{156}$