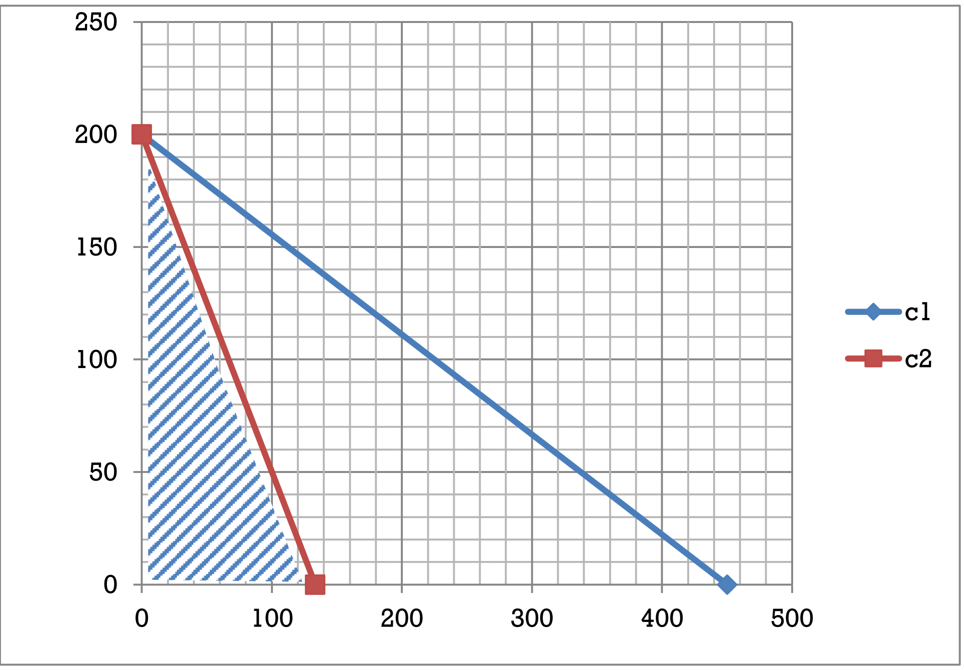

$(133.33, 0)$

$$z(0,0) = 12(0) + 8(0) = 0$$

$$z(0,200) = 12(0) + 8(200) = 1600$$

$$z(133.33,0) = 12(133.33) + 8(0) = 1600$$

Chapter Three

2.4 Solving linear programming problems: the simplex method

2.5 The Simplex method:-

The simplex method is an algebraic procedure. However, its basic concepts are geometric. Understanding these geometric concepts provides a strong intuitive feeling for how the simplex method operates and what makes it so efficient.

1. **Steps to the simplex method in tabular form:**
   Step1- standard form:
   Step2-A-Normalize restrictions: converts all the constraints to equality by adding slack, artificial variables, surplus as follow:

Constraint type	Variable to be added
≤	+slack (s)
=	+Artificial (R)
≥	-Surplus(s)+artificial(R)

Definition slack or surplus: The term "slack" applies to less than or equal constraints, and the term "surplus" applies to greater than or equal constraint. If a constraint is binding, then the corresponding slack or surplus value will equal zero. When a less than or equal constraint is not binding, then there is some unutilized, or slack, resource. The slack value is the amount of the resource, as represented by the less than or equal constraint, that is not being used. When greater than or equal constraint is not binding, then the surplus is the extra amount over the constraint that is being produced or utilized. The units of the slack or surplus values are same as the units of the corresponding constraints.

Definition Artificial variable: Artificial variables are added to those constraints with equality (=) and greater than or equal to (≥)sign. An artificial is added to the constraints to get an initial solution to an LP problem.

B- Constructs the initial simplex tableau.

Step3- test for optimality:

Case 1: Maximization problem

The current solution is optimal if every coefficient in the objective function row is non-negative.

Case 2: Minimization problem

The current basic function solution is optimal if every coefficient in the objective function row is non-positive.

Step 4: Iteration

Step 1; determine the **entering basic variable** by selecting the variable with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the z-row. Put a box around the column below this variable, and call it the "pivot column"

Step 2: Determine the **leaving basic variable by applying the minimum ratio test as following:**

1. Pick out each coefficient in the pivot column that is strictly (>0)

2. Divide each of these coefficients into the right hand side entry for the same row.

3. Identify the row that has the smallest of these ratios.

4. The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simples tableau. Put a box around this row and call it the "pivot row"

Step 3: solve for the new BF solution by using elementary row operations (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:

1. Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column).

2. For each other row that has a negative coefficient, add to this row the product of the absolute value of this coefficient and the new pivot row.

3. For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new "pivot row".

Example:

$$z_{max} = 3x_1 + 5x_2$$

Subject to

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

Solution:-1. Standard form

$\mathbf{z_{maximize}}$,

$$\boldsymbol{z - 3x_1 - 5x_2 + 0s_1 + 0s_2 + 0s_3 = 0}$$

Subject to

$$\boldsymbol{x_1 + s_1 = 4}$$
$$\boldsymbol{2x_2 + s_2 = 12}$$

$$\boldsymbol{3x_1 + 2x_2 + s_3 = 18}$$

$$\boldsymbol{x_1, x_2, s_1, s_2, s_3 \geq 0}$$

2. Initial tableau:

Basic	x_1	x_2	s_1	s_2	s_3	RHS	ratio
Z	-3	-5	0	0	0	0	
s_1	1	0	1	0	0	4	None
s_2	0	2	0	1	0	12	6
s_3	3	2	0	0	1	18	9

most negative	least ratio
-5	6

Table 1

Entering variable	leaving variable	
pivot column	pivot row	pivot number
x2	s2	2

3. Optimality test:

- By investing the z-row of the initial tableau, we find that there are some negative numbers. Therefore, the current solution is not optimal
- Step 1: Determine the entering variable by selecting the variable with the most negative in the z-row
- From the initial tableau, in the z-row, the coefficient of x_1 is -3 and the coefficient of x_2 is -5; therefore, the most negative is -5. Consequently, x_2 is the **entering variable.**
- x_2 is surrounded by a box and it is called the pivot column.
- Step 2: determining the leaving variable by using the minimum ratio test.

4. Solving for the new solution by using the eliminatory row operations as following:

1-New pivot row= old pivot row ÷ pivot number

2-New row= old row+ the coefficient of this row in the pivot column*(new pivot row)

Table 2

Entering variable	leaving variable	
pivot colum	pivot row	pivot number
x2	s2	3

most negative	least ratio
-3	2

1-New pivot row= old pivot row ÷ pivot number

2-New row= old row+ the coefficient of this row in the pivot column*(new pivot row)

Basic	x_1	x_2	s_1	s_2	s_3	RHL	Ratio
Z	-3	0	0	2.5	0	30	
s_1	1	0	1	0	0	4	4
x_2	0	1	0	0.5	0	6	None
s_3	3	0	0	-1	1	6	2

Basic	x1	x2	s1	s2	s3	RHL
Z	**0**	**0**	**0**	**1.5**	**1**	**36**
s1	**0**	**0**	**1**	**0.33333**	**-0.3333**	**2**
x2	**0**	**1**	**0**	**0.5**	**1**	**6**
x1	**1**	**0**	**0**	**-0.3333**	**0.33333**	**2**

Table 3

This solution is optimal; since there is no negative solution in the z-row: basic variables are $x_1 = 2,\ x_2 = 6$ and $s_1 = 2$; the non-basic variables are $s_1 = s_3 = 0$, z=36

$$z = 3 \times 2 + 5 \times 6$$

$$z = 6 + 30$$

$$z = 36$$

2.6 Artificial variable

- Artificial variable is added to the LHS of an equation of a $\geq$ and = (are called M-method) constraint in order to convert an equation (=).
- Artificial variable cannot take negative value just like other variables and are assigned non-negativity restrictions, i.e. $R_i \geq 0$.
- An artificial variable is fictitious and do not have any physical meaning.
- **Assign –M to an artificial variable in the objective function of a maximization** problem and **assign +M to an artificial variable in the objective function of a minimization problem,** where M is a big penalty or large coefficient.

  Steps of the big M-method:

  Step1: express the LP problem in the standard form by introducing slack variable.

  Step2: add non-negative artificial variable (R) corresponding to constraints having "≥"and"="equations.

  Step3: set up the initial solution.

  Step4: 21.

Example1:

$$Z_{\max} = 4x_1 + 5x_2$$

Subject to

$$2x_1 + 5x_2 \leq 6$$

$$3x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Sol:

$$Z_{\max} = 4x_1 + 5x_2 - MR1$$

$$2x_1 + 5x_2 + S_1 = 6$$

$$3x_1 + x_2 - S_2 + R_1 = 3$$

$$x_1, x_2 \geq 0$$

$$Z - 4x_1 - 5x_2 + 0S1 + 0S_2 + MR1 = 0$$

Example2:(H.W)

$$Z_{\max} = 3x_1 - x_2$$

Subject to

$$2x_1 + x_2 \leq 2$$

$$x_1 + 3x_2 \geq 3$$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution:

Ex : find the optimal solution for the following LPP.

Max z= 5x1+12x2+4x3

$X1+2x_2+x_3\leq 5$

$2x_1+x_2+3x_3=2$

$X1\geq 0$ $x2\geq 0$ $x3\geq 0$

Convert to the standard form: $Z=5x_1+12x_2+4x_3-MR$

$x_1+2x_2+x_3+s_1=5$

$2x_1-x_2+3x_3+R=2$

$R=2-2x_1+x_2-3x_3$

Substitute in Z $Z = 5X_1+12X_2+4X_3-M(2-2X_1+X_3-3X_3)$

$=(5+2M)X_1+(12-M)X_2+(4+3M)X_3-2M$

$Z-(5+2M)X_1-(12-M)X_2-(4+3M)X_3=-2M$

$X1+2X_2+X_3+S1=5$

$2X1-X_2+3X_3+R=2$

Basic var.	X_1	X_2	X_3	S_1	R	SOLU
Z	-5-2M	-12+M	-4-3M	0	0	-2M
S1	1	2	1	1	0	5
R	2	-1	3	0	1	2
Z	-7/3	-40/3	0	0	4/3+M	8/3
S1	1/3	7/3	0	1	-1/3	13/3
X3	2/3	-1/3	1	0	1/3	2/3

Z	-3/7	0	0	40/7	-4/7+M	192/7
X2	1/7	1	0	3/7	-1/7	13/7
X3	5/7	0	1	1/7	2/7	9/7
Z	0	0	3/5	29/5	-2/5+M	141/5
X2	0	1	-1/5	2/5	-1/5	3/5
X1	1	0	7/5	1/5	2/5	9/5

EX: Find the optimal solution for the following LPP.

Minimization case of simplex method

For most part of finding solution for minimization problem using simplex method are handled in the same fashion as maximization problem. The three key exceptions are:

- The coefficients of artificial (M) have positive sign in the objective function.
- The selection of pivot column (entering variables) is based on the largest positive number in the (Δj) row.
- The solution is optimal when all values in the (Δj) row are non-positive.

The other alternative method of solving minimization problem is by converting it to maximization.i.e., Min Z =Max (-Z).

Example: $z_{min} = 12x_1 + 20x_2$

subject to:

$$6x_1 + 8x_2 \geq 100$$

$$7x_1 + 12x_2 \geq 120$$

$$x_1, x_2 \geq 0$$

Standard form: $Z_{Min} = 12x_1 + 20x_2 + 0S_1 + 0S_2 + MR_1 + MR_1$

subject to:

$$6x_1 + 8x_2 - s_1 + R_1 = 100$$

$$7x_1 + 12x_2 - s_2 + R_2 = 120$$

$$x_1, x_2, s_1, s_2, R_1R_2, \geq 0$$

BV	x1	x2	s1	s2	R1	R2	RHS
Cj	12	20	0	0	M	M	0
R1	6	8	-1	0	1	0	100
R2	7	12	0	-1	0	1	120
Z	-12	-20	0	0	-M	-M	0

Table1

BV	x1	x2	s1	s2	R1	R2	RHS
Z	13M- 12	20M-20	(-M)	(-M)	0	0	220M
R1	(4/3)	0	-1	(2/3)	1	0	20
X2	(7/12)	1	0	(-1/12)	0	1	10

Table2

BV	x1	x2	s1	s2	R1	R2	RHS
Z	4/3M-1/3	0	(-M)	2/3M-5/3	0	0	220M
R1	(4/3)	0	-1	(2/3)	1	0	20
X2	(7/12)	1	0	(-1/12)	0	1	10

Table3

BV	x1	x2	s1	s2	RHS
Z	0	0	(-1/4)	-9	220
x1	1	0	(-3/4)	(1/2)	15
x2	0	1	(7/16)	(-3/4)	(5/4)

Example: (H.W)

$$\begin{aligned} \min z &= 2x_1 + 3x_2 \\ \text{st} \quad & 0.5x_1 + 0.25x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2, > 0 \end{aligned}$$

MIN z=4x1+x2

S.T

$$\begin{aligned} & 3x1+x2=3 \\ & 4x1+3x2\geq 6 \\ & X1+2x2\leq 4 \\ & X1\geq 0,x2\geq 0 \end{aligned}$$

Example: $z_{min} = 2x_1 + x_2$

subject to: $x_1 + 3x_2 \geq 30$

$4x_1 + 2x_2 \geq 40$

$x_1, x_2 \geq 0$

Standard form: $Z_{Min} = 2x_1 + x_2 + MR_1 + MR_2$

subject to: $x_1 + 3x_2 - s1 + R1 = 30$

$4x_1 + 2x_2 - s2 + R2 = 40$

$x_1, x_2, s_1, s_2, R_1R_2, \geq 0$

BV	x1	x2	s1	s2	R1	R2	RHS
Z	(-2+5M)	(-1+5M)	-M	-M	0	0	70M
R1	1	3	-1	0	1	0	10
R2	4	2	0	-1	0	1	20

BV	x1	x2	s1	s2	R1	R2	RHS
Z	(-2+5M)	1	-M	-M	0	0	10+20M
X2	1/3	0	-1/3	0	1/3	0	10
R2	10/3	0	2/3	-1	-2/3	1	20

BV	x1	x2	s1	s2	R1	R2	RHS
Z	0	0	0	-1/2	-M	(1/2-M)	20
X2	0	1	-1/3	0	1/3	0	8
X1	1	0	1/5	-3/10	-1/5	3/10	6

Dual problem:

Associated with any linear programming problem is another linear problem, called the dual problem. Now we explain how to find the dual problem to the given linear programming problem, we discuss the economic interpretation of the dual problem and we discuss the relations that exist between an linear programming problem (called primal) and its dual problem. We consider the linear programming problem with normal form:

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

$x_j \ge o$(j=1, 2,$\cdots$, n)

The original problem is called the primal.

The dual problem is defined as follows:

$$\text{Min}\, z = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

S.to

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \le c_1$$

$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \le c_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}x_m \le c_n$$

$y_i \ge o$(i=1, 2,$\cdots$, m)

Rules for construction of dual:

The general conclusion from the preceding example is that the variables and constraints in the primal and dual problems are defined by rules in the following table:

Maximization	$\leftrightarrow$	**minimization**
Dual program	$\leftrightarrow$	primal program
Primal program	$\leftrightarrow$	dual program

Constraint		variables
$\geq$	$\leftrightarrow$	$\leq$ **0**
$\leq$	$\leftrightarrow$	≥ 0
$=$	$\leftrightarrow$	free
Variables		**constraint**
≥ 0	$\leftrightarrow$	$\geq$
$\leq$ **0**	$\leftrightarrow$	$\leq$
Free	$\leftrightarrow$	$=$

Example 1:

Primal problem:

$$z_{\min} = 2x_1 + x_2$$

$$3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$[x_1 + 2x_2 \leq 3] \times -1$$

$$x_1, x_2 \geq 0$$

Primal problem:

$$z_{\min} = 2x_1 + x_2$$

$$3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$-x_1 - 2x_2 \geq -3$$

$$x_1, x_2 \geq 0$$

Dual problem:

$$z_{max=}3y_1 + 6y_2 - 3y_3$$

$$3y_1 + 4y_2 - y_3 \leq 2$$

$$y_1 - 3y_2 - 2y_3 \leq 1$$

$$y_1, y_2, y_3 \geq 0$$

Example 2:

Primal problem:

$$z_{max} = 7x_1 + 5x_2$$

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + 3x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Dual problem:

$$z_{min} = 4y_1 + 6y_2 + y_3$$

$$y_1 + 2y_2 - y_3 \geq 7$$

$$2y_1 + 3y_2 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

Example 3:- Primal problem:

$$z_{max} = 2x_1 + 4x_2 + 3x_3$$

$$2x_1 + 2x_2 + 3x_3 \leq 6$$

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

Dual problem:

$$z_{min} = 6y_1 + 4y_2$$

$$2y_1 + y_2 \geq 2$$

$$2y_1 + 2y_2 \geq 4$$

$$3y_1 + y_2 \geq 3$$

$$y_1, y_2 \geq 0$$

Solution the primal problem by simplex method:

$$z - 2x_1 - 4x_2 - 3x_3 + 0s_1 + 0s_2$$

$$2x_1 + 2x_2 + 3x_3 + s_1 = 6$$

$$x_1 + 2x_2 + x_3 + s_2 = 4$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

basic	x1	x2	x3	s1	s2	RHS	Ratio
z	-2	-4	-3	0	0	0	
s1	2	2	3	1	0	6	3
s2	1	2	1	0	1	4	2

Table1

entering var	leaving var	
piovt column	pivot row	pivot element
x2	s2	2

most negative	least ratio
-4	2

Basic	x1	x2	x3	s1	s2	RHS	Ratio
Z	0	0	-1	0	2	8	
s1	1	0	2	1	-1	2	1
x2	0.5	1	0.5	0	0.5	2	4

Table2

most negative	least ratio
-1	1

entering var	leaving var	
piovt column	pivot row	pivot element
x3	s1	2

basic	x1	x2	x3	s1	s2	RHS
Z	0.5	0	0	0.5	1.5	9
x1	0.5	0	1	0.5	-0.5	1
x2	0.25	1	0	-0.25	0.75	1.5

Table3

Since z=9,$x_1 = 1$, $x_2 = 1.5$

$$2 \times 0 + 4 \times 1.5 + 3 * 1$$

$$6 + 3 = 9$$