زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Matrices in Graph Theory

Research Project

Submitted to the Department of Mathematic in partial fulfillment of the requirements for the degree of BSc. In Mathematics

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Certification of the supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / University of Salahaddin in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Scientific grade: **Lecturer**

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In view of the available recommendations, I forward this report for debate by the examining committee.

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Acknowledgment

Have a lot of thanks to Allah for giving me the power and patience to perform this study.

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Asuda Mukhls

Abstract

As computer are more adept at manipulating numbers than at recognizing pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. In work we study some types of matrices associated with a graph, also we find some properties of this matrices.

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INTRODUCTION

Since a graph is completely determined by specifying either it is adjacency structure, these specifications provide far more efficient ways of representing a large or complicated graph than a pictorial representation.

Graph theory is an applied branch of the mathematics which deals the problems, with the help of graphs. There are many applications of graph theory to a wide variety of subjects which include operations Research, Physics, Chemistry, Economics, Genetics, Sociology, Computer Science, Engineering Mechanical Engineering and the other branches of science.

Many of the early concepts and theorems of graph theory came about quite indirectly, often from recreational mathematics, through puzzles, or games or problems that, as were seen later, could be phrased in terms of graphs. The very first of these was a problem called the Konigsberg Bridge Problem, which was not only solved by one of the most famous mathematicians of all time but whose solution is considered the origin of graph theory and would lead to an important class of graphs.

Our work consists of two chapters:

Chapter one we present fundamental concepts of graphs.

Chapter two we present some types of matrices.

CHAPTER ONE

Basic Notations and Definitions

Definition 1.1[(Vasudev, 2006)**]:** A graph G consists of a set of objects $V = \{v_1, v_2, v_3, \dots\dots\}$ Called vertices (also called points or nodes) and other set $E = \{e_1, e_2, e_3, \dots.\}$ whose elements are called edges (also called lines or arcs).

The set $V(G)$ is called the vertex set of G and $E(G)$ is the edge set, usually the graph is denoted by $G = (V, E)$.

Definition 1.2[(Zhang, 2012)**]** The number of vertices in G is often called order of G.

Definition 1.3[(Zhang, 2012)**]:** The number of edges in G is often called size of G.

Definition 1.4[(Wilson, 1972)**]:** The degree of a vertex is the number of edges which have that vertex set as an end point.

Definition 1.5[(Vasudev, 2006)**]:** An edge of a graph that joins a node to itself is called loop or self-loop.

Definition 1.6[(Wilson, 1972)**]:** A graph in which any two vertices are connected by a path is called a connected graph.

Definition 1.7[(Vasudev, 2006)**]:** A graph with finite number of vertices as well as a finite number of edges is called a finite graph.

Definition 1.8[(Zhang, 2012)**]:** A multigraph M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges. If two or more edges join the same pair of distinct vertices, then these edges are called parallel edge or multigraph.

Definition 1.9[(Zhang, 2012)**]:** A graph H is called a subgraph of a graph G, written $H \subset G$, if $V\ (H) \subset V\ (G)$ and $E\ (H) \subset E\ (G)$.

Definition 1.10[(Zhang, 2012)**]:** If a subgraph of a graph G has the same vertex set as G, then it is a spanning subgraph of G.

Definition 1.11[(Biggs, 1985)**]:** A walk in a graph G is a sequence of vertices $v_1, v_2, v_3, \dots, v_k$, such that $v_i$and $e_{i+1}$are adjacent $(1 \leq i \leq k-1)$, if all it is vertices are distinct, a walk is called a path.

Definition 1.12[(Gray Chartrand & Linda Lesniak& Ping Zhang, 2016) (G. Chartrand & L.Lesniak, 1996)**]:** A walk in a graph G in which no edge is repeated is a trail in G.

Definition 1.13[(Biggs, 1985)**]:** A walk $v_1, v_2, \dots., v_{r+1}$ whose vertices are all distinct except that $v_1 = v_{r+1}$ is called cycle.

Definition 1.14[(Lesniak, 1996)**]:** For a connected graph G, we define the distance $d(u, v)$ between two vertices u and v as the minimum of the lengths of the $u - v$ paths of G.

Definition 1.15[(Lesniak, 1996)**]:** A vertex u is said to detour dominate a vertex v if $u = v$ or u is a detour neighbor of v.

Definition 1.16[(Bapat, 2010)**]:** An $m \times n$ matrix is called a square matrix if $m = n$.

Definition 1.17[(Bapat, 2010)**]:** Operations of matrix addition, scalar multiplication and matrix multiplication are basic and will not be recalled here.

The transpose of the $m \times n$ matrix A is denoted by A'

Definition 1.18[(Bapat, 2010)**]:** A square matrix A is called symmetric if $A = A'$

CHAPTER TWO

Some Types of Matrices

1. Adjacency matrix:[(Ray, 2013)]

The adjacency matrix of a graph G with n vertices and no parallel edge is an n by n symmetric binary matrix $X = \left(x_{ij}\right)_{n\times n}$ defined over ring of integer such that

$$x_{ij} = \begin{cases} 1 & \text{, if there is an edge between ith and jth vertices} \\ 0 & \text{, if there is no edge between them.} \end{cases}$$

Example 2.1 Consider the graph

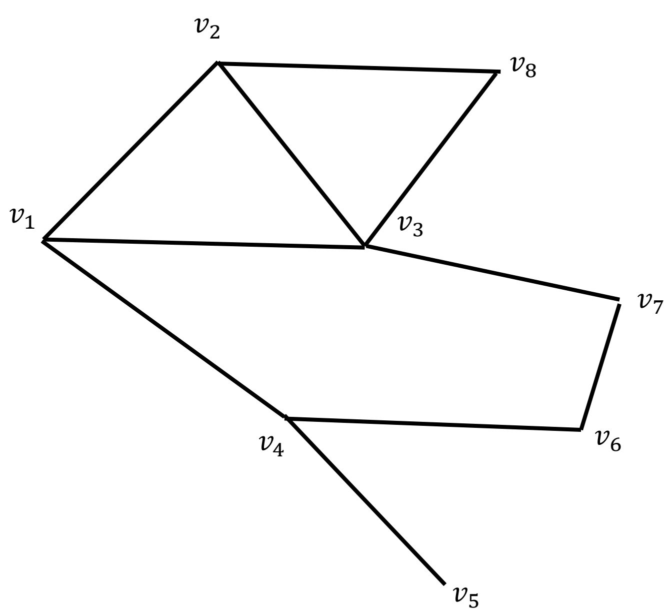

Figure 2.1 the graph G

The adjacency matrix of a graph G is

$$X_{ij} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The adajcency matrix contains only two types of elements, 0and 1.

This clearly is a binary matrix or a (0,1)-matrix.

We have the following observation about the adjacency matrix $X(G)$ of a graph G.

a. The entries along the principle diagonal of $X(G)$ are all zeros if and only if the graph has no self-loops.
b. If the graph has no self-loops, the degree of a vertex equals the number of ones in the corresponding row or column of X(G).
c. The $X(G)$ is square matrix and it is a symmetric.

2. **The edge adjacency matrix:[** (Anon., n.d.)]

The edge adjacency matrix denoted by A^e, of an edge labeled connected graph G is a square $E \times E$ matrix which is determined by the adjacencies of edges

$$A^e_{ij} \begin{cases} 1 & , if\ edge\ i\ and\ j\ are\ adjacent \\ 0 & , \qquad otherewise. \end{cases}$$

Example 2.2 Consider the graphs G

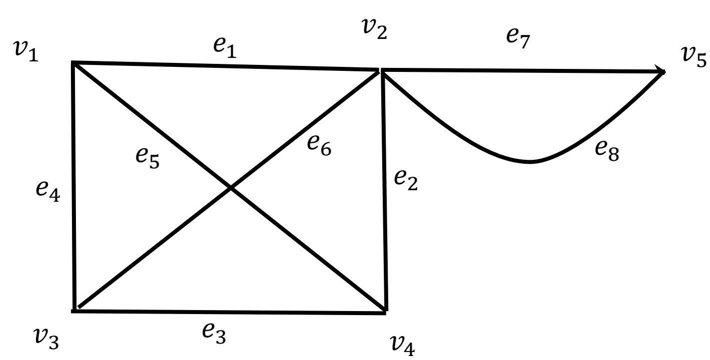

Figure 2.2 the graph G

The edge adjacency matrix of a graph G is

$$A^e_{ij} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The edge adacency matrix contains only two types of elements, 0and 1.

This clearly is a binary matrix or a (0,1)-matrix.

We have the following observation about the edge adjacency matrix E(G) of a graph G.

a. The A^e matrix is squre and symmetric matrix
b. The entries along the principle diagonal of A^e are all zeros

3. Incident matrix:[(Ray, 2013)]

Let G be a graph with *n* vertices, *e* edges, and no self-loop, we define a matrix A= $(a_{ij})_{n\times e}$ where *n* rows correspond to the *n* vertices and *e* the columns correspond to the *e* edges, as follows:

$$A_{ij} = \begin{cases} 1, if\ jth\ edge\ e_j\ \ is\ incident\ on\ ith\ vertex\ v_i \\ 0\ , otherewise \end{cases}$$

This matrix A is called **incidence matrix** of G. Sometimes it is written as $A(G)$

Example 2.3 Consider the graphs given

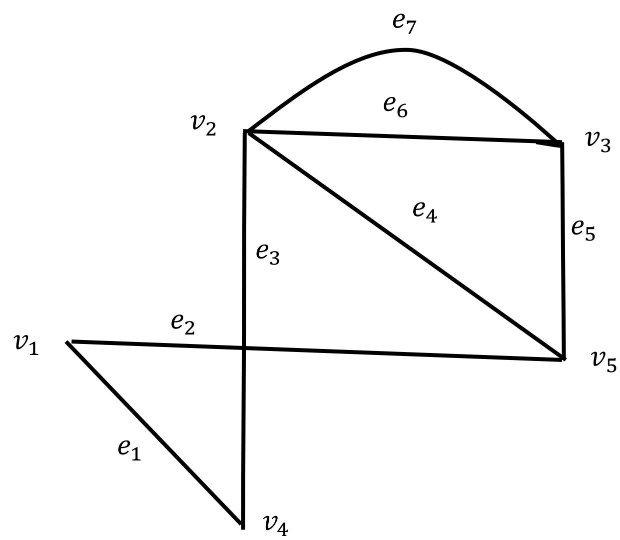

Figure 2.3 the graph G

The incidence matrix of a graph G is

$$A_{ij} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The incidence matrix contains only two types of elements, 0and 1.

This clearly is a binary matrix or a (0,1)-matrix.

We have the following observation about the adjacency matrix X(G) of a graph G.

a. Since every edge is incident on exactly two vertices, each column of I has exactly two ones.
b. The number of ones in each row equals the degree of the corresponding vertex.

4. Cycle matrix:[(Ray, 2013)]

Let the number of different circuits in a graph G be k and the number of edges in G be e .Then a circuit matrix B= $(b_{ij})_{k\times e}$ is a binary matrix defined where k rows correspond to the k circuits and e the columns correspond to the e edges, as follows:

$$B_{ij} = \begin{cases} 1, & if\ ith\ circut\ includes\ jth\ edge \\ 0, & otherewise. \end{cases}$$

This matrix B is called **cycle matrix** of G, and written as $B(G)$.

Example 2.4 Consider the given graph G

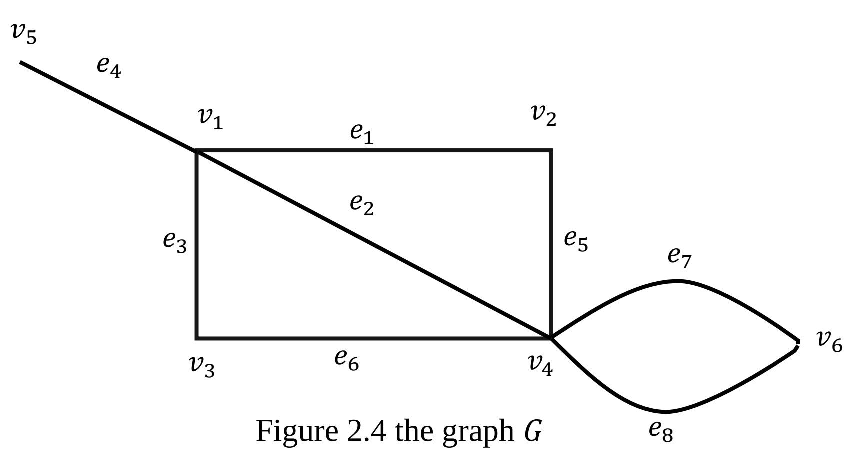

Figure 2.4 the graph G

The graph G has four different cycle

k_1={e_1, e_2 , e_5} , k_2={e_2, e_3 , e_6} , k_3={e_1, e_3, e_5, e_6} , k_4={e_7, e_8}

The cycle matrix of a graph G is

$$\begin{array}{cc} & \begin{array}{cccccccc} e_1 & & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} k_1 \\ k_2 \\ k_3 \\ k_4 \end{array} & \end{array}$$

$$B_{ij}= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The cycle matrix contains only two types of elements, 0and 1.

This clearly is a binary matrix or a (0,1)-matrix.

We have the following observation regarding the cycle matrix B(G) of a graph G.

a. A column of all zeros corresponds to a non cycle edge, that is, an edge which does not belong to any cycle.
b. The number of ones in a row is equal to the number of edges in the corresponding cycle.

5. **Path matrix:[** (Bapat, 2010)**]**

Let G be a graph with vertex set $V(G) = \{1,2,\ldots,n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Given a path P in G, the incidence vector of P is an $m\times1$ vector defined as follows. The entries of the vector are indexed by $E\ (G)$. if $e_i \epsilon E(G)$ then the *j*th element of the vector is 0 if the path does not contain e_i .

If the path contains e_i then the entry is 1 or -1, according as the direction of the path agrees or disagree, respectively, with e_i . where p rows correspond to the p paths and *e* the columns correspond to the *e* edges.

Example 2.5 Consider the graph G

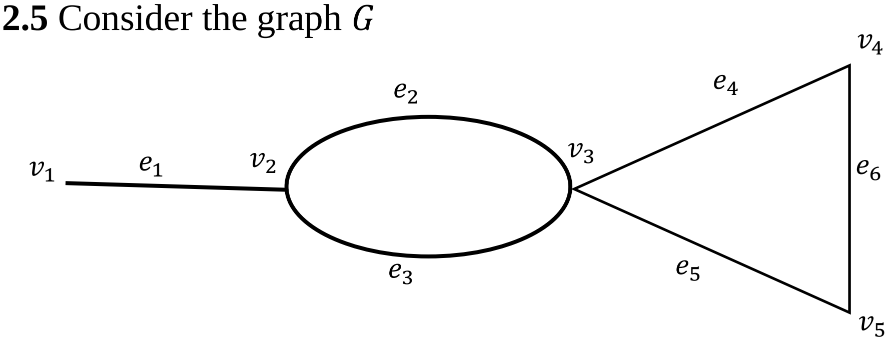

Figure 2.5 the graph G

The different paths between the vertices v_2 and v_4 are

$p_1=\{e_2, e_4\}$, $p_2=\{e_3, e_4\}$, $p_3=\{e_3, e_5, e_6\}$

The path matrix for v_2 and v_4 is given by

$$\begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} & \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix} \end{array}$$

The path matrix contains only two types of elements, 0and 1.

This clearly is a binary matrix or a (0,1)-matrix.

We have the following observation about the path matrix.

a. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
b. A column of all zeros corresponds to an edge that lies in every path between u and v.
c. There is no row with all zeros.

6. **Detour Matrix:[** (Ali Reza Ashrafi & Modjtaba Ghorbani & Maryam jalali, 4 december 2008)**]**

Let G be a graph the matrix $DD = [dd_{ij}]$, in which dd_{ij} is defined as the length of the longest path between vertices $i$and j is called the detour matrix of G.

Example 2.6 Consider the graph G

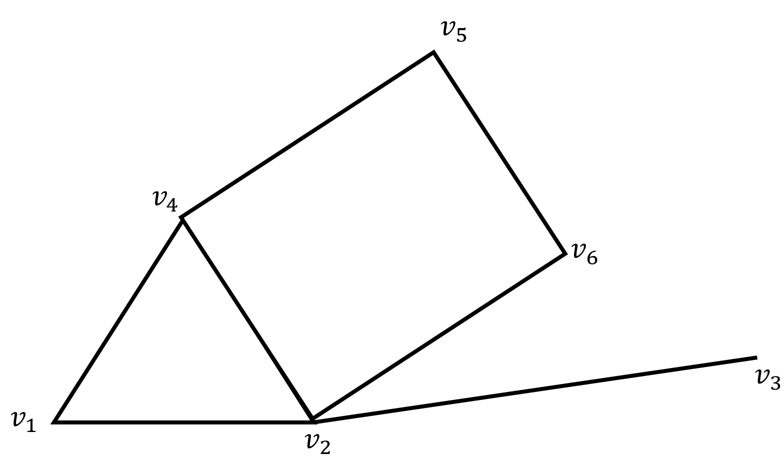

Figure 2.6 the graph of G

The detour matrix of a graph G is

$$dd_{ij}= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 2 & 5 & 2 & 3 & 3 \\ 2 & 0 & 1 & 3 & 2 & 3 \\ 5 & 1 & 0 & 4 & 3 & 2 \\ 2 & 2 & 4 & 0 & 3 & 2 \\ 4 & 3 & 4 & 3 & 0 & 3 \\ 4 & 3 & 4 & 2 & 3 & 0 \end{bmatrix} \end{array}$$

We have the following observation about the detour matrix.

a. The entries along the principle diagonal of $DD(G)$ are all zeros.
b. The $DD(G)$ is square matrix and it is a symmetric.
c. The detour matrix of complete graph has a simple form, that's mean, all off-diagonal elements are equal to the degree of a vertex, while the diagonal elements are, of course, equal to zero.

7. Degree Matrix [(Ray, 2013)**]**

The degree matrix of a graph is a diagonal matrix where the rows and columns are indexed by the set of vertices (in the same order), and each diagonal entry gives the degree of the corresponding vertex.

Where $V(G)=\{v_1, v_2, \dots, v_p\}$,

$D(G)=[d_{ij}]_{p\times p}$ with $d_{ij}=\begin{cases} deg v_i & i=j \\ 0 & i\neq j \end{cases}$

Example 2.7 Consider the graph G

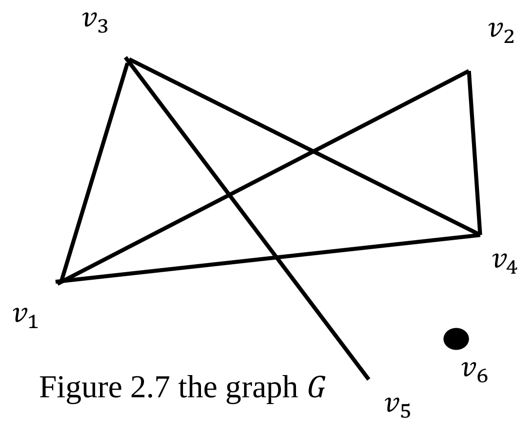

Figure 2.7 the graph G

The degree matrix of a graph G is

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	**3**	**0**	**0**	**0**	**0**	**0**	**0**
v_2	**0**	**3**	**0**	**0**	**0**	**0**	**0**
v_3	**0**	**0**	**3**	**0**	**0**	**0**	**0**
v_4	**0**	**0**	**0**	**4**	**0**	**0**	**0**
v_5	**0**	**0**	**0**	**0**	**4**	**0**	**0**
v_6	**0**	**0**	**0**	**0**	**0**	**1**	**0**
v_7	**0**	**0**	**0**	**0**	**0**	**0**	**0**

We have the following observation about the degree matrix.

a. The entries along the principle diagonal of $D(G)$ are not all zero it is equal to a number of edge incident on vertex.
b. All column and row of matrix are zero except main diagonal.

References

1) Ali Reza Ashrafi & Modjtaba Ghorbani & Maryam jalali, 4 december 2008 . Detour Matrix and Detour Index of Some Nanotubes. *Digest Jornal of Nanomaterials and Biostructures,* pp. 245-250.

2) Bapat, R. B., 2010. *Graphs and Matrices.* New Delhi: Hindustan Book Agency.

3) Biggs, N. L., 1985. *Discrete Mathematics.* United States : Oxford University Press.

4) Gary Chartrand & Linda Lesniak & Ping Zhang, 2016. *Graphs and Digraphs.* sixth edition, New York: Taylor & Francis Group.

5) Lesniak, G. C. a. L., 1996. *Graphs and digraphs.* s.l.:Chapman and Hall/CRC.

6) Ray, S. S., 2013. *Graph theory with Algorithms and Applications.* New Delhi Heidelberg New York London: s.n.

7) Vasudev, C., 2006. *Graph Theory with Applications.* New Delhi: New Age International(P) Ltd.

8) Wilson, R. J., 1972. *Introduction to Graph Theory.* United States and Canada: Academic Press, INC.

9) Zhang, G. C. a. P., 2012. *A First Course in Graph theory.* INC. Mineola, New York: s.n.

10) https://www.cmm.ki.si/~FAMNIT-knjiga/wwwANG/The_Adjancency_Matrix-5.htm

پوختە

کۆمپیوتەر بەتواناترە لە بەکارهێنانی دەست یاخود دەست بەکارهێنان هەروەها بۆ ناسینەوەی وێنەکان و ژمارە پێوانەییەکان و کرداری پەیوەندی کردن بە تایبەتمەندی هێڵکاری بە کۆمپیوتەر لەناو ڕیزکراوەکان لەم بەدواداگەڕانەدا ئێمە هەندێک جۆری ڕیزکراوەکان پەیوەندی دار دەخوێنین لەگەڵ هێڵکاری هەروەها ئێمە هەندێک تایبەتمەندی دەدۆزینەوە لەم بەتواناترە.

زانکۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

ڕیزکراوەکان لە گراف تیۆری

نامەیەکە

پێشکەش بە ئەنجومەنی کۆلێژی پەروەردەی - زانکۆی سەلاحەددین- هەولێر کراوە وەك بەشێك لەپێداویستیەکانی بەدەست هێنانی پلەی بەکالۆریۆس لە زانستی ماتماتیک

لەلایەن:

ئاسودە مخلص قادر

بەسەرپەرشتی:

م. ایڤان دلێر عەلی

گوڵان-٢٠٢٣