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Degree Sequence in Graph and Digraph

Research Project

Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of BSc. in (Mathematics)

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Certification of the Supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Date: 6 / 4 / 2023

In view of the available recommendations, I forward this work for debate by the examining committee.

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Date: 6 / 4 / 2023

Acknowledgment

To

- ALLAH who gave me everything.
- My supervisor,
- To Salahaddin University College of Education Mathematics Department.
- All the staff of the department
- The noble spirits of my Parents.
- My sisters and brothers

Zhyan Muhammad amin

CONTENTS

Abstract

The degree of a vertex v in a graph G is the number of edges incident with it and denoted by $\deg v$. In this work, we investigate the concept of degree in graphs in more detail. A sequence $d_1, d_2, \dots, d_p$ of a nonnegative integer is called a degree sequence of a simple graph G if the vertices of G can labeled $v_1, v_2, \dots, v_p$ so that $\deg v_i = d_i$ for all i.

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Introduction

In graph theory, the degree (or valency) of a vertex of a graph is the number of edges that are incident to the vertex; in a multigraph, a loop contributes 2 to a vertex is degree, for the two ends of the edge. The degree of a vertex v is denoted $deg(v)$ or $degv$. The maximum degree of a graph G. denoted by $\Delta(G)$, and the minmum degree of a graph, denoted by $\delta(G)$. The degree sequence of an undirected graph is the non-increasing sequence of it is vertex degrees. The degree sequence is a graph invariant, so isomorphic graphs have the same degree sequence. However, the degree sequence does not, in general, uniquely identify a graph; in same degree sequence.

The degree sequence problem is the problem of finding some or all graphs with the degree sequence being a given non-increasing sequence of positive integers. (Trailing zeroes may be ignored since they are trivially realized by adding an appropriate number of isolated vertices to the graph.) A sequence which is the degree sequence of some graph, i.e. for which the degree sequence problem has a solution, is called a graphic or graphical sequence. As a consequence of the degree sum formula, any sequence with an odd sum, such as $(3,3,1)$, con not be realized as the degree sequence of a graph. The inverse is also true: if a sequence has an even sum, it is the degree sequence of a multigraph.

Our work consists of two chapters:

Chapter one we present fundamental concepts of graphs.

Chapter two consists of two sections in section one we present degree sequence of graph and some theorems and in section two we present degree sequence of digraph and some theorems.

Chapter One

Basic Notations and Definitions

Definition 1.1[(ray, 2013)**]:**

A **Graph** $G = (V(G), E(G))$ or $G = (V, E)$ consists of two finite sets $V(G)$ or V the vertex set of the graph ,which is a non-empty set of elements called vertices and $E(G)$ or E ,the edge set of graph, which is a possibly empty set of elements called edges.

Definition 1.2[(ray, 2013)**]:**

Order and size we define $|V|$=n to be the order of G and $|E|$=m to be the size of G.

Definition 1.3[(Taylor&Francis group, 2016)**]:**

Degree of the vertex V in a graph G is the number of vertices in G that are adjacent to V.

Definition 1.4[(ray, 2013)**]:**

The definition of a graph allows the possibility of the edge E having identical end vertices such an edge having the same vertex as both of it end vertices is called a **self-loop**.

Definition 1.5[(ray, 2013)**]:**

A graph that has neither self-loops nor parallel edge is called a **simple graph.**

Definition 1.6[(ray, 2013)**]:**

A multi graph G is an ordered pair $G = (V, E)$ with V a set of vertices or nodes and E a multi set of unordered pair of vertices called edges.

Definition 1.7[(ray, 2013)**]:**

Let H be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly let G be a graph with vertex set V(G) and edge set E(H), then we say that H is sub graph of G if $V(H)$ $V(G)$ and $E(H)$ $E(G)$.

Definition 1.8[(ray, 2013)**]:**

A **complete graph** is a simple graph in which each pair of distinct vertices is joined by an edge.

Definition 1.9[(ray, 2013)**]:**

A graph G is called **connected** if every two of its vertices are connected.

Definition 1.10[(C.Vasudev, 2006)**]:**

A **walk** in a graph G is a finite sequence $W \equiv v_0\, e_1 v_1 e_2 \ldots v_{k-1} e_k v_k$

whose term are alternately vertices and edges such that for $1 \leq i \leq k$, the edge ei (graph and digraph six edition , 1972) (Taylor&Francis group, 2016)has end $v_{i-1}$and v_i.

Definition 1.11[(introduition of graph theory, 1972)**]:**

A **digraph** D is defined to be a pair $(V(D), A(D))$.where $V(D)$ is non-empty finite set of elements called vertices. And $A(D)$ is a finite family of ordered pairs of elements of $V(D)$ called **arces**, $V(D)$ and $A(D)$ are the vertex set and arc-family of D.

Definition 1.12[(graph and digraph in third edition , 1996)**]:**

The cardinality of the vertex set of a digraph D is called the order of D and is denoted by $n(D)$,or simply n. the size $m(D)$or m of D is the cardinality of its arc set. An (n, m) digraph is a digraph of order n and size m.

The outer degree od V of a vertex V of a digraph D is the number of vertices of D that are adjacent from V. the in degree id V of V is the number of vertices of D adjacent to V, the degree $degV$ of a vertex V of D is defined by

$Degv = od\ V \neq\ idV$.

Chapter Two

2.1 Degree sequence in graph[(Wadsworth, 1986)]:

In this section, we investigate the concept of degree in graphs and digraphs in more detail.

A sequence $d_1, d_2, \dots, d_p$ of nonnegative integer is called a degree sequence of a graph G can be labeled $v_1, v_2, \dots, v_p$ so that $degv_i = d_i$ for $all\ \ i$. Let $s\colon d_1, d_2, \dots, d_p$ a sequence of a nonnegative integer, if the graph of s is called a graphical sequence. Certainly the conditions $d_i \leq p-1$ for all i and $\sum_{i=1}^{p} d_i$ is even are necessary for a sequence to be graphical but these conditions are not sufficient. The sequence 4,4,4,1 is not graphical for example. Two graphs with the same degree sequence are said to be degree equivalent.

Example 2.1.1 A degree sequence of the graph G of the Figure 2.1 is 3,3,4,4,3,3,1,1 or 4,4,3,3,3,1,1 or 1,1,3,3,3,3,4,4 where $p = 8$ and $d_i \leq p-1$

$\sum_{i=1}^{p} d_i = \sum_{i=1}^{8} d_i = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 = 4+4+3+3+3+3+1+1 = 22$ is even and the graph of the degree sequence of graph G is graphical.

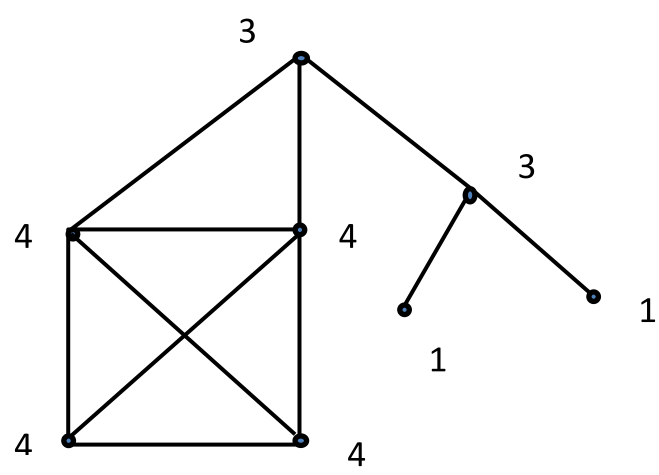
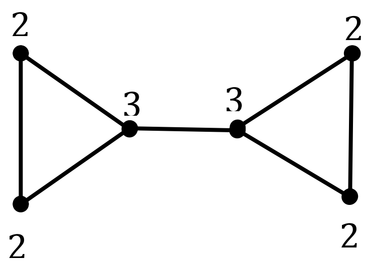
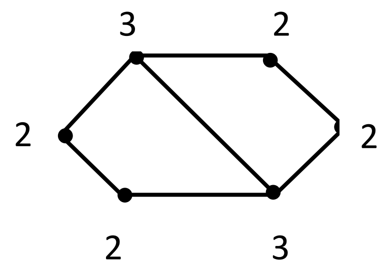


Figure 2.1.1 Graph G

Example 2.1.2. The Figures 2.2 shows that two different graphs G and H, the two graph G and H are equivalent because have the same degree sequence 3,3,2,2,2,2



The graph G



The graph H

Figure 2.1.2

A necessary and sufficient condition for a sequence to be graphical was found by Havel and later rediscovered by Hakimi.

Theorem 2.1.3 (Havel-Hakimi Theorem) **)[** (Wadsworth, 1986)**] :**

A sequence $s: d_1, d_2, \ldots, d_p$ of a nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_p$, $p \geq 2$, $d_1 \geq 1$ is graphical if and only if the sequence $s_1: d_2 - 1, d_3 - 1, \ldots$, $d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$ is graphical.

With the aid of theorem 2.1, we may now present an algorithm that allows us to determine whether a finite sequence of nonnegative integers is graphical.

Algorithm (Havel 1955 – Hakimi 1962) **[** (Wadsworth, 1986)**]:** Given a sequence of $p(\geq 1)$ nonnegative integers:

1. If some integer in the sequence exceeds $p - 1$, then the sequence is not graphical. Otherwise, continue to step 2.

2. If all integers in the sequence are 0, then the sequence is graphical. If the sequence continues a negative integer, then the sequence is not graphical. Otherwise continue to step 3.
3. Reorder the numbers in the current sequence, if necessary, so that it is a non increasing sequence.
4. Delete the first number, say n , from the sequence , and subtract 1 from the next n numbers in the sequence. Return to step 2.

We illustrate the algorithm of Havel-Hakimi by the following examples.

Example 2.1.4. Is the degree sequence $S = 4,4,3,2,1$ graphical?

Solution: since $p = 5$ and $d_1 = 4,\ d_2 = 4,\ d_3 = 3,\ d_4 = 2, and\ d_5 = 1$

1. $d_1 \leq p - 1$ then $4 \leq 5 - 1$
2. $\sum_{i=1}^{p} d_i = \sum_{i=1}^{5} d_i = d_1 + d_2 + d_3 + d_4 + d_5 = 4 + 4 + 3 + 2 + 1 = 14$

Is even. Now by Havel-Hakimi we have

$s = 4,4,3,2,1$

$s_1 = 4 - 1,3 - 1,2 - 1,1 - 1$

$s_1 = 3,2,1,0$

$s_2 = 2 - 1,1 - 1,0 - 1$

$s_2 = 1,0,-1$

Since the sequence s_2 contains a negative integer, then is not graphical.

Example2.1.5. Construct a graph with a degree sequence $s = 1,3,4,2,4,2,4$ by using Havel-Hakimi algorithm.

Solution:

$s = 4,4,4,3,2,2,1$

1. $d_1 \leq p - 1$ then $4 \leq 7 - 1$
   $4 \leq 6$
2. $\sum_{i=1}^{p} d_i = \sum_{i=1}^{7} d_i = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$
   $= \sum_{i=1}^{7} d_i = 4 + 4 + 4 + 3 + 2 + 2 + 1$
   $= \sum_{i=1}^{7} d_i = 20$ is even

Now by Havel-Hakimi Theorem we have:

$s = 4,4,4,3,2,2,1$

$s_1 = 4 - 1,4 - 1,3 - 1,2 - 1,2,1$

$s_1 = 3,3,2,1,2,1$

$s_1 = 3,3,2,2,1,1$

$s_2 = 3 - 1,2 - 1,2 - 1,1,1$

$s_2 = 2,1,1,1,1$

$s_3 = 1 - 1,1 - 1,1,1$

$s_3 = 0,0,1,1$

$s_3 = 1,1,0,0$

$s_4 = 1 - 1,0,0$

$s_4 = 0,0,0$

Since all integers in the sequence are 0, then the sequence is graphical.

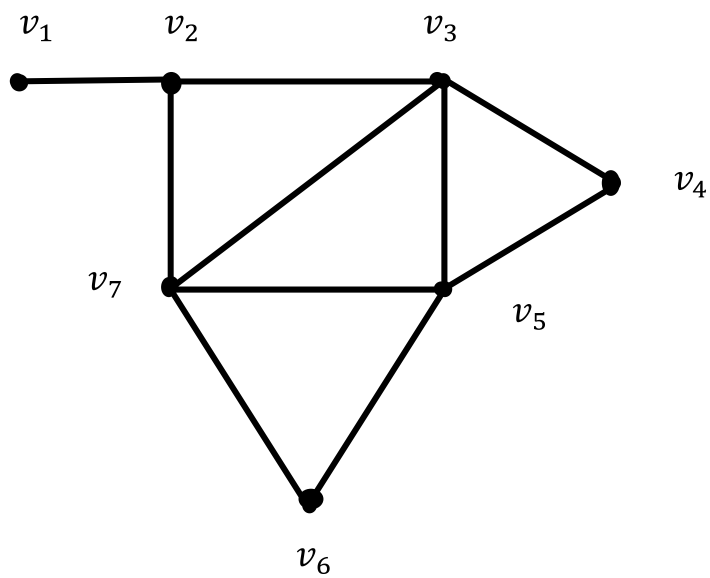


Figure 2.1.3 The graph of the degree sequence S

Another result that determines which sequence are graphical comes from Erdos and Gallai [EG2].

Theorem 2.1.6 (Erdos and Gallai)[(Wadsworth, 1986)]:

A sequence $d_1, d_2, \dots, d_p (p \geq 2)$ of a nonnegative integers with $d_1 \geq d_2 \geq, \dots, \geq d_p$, is graphical if and only if $\sum_{i=1}^{p} d_i$ is even and for each integer n.

$$1 \leq n \leq p - 1$$

$$\sum_{i=1}^{n} d_i \leq n(n-1) + \sum_{i=n+1}^{p} \min\{n, d_i\}$$

Example 2.1.7. is the degree sequence $D = 5,4,3,3,2,2,1$ graphical?

Solution: Since $p = 7$ and

$d_1 = 5, d_2 = 4, d_3 = 3, d_4 = 3, d_5 = 2, \; d_6 = 2, d_7 = 1$

1. $d_1 \leq p - 1$ then $5 \leq 7 - 1$
2. $\sum_{i=1}^{p} d_i = \sum_{i=1}^{7} d_i = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$

   $= \sum_{i=1}^{7} d_i = 5 + 4 + 3 + 3 + 2 + 2 + 1$

   $= 20$ is even

Now, by Erdos and Gallai theorem we have $1 \leq n \leq p - 1$ then $1 \leq n \leq 7 - 1$ for each integer number n

$$\sum_{i=1}^{n} d_i \leq n(n-1) + \sum_{i=n+1}^{7} \min\{n, d_i\}$$

Now when $n = 1$ then $\sum_{i=1}^{1} d_i \leq 1(1-1) + \sum_{i=1+1}^{7} \min\{1, d_i\}$

$d_1 \leq \min\{1, d_2\} + \min\{1, d_3\} + \min\{1, d_4\} + \min\{1, d_5\} + \min\{1, d_6\} + \min\{1, d_7\}$

$5 \leq \min\{1,4\} + \min\{1,3\} + \min\{1,3\} + \min\{1,2\} + \min\{1,2\} + \min\{1,1\}$

$5 \leq 1 + 1 + 1 + 1 + 1 + 1$

$5 \leq 6$

When $n = 2$ then $\sum_{i=1}^{2} d_i \leq 2(2-1) + \sum_{i=2+1}^{7} \min\{2, d_i\}$

$d_1 + d_2 \leq 2 + \min\{2, d_3\} + \min\{2, d_4\} + \min\{2, d_5\} + \min\{2, d_6\} + \min\{2, d_7\}$

$5 + 4 \leq 2 + \min\{2,3\} + \min\{2,3\} + \min\{2,2\} + \min\{2,2\} + \min\{2,1\}$

$9 \leq 2 + 2 + 2 + 2 + 2 + 1$

$9 \leq 11$

When $n = 3$ then $\sum_{i=1}^{3} d_i \leq 3(3-1) + \sum_{i=3+1}^{7} \min\{3, d_i\}$

$d_1 + d_2 + d_3 \leq 6 + \min\{3, d_4\} + \min\{3, d_5\} + \min\{3, d_6\} + \min\{3, d_7\}$

$5 + 4 + 3 \leq 6 + \min\{3,3\} + \min\{3,2\} + \min\{3,2\} + \min\{3,1\}$

$12 \leq 6 + 3 + 2 + 2 + 1$

$12 \leq 14$

When $n = 4$ then $\sum_{i=1}^{4} d_i \leq 4(4-1) + \sum_{i=4+1}^{7} \min\{4, d_i\}$

$d_1 + d_2 + d_3 + d_4 \leq 12 + \min\{4, d_5\} + \min\{4, d_6\} + \min\{4, d_7\}$

$5 + 4 + 3 + 3 \leq 12 + \min\{4,2\} + \min\{4,2\} + \min\{4,1\}$

$15 \leq 12 + 2 + 2 + 1$

$15 \leq 17$

When $n = 5$ then $\sum_{i=1}^{5} d_i \leq 5(5-1) + \sum_{i=5+1}^{7} \min\{5, d_i\}$

$d_1 + d_2 + d_3 + d_4 + d_5 \leq 20 + \min\{5, d_6\} + \min\{5, d_7\}$

$5 + 4 + 3 + 3 + 2 \leq 20 + \min\{5,2\} + \min\{5,1\}$

$17 \leq 20 + 2 + 1$

$17 \leq 23$

When $n = 6$ then $\sum_{i=1}^{6} d_i \leq 6(6-1) + \sum_{i=6+1}^{7} \min\{6, d_i\}$

$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \leq 30 + \min\{6, d_7\}$

$5 + 4 + 3 + 3 + 2 + 2 \leq 30 + \min\{6,1\}$

$19 \leq 30 + 1$

$19 \leq 31$ Therefore the degree sequence is graphical.

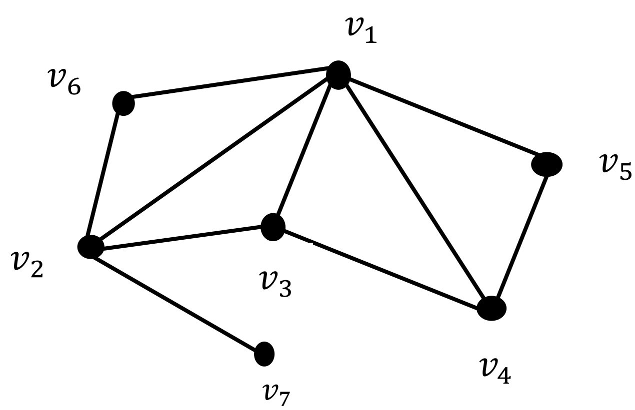


Figure 2.1.4 the graph of degree sequence D

Application 2.1.8 Sociological – Acquaintance Networks: In an acquaintance network, the vertices represent persons, such as the students in a college class. An edge joining two vertices indicates that the corresponding pair of persons knew each other when the course began. The simple graph in Figure 2.1.8 shows a typical acquaintance network. Including the Socratic concept of self – knowledge would require the model to allow self – loops. For instance, a self – loop drawn at the vertex representing Slim might mean that she was " in touch " with herself .

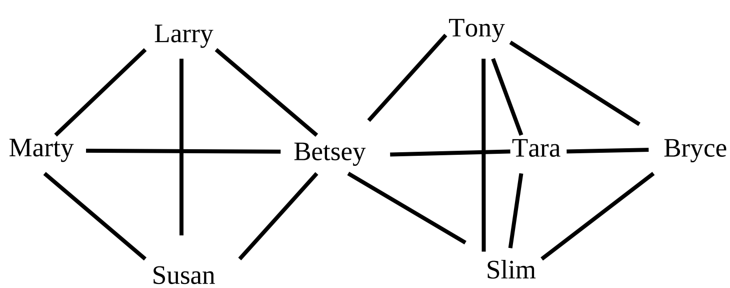


Figure 2.1.8 An acquaintance network

Every group of two or more persons must contain at least two who know the same number of persons in the group. The acquaintance network of Figure 2.1.8 has degree sequence $3,3,3,3,4,4,4,6$.

2.2. Degree sequence in digraph [(Wadsworth, 1986)**]:**

In this section, we investigate the concept of degree in graphs and digraphs in more detail.

When considering degree sequences for digraphs, it is necessary to account for both the outdegree and indegree of each vertex. A sequence $(s_1,t_1),(s_2,t_2),\ldots,(s_p,t_p)$ of ordered pairs of nonnegative integers is called a degree sequence of a digraph D if the vertices of D can be labeled $v_1,v_2,\ldots,v_p$ so that $od\ v_i=s_i$ and $id\ v_i=t_i$ all i. A sequence $(s_1,t_1),(s_2,t_2),\ldots,(s_p,t_p)$ of ordered pairs of nonnegative integers is called a digraphical sequence if it is a degree sequence of some digraph. Clearly if the sequence $(s_1,t_1),(s_2,t_2),\ldots,(s_p,t_p)$ is digraphical, then $\sum_{i=1}^{p} s_i=\sum_{i=1}^{p} t_i$. And we have $s_i\le p-1$ and $t_i\le p-1$ for all i.that these condition are not sufficient for a sequence to be digraphical is illustrated by the sequence $(1,1),(0,0)$. However, necessary and sufficient conditions for a sequence to be digraphical were discovered independently by Fulkerson and Ryser.

Theorem 2.2.1 (Fulkerson-Ryser)[(Wadsworth, 1986)**]:**

A sequence $(s_1,t_1),(s_2,t_2),\ldots,(s_p,t_p)$ of ordered pairs of nonnegative integers with $s_1\ge s_2\ge\cdots\ge s_p$ is digraphical if and only if

a. $s_i\le p-1\ and\ t_i\le p-1$ for $1\le i\le p$

b. $\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} t_i$ and

c. $\sum_{i=1}^{n} s_i \leq \sum_{i=1}^{n} \min\{n-1, t_i\} + \sum_{i=n+1}^{p} \min\{n, t_i\} \quad for \quad 1 \leq n \leq p$

Example 2.2.2. Construct a digraph with a degree sequence D is $(2,1), (2,0), (1,2), (0,2)$ by using Fulkerson-Ryser theorem.

Solution: The degree sequence of a digraph D is $(2,1), (2,0), (1,2), (0,2)$ where

$p = 4$ and $s_1 = 2, s_2 = 2, s_3 = 1, s_4 = 0$ and $t_1 = 1, t_2 = 0, t_3 = 2, t_4 = 2$ then the sequence is graphical if and only if

1- $s_i \leq 4 - 1$ and $t_i \leq 4 - 1$ for $1 \leq i \leq 4$

2- $\sum_{i=1}^{p} s_i = \sum_{i=1}^{4} s_i = s_1 + s_2 + s_3 + s_4 = 2 + 2 + 1 + 0 = 5$ and

$\sum_{i=1}^{p} t_i = \sum_{i=1}^{4} t_i = t_1 + t_2 + t_3 + t_4 = 1 + 0 + 2 + 2 = 5$ then $\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} t_i$.

3-$\sum_{i=1}^{n} s_i \leq \sum_{i=1}^{n} \min\{n-1, t_1\} + \sum_{i=n+1}^{p} \min\{n, t_i\}$

When $n = 1$

$\sum_{i=1}^{n} s_i \leq \sum_{i=1}^{1} \min\{1-1, t_i\} + \sum_{i=1+1}^{4} \min\{1, t_i\}$

$s_1 \leq \min\{0, t_1\} + \min\{1, t_2\} + \min\{1, t_3\} + \min\{1, t_4\}$

$2 \leq \min\{0,1\} + \min\{1,0\} + \min\{1,2\} + \min\{1,2\}$

$2 \leq 0 + 0 + 1 + 1$

$2 \leq 2$

When $n = 2$

$\sum_{i=1}^{2} s_i \leq \sum_{i=1}^{2} \min\{2-1, t_i\} + \sum_{i=3}^{4} \min\{2, t_i\}$

$s_1 + s_2 \leq \min\{1, t_1\} + \min\{2, t_2\} + \min\{2, t_3\} + \min\{2, t_4\}$

$2 + 2 \leq \min\{1,1\} + \min\{2,0\} + \min\{2,2\} + \min\{2,2\}$

$4 \leq 1 + 0 + 2 + 2$

$4 \leq 5$

When $n = 3$

$\sum_{i=1}^{3} s_i \leq \sum_{i=1}^{3} \min\{3, t_i\} + \sum_{i=4}^{4} \min\{3, t_i\}$

$s_1 + s_2 + s_3 \leq \min\{2, t_1\} + \min\{3, t_2\} + \min\{3, t_3\} + \min\{3, t_4\}$

$2 + 2 + 1 \leq \min\{2,1\} + \min\{3,0\} + \min\{3,2\} + \min\{3,2\}$

$5 \leq 1 + 0 + 2 + 2$

$5 \leq 5$ Therefore the degree sequence is digraphical.

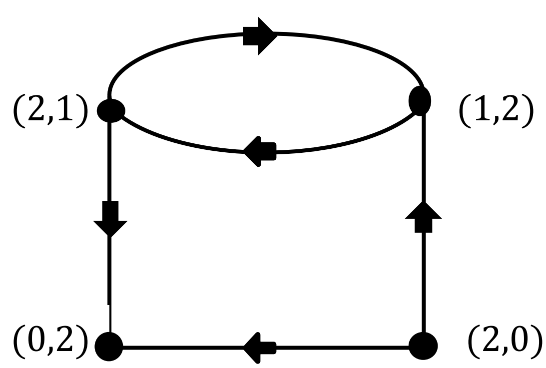


Figure 2.2.1 A digraph *D*

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کورتە

دووری$d(u,v)$ بۆ دوو u,v لە گرافی بەیەک گەینراو G بریتیە لەکورترین رێڕەو P لە نێوان دوو خاڵی (u,v) . لەم پرۆژەدا بیروکە یان چەمکی (پلە) لە گراف دەپشکنین وە بە شێوەیەکی درێز باسی دەکەین. یەك بە دوای یەکی $d_1,d_2,\dots,d_p$ زمارەی تەواوی نا سالب پێ دەوترێت یەك بە دوای یەکی پلەی گرافی سادە ئەگەر توانیمان سەرەکانی G بە $v_1,v_2,\dots,v_p$ دابنێین کاتێك $degv_1 = d_i$ بۆ هەموو i .

زانكۆی سەلاحەددین هەولێر

یەك بەدوای یەكی پلەیی بو گراف و گرافی ئاراستەكراو

نامەیەكە

پێشكەش بە ئە نجومەنی كۆلێژی پەروەردەی - زانكۆی سەلاحەددین- هەولێر كراوە وەك بەشێك لە پێداویستیەكانی بەدەست هێنانی پلەی بكالۆریۆس لە زانستی ماتماتیك (ماتماتیك)

لەلایەن:

یان محمد أمین دەروێش

بەسەرپەرشتی:

مامۆستا ایڤان دلێر علی

گولان 2023