زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Binary Operations on Graph Theory

Research project

Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of BSc. in (Graph Theory)

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Scientific grade: Lecturer

Date: / 4 / 2023

In view of the available recommendations, I forward this report for debate by the examining committee.

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Date: / 4 / 2022

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Acknowledgment

To

- My God who gave me everything.
- My supervisor, Lecturer Ivan
- To Salahaddin University College of Education Mathematics Department.
- All the staff of the department
- The noble spirits of my Parents.

Razhan Osman raza

Abstract

There are many ways of combining graphs to produce new graphs. In this work some operation containing, union, join, some kind of product, disjunction, and symmetric difference of graph will be present.

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Introduction

Graph theory is a branch of discrete mathematics. Graph theory is the study of graphs which are mathematical structures used to model pair wise relation between objects. A graph is made up of vertices V (nodes) and edges E (lines) that connect them. A graph is an ordered pair $G = (V, E)$ consisting a set of vertices V with a set of edges E.

Graph theory is originated with the problem of Koinsberg Bridge in 1735. This problem escort to the concept of Eulerian Graph. Euler studied the problem Koinsberg Bridge and established a structure to resolve the problem called Eulerian graph.

The operation of adding and deleting vertices and edges of graphs are regarded as primary operations, because they are the foundation for other operations, which many be called secondary operations.

Our work consists of two chapters:

Chapter one we present fundamental concepts of graphs.

Chapter two consists of two sections in section one we present some type of operations and in section two we present some properties of these operations.

Chapter One

Some basic concept in graph theory

Definition 1.1[(Chartrand, Lesniak, & Zhang, 2016)**]**A graph G consists of a finite nonempty set V of objects called vertices (the singular is vertex) and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G, respectively. So a graph G is a pair (actually an ordered pair) of two sets V and E. For this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. Although G is the common symbol to use for a graph, we also use F and H, as well as G', G'' and G_1, G_2, etc. Vertices are sometimes called points or nodes and edges are sometimes called lines.

Definition1.2[(Chartrand, Lesniak, & Zhang, 2016)**]**If uv is an edge of G, then u and v are said to be adjacent in G.

Definition 1.3[(Chartrand, Lesniak, & Zhang, 2016)**]** The vertex u and the edge uv are said to be incident with each to other.

Definition 1.4[(Chartrand, Lesniak, & Zhang, 2016)**]**The number of vertices in G is often called the order of G, while the number of edges is its size. Since the vertex set of every graph is nonempty, the order of every graph is at least 1.

Definition 1.5[(Chartrand & Zhang, 2012)**]**An edge having the same vertex as both of its end vertices is called a self-loop (or simply a loop).

Definition1.6[(Chartrand & Zhang, 2012)] Multigraph M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges (possibly zero). If two or more edges join the same pair of (distinct) vertices, then these edges are called parallel edges.

Definition1.7[(Chartrand & Zhang, 2012)**]** A graph, that has neither self-loops nor parallel edges, is called a simple graph.

Definition1.8[(Chartrand & Zhang, 2012)**]**A graph with a finite number of vertices as well as finite number of edges is called a finite graph; otherwise it is an infinite graph.

Definition1.9[(Chartrand, Lesniak, & Zhang, 2016)**]**The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $deg_G\ v$ or simply by deg v.

Definition 1.10[(Chartrand, Lesniak, & Zhang, 2016)**]**A graph H is called a subgraph of a graph G, written $H \subseteq G$,
if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.We also say that G contains H as a subgraph.

Definition 1.11[(Chartrand & Zhang, 2012)**]** $u - v$ walk W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W = (u = v_1, v_2, \ldots, v_k = v)$where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, \ldots, k - 1$

Definition 1.12[(Chartrand, Lesniak, & Zhang, 2016)**]**A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ path.

Definition 1.13[(Chartrand & Zhang, 2012)**]**If G contains a $u - v$ path, then u and v are said to be connected and u is connected to v (and v is connected to u).

Definition 1.14[(Chartrand, Lesniak, & Zhang, 2016)**]**A graph G is complete if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.

Definition 1.15[(Chartrand, Lesniak, & Zhang, 2016)**]**A graph G is bipartite if $V(G)$ can be partitioned into two sets U and W (called partite sets) so that every edges of G joins a vertex of U and a vertex of W.

Definition 1.16[(Chartrand, Lesniak, & Zhang, 2016)**]** A nontrivial closed path is called a cycle.

Definition 1.17[(Ivan & Herish , 2017)**]** wheel w_n for $n \geq 4$, is a graph of order n consisting of a cycle c_{n-1} together with a vertex adjacent to every vertex of c_{n-1}.

Chapter Two

Binary operation

2.1.Some Common Binary Operation in Graph Theory

In this section we describe some common binary operations defined in graph theory. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

1:union (C.vasudev, 2006)

The union $G = G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

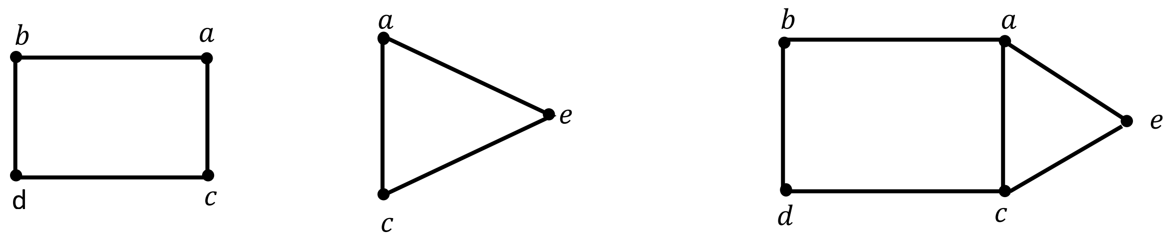


Figure 2.1. The Union of graphs

2 Join[(C.vasudev, 2006)**]**

If the graph G_1 and G_2 such that $V(G_1) \cup V(G_2) = \emptyset$, then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting those edges, which are in G_1 *and in* G_2 and the edges obtained by joining each vertex of G_1 *to each vertex of* G_2. That's mean

$V(G) = V(G_1) + V(G_2)$

$E(G) = E(G_1) + E(G_2) \cup \{uv; u \in V(G_1, v \in V(G_2)\}$

Example 2.2. The join of two graphs $k_1 + C_6 = W_7$ when k_1 is complete graph, C_6 is a cycle and W_7 is a wheel.

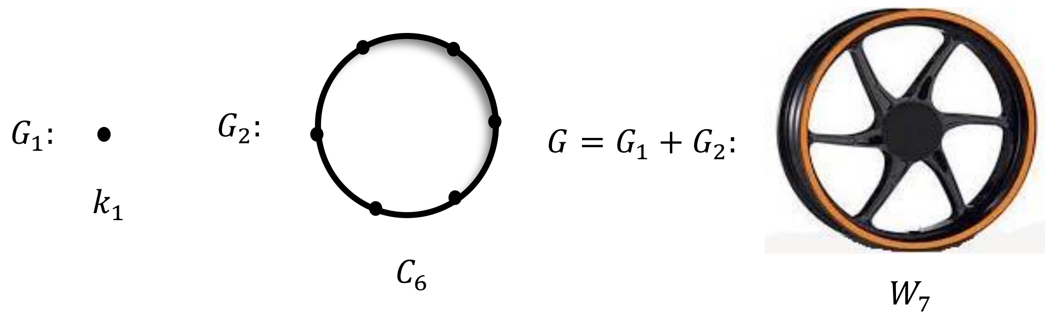


Figure 2.2. The join of graphs

Example2.3. The join of k_1 and P_n is F_n Where $k_1$is complete with order 1, P_n is path with order n and F_n is Fan graph with order n.

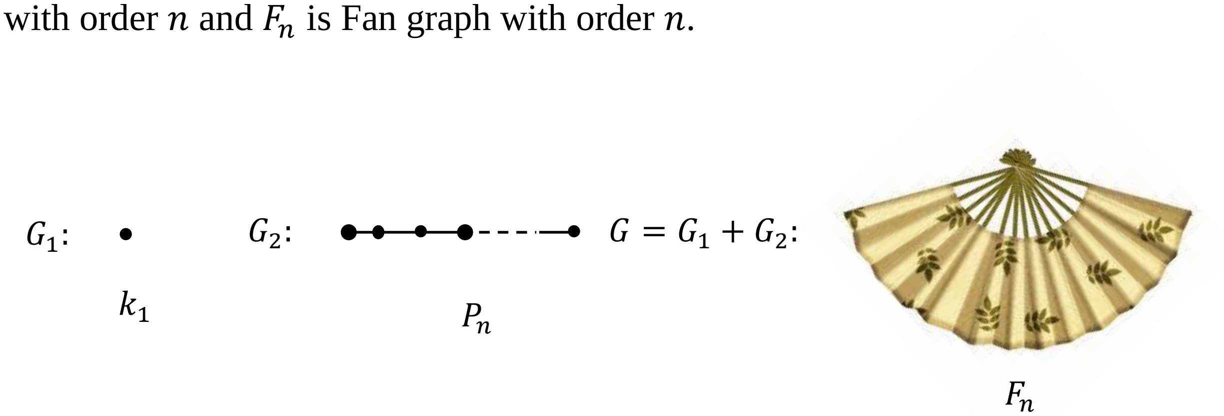


Figure 2.3. The join of graphs

3 Box product[(C.vasudev, 2006)**]**

$G = G_1 \blacksquare G_2$

$V(G) = V(G_1) \times V(G_2)$

$E(G)\ if\ (u_1, u_2)\ adjacent\ (v_1, v_2) then\ u_1 = v_1\ and\ \ u_2\ adjacent\ v_2 \in E(G_2)\ OR\ u_2 = v_2\ \ and\ u_1\ adjacent\ \ v_1 \in E(G_1)$

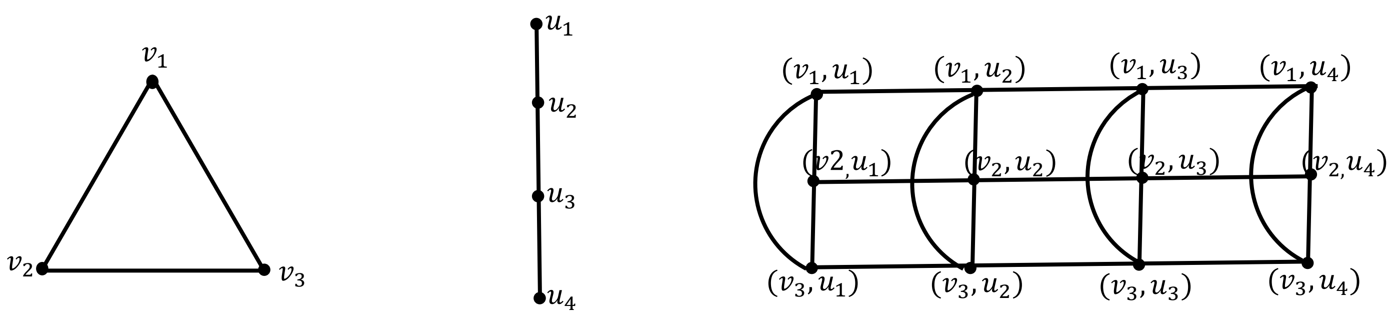


Figure 2.4. The box product of graphs

2. The box product of k_2 and P_n graph is a ladder graph L_n :

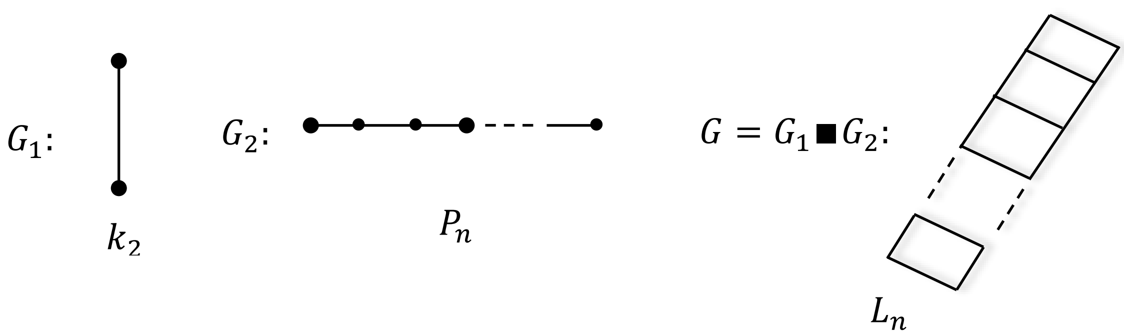


Figure 2.5. The box product of graphs

4. Tensor Product[(Kiran , 2017)**]**

The tensor product $G_1 \times G_2$ of graph G_1 and G_2 is a graph such that

1.The vertex set of $G_1 \times G_2$ is the box product $V(G_1) \times V(G_2)$.

2. Distance vertices $(u, \acute{u})$ and $(v, \acute{v})$ are adjacent in $G_1 \times G_2$ if and only if

- u is adjacent to v and
- $\acute{u}$ is adjacent to $\acute{v}$

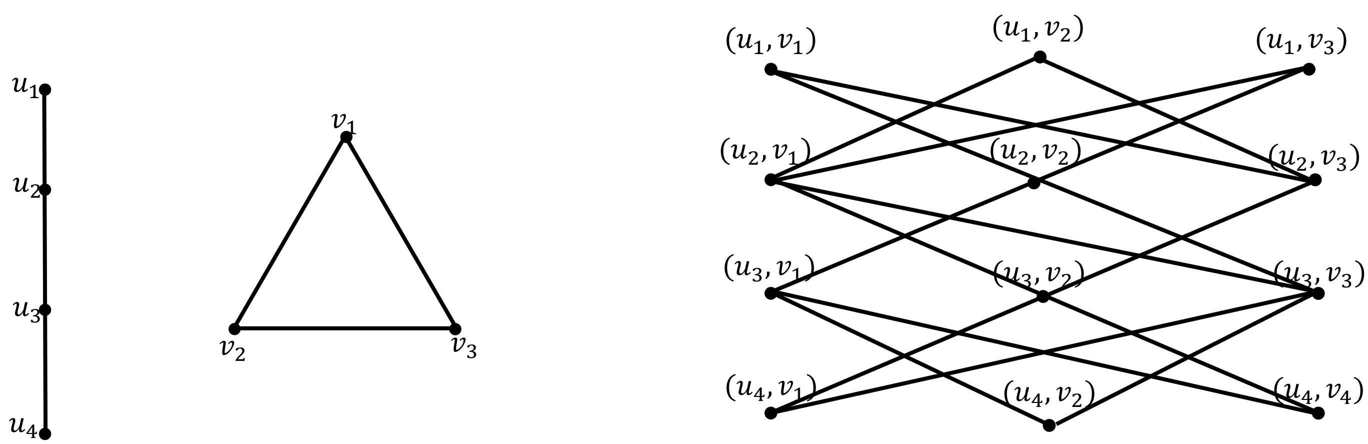


Figure 2.6. The tensor product of graphs

The tensor product is also called

- Direct product
- Categorical product
- Relational product
- Cardinal product
- Kronecker product

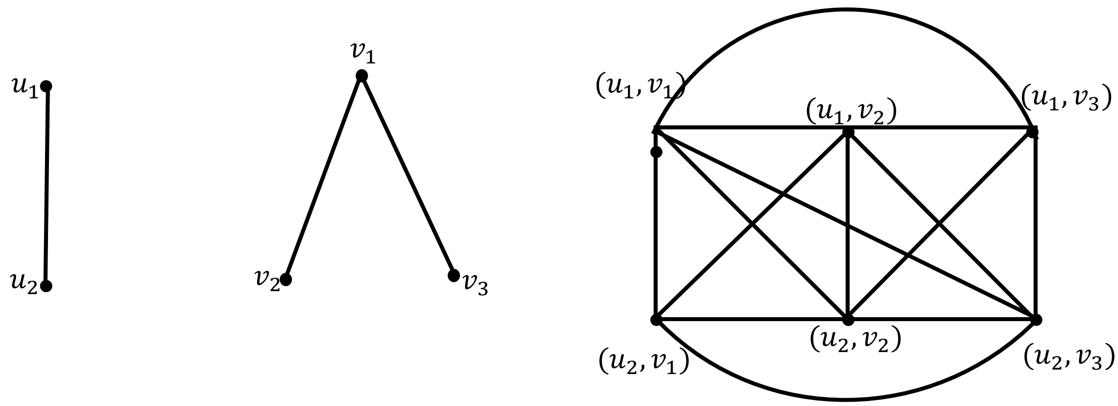
5.The disjjuction [(Chartrand, Lesniak, & Zhang, 2016)**]**

$G = G_1 \vee G_2$

$\mathrm{V(G)} = \mathrm{V}(G_1) \times V(G_2)$

$\mathrm{E(G)} = u_1u_2 \in E(G_1)$

Or $v_1v_2 \in E(G_2)$



6. Strong Product[(Kiran , 2017)**]**

The strong product $G = G_1 \otimes G_2$of graphs G_1 and G_2 is a graphs such that

1. The vertex set of $G_1 \otimes G_2$ is the box product $V(G_1) \times V(G_2)$.

2. Distance vertices $(u, \acute{u})$ and $(v, \acute{v})$ are adjacent in $G_1 \otimes G_2$ if and only if

- $u = v$ and $\acute{u}$ is adjacent to $\acute{v}$ or
- $\acute{u} = \acute{v}$ and u is adjacent to v or
- u is adjacent to v and $\acute{u}$ is adjacent to $\acute{v}$.

It is the union of box product and tensor product.

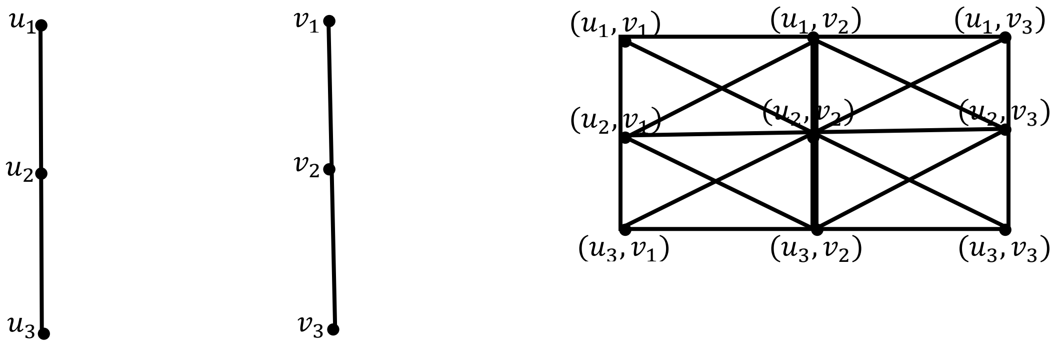


Figure 2.7. The strong product of graphs

The strong product is also called

- Normal product
- AND product

6. Composition Product[(Kiran , 2017)**]**

The composition product $G = G_1[G_2]$ of a graphs G_1 and G_2 with disjoint vertex sets V_1 and $V_2$and edge set E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and

$u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2)

Or

($u_1 = u_2$ and v_1 is adjacent with v_2)

The composition product, also known as the lexicographic product.

Example 6.1. The composition of two graphs is

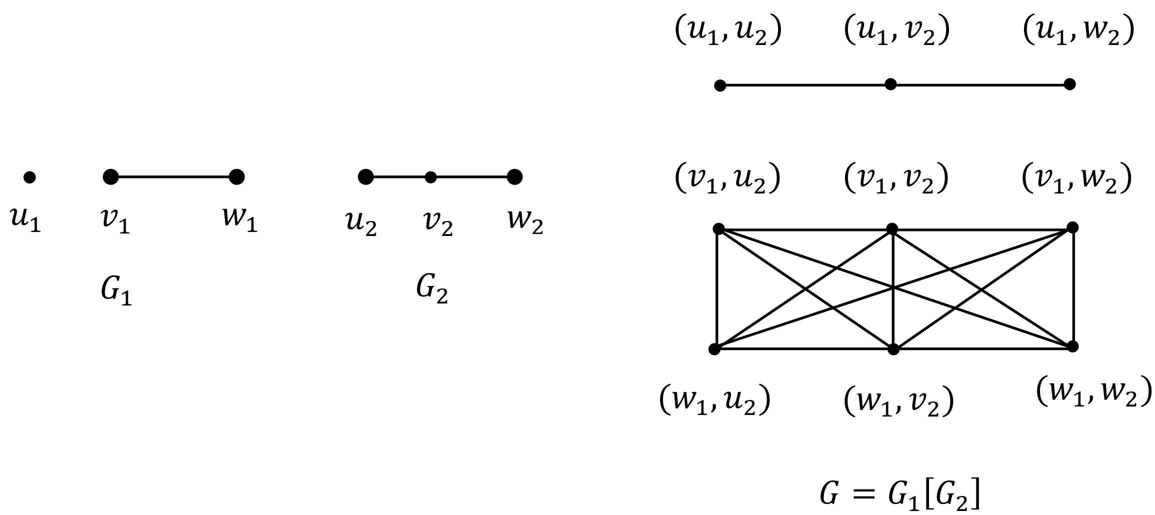


Figure 2.8. The composition product of graphs

8. The Symmetric Difference[(Kiran , 2017)**]**

The symmetric difference $G = G_1 \oplus G_2$of two graphs G_1 and G_2 is the graphs with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set

$E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) | u_1 v_1 \in E(G_1)$or $u_2 v_2 \in E(G_2)$ but not both}.

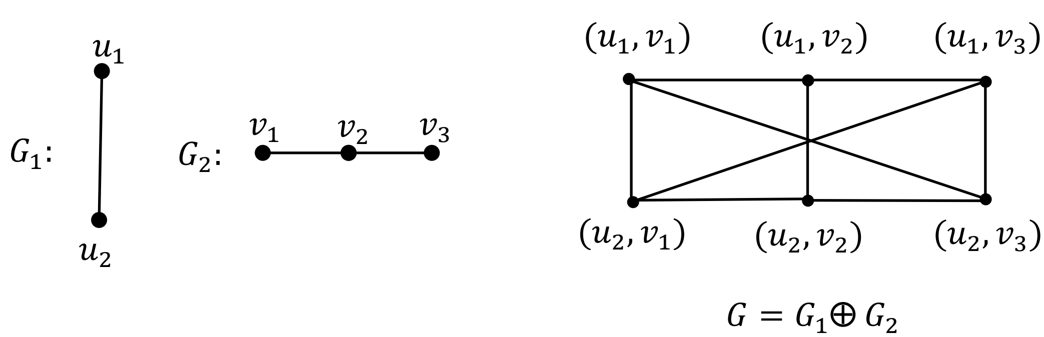


Figure 2.9. The symmetric difference of gra

2.2 Some Properties of Operation in Graph Theory[]

In this section we present some properties of the following graph operations:

(a) Let p_1 and p_2 are the orders and q_1 and q_2 are the sizes of G_1 and G_2 respectively then the order and size of the operations as the following:

1. Union

- $V(G_1 \cup G_2 = |V(G_1)| + |V(G_2)| = p_1 + p_2$
- E $(G_1 \cup G_2) = |E(G_1)| + |E(G_2)| = q_1 + q_2$

2. Join

- $V(G_1 + G_2) = |V(G_1)| + |V(G_2)| = p_1 + p_2$
- $E(G_1 + G_2) = |E(G_1)| + |E(G_2)| + |V(G_1)||V(G_2)| = q_1 + q_2 + p_1p_2$

3. Box Product

- $V(G_1 \blacksquare G_2) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1 \blacksquare G_2) = |E(G_1)| + |V(G_2)| + |V(G_1)||E(G_2)| = q_1p_2 + p_1q_2$

4. Tensor Product

- $V(G_1 \times G_2) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1 \times G_2) = 2|E(G_1)||E(G_2)| = 2q_1q_2$

5- Strong Product

- $V(G_1 \otimes G_2) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1 \otimes G_2) = |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)| + 2|E(G_1)||E(G_2)|$

$$= p_1 q_2 + p_2 q_1 + 2q_1 q_2$$

6- Composition Product

- $V(G_1[G_2]\,) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1[G_2]) = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)| = q_1 {p_2}^2 + q_2 p_1$

7- Disjunction

- $(G_1 \mathrm{v}\, G_2) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1 \mathrm{v}\, G_2) = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 2|E(G_1)||E(G_2)|$

$$= q_1 {p_2}^2 + q_2 {p_1}^2 - 2q_1 q_2$$

8- Symmetric Difference

- $(G_1 \oplus G_2) = |V(G_1)||V(G_2)| = p_1 \cdot p_2$
- $E(G_1 \oplus G_2) = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 4|E(G_1)||E(G_2)|$

$$= q_1 {p_2}^2 + q_2 {p_1}^2 - 4q_1 q_2$$

(b) The union, join, box product, tensor product, strong product, composition, disjunction and symmetric difference of the graphs are associative and all of them are commutative except composition operation on the graph.

To show that the all operations are associative we choose one of these operations let be join. We must show that $(G_1 + G_2) + G_3 = G_1 + (G_2 + G_3)$

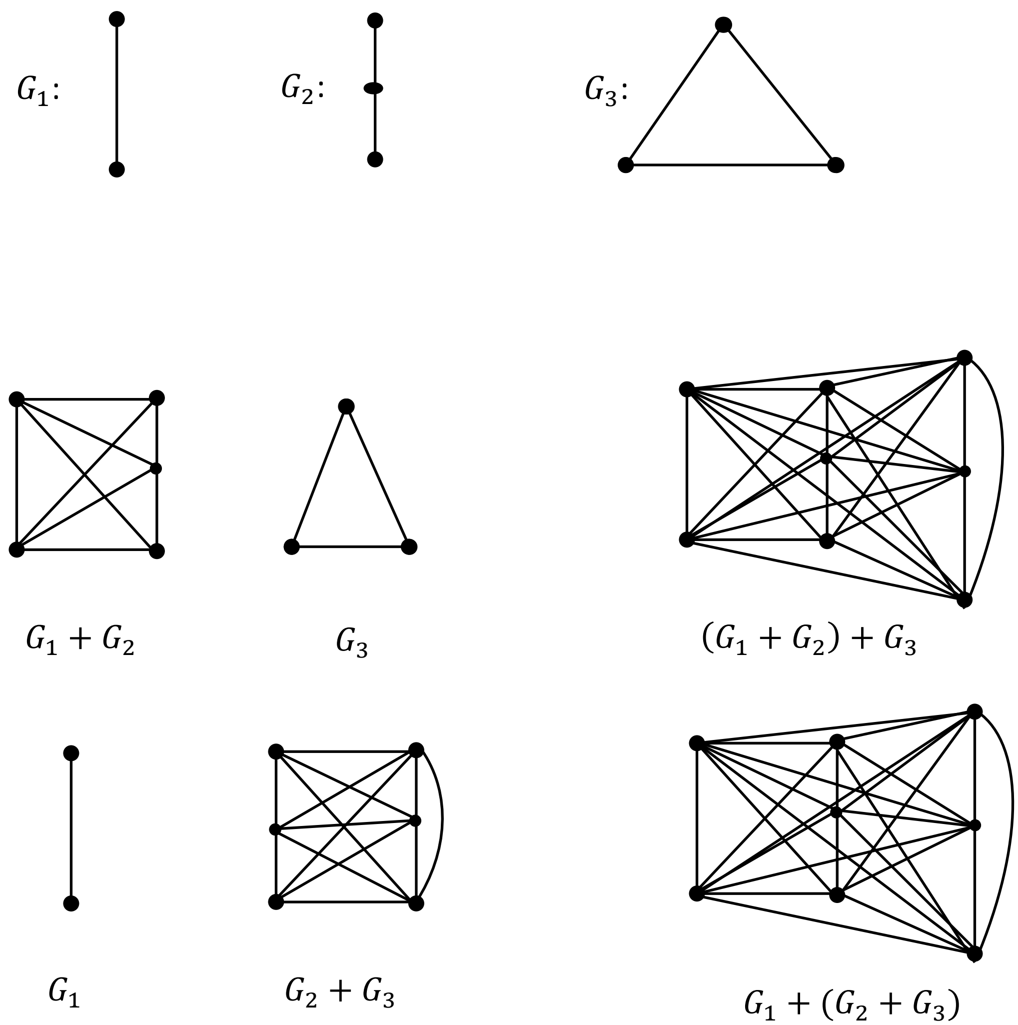


Figure 2.2.1 The associative of graphs

$$\therefore (G_1 + G_2) + G_3 = G_1 + (G_2 + G_3)$$

Two show that it's commutative let G_1 and $G_2$are two graphs then we have to show that $G_1 + G_2 = G_2 + G_1$

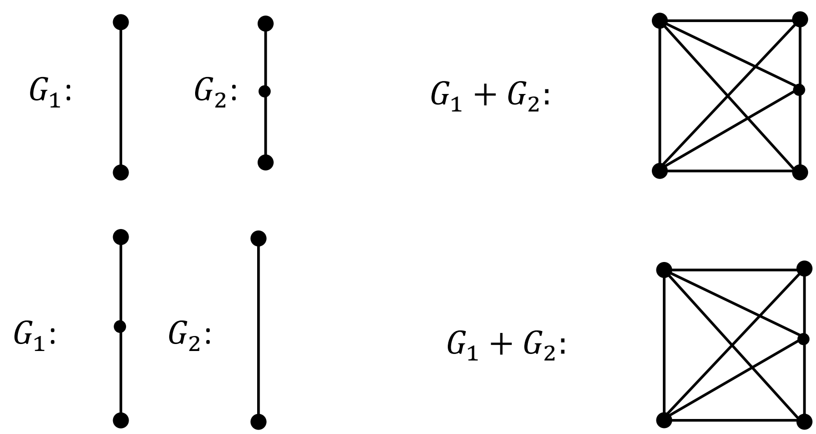
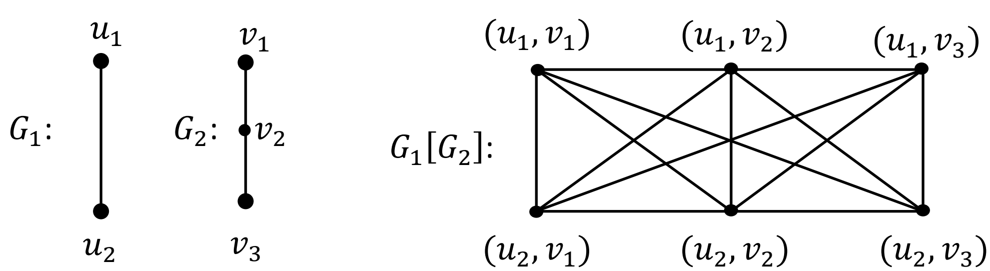


Figure 2.2.2 The commutative of the graphs

$\therefore G_1 + G_2 = G_2 + G_1$

The join of the graphs is associative and commutative, its similar to all anther operations except the commutative of competition operation.

Now to show that the competition operation is not commutative let G_1 and $G_2$are two graphs then we must show that $G_1[G_2] \neq G_2[G_1]$



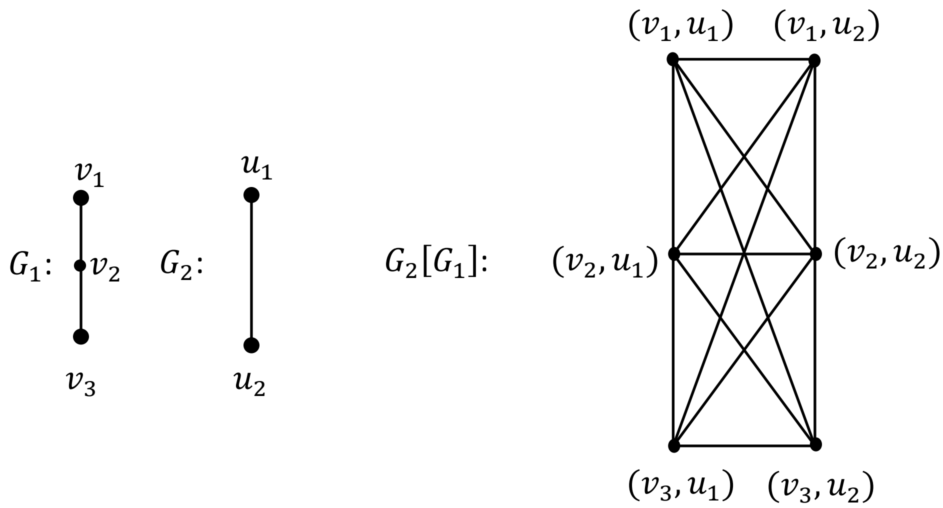


Figure 2.2.3. The non-commutative of the graphs

$\therefore\ G_1[G_2] \neq G_2[G_1]$

Therefor the composition operation is not commutative.

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پوختە

ڕێگای زۆر هەیە بۆ تێکەڵ کردنی گراف بۆ دروست کردنی گرافی نوێ. لەم کارەدا هەندێ کردار وە یەکگرتن, یەکخستن, هەندێ جۆر لە لێکدان, جیابوونەوە, جیاوازی هاوجێ بۆ گراف پێشکەش دەکرێ .

زانکۆی سەلاحەددین هەولێر

کرداری دوانەیی باو لە گراف تیۆری

نامەیەکە

پێشکەش بە ئەنجومەنی کۆلێژی پەروەردەی - زانکۆی سەلاحەددین- هەولێر کراوە وەك بەشێك لە پێداویستیەکانی بەدەست هێنانی پلەی بکالۆریۆس لە زانستی ماتماتیك (تیۆری گراف)

لەلایەن

راژان عوسمان

بەسەرپەرشتی

مامۆستایی ایڤان دلێر علی

گولان 2023