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Chapter One

This chapter reviews the basic ideas you need to start calculus.

A- Classifing the number

1- N = { 1,2,3,…} is called a set of Natural numbers and denoted by N .

2- Z= { 1,2,3,…}∪ { 0}∪ { -1,-2,-3,…} is called a set of Integer numbers and denoted by (Z) .

3- Q={a/b; a,b∈Z and b≠0} is called a set of rational numbers and denoted by Q.

4- Irrational number (I_{rr}) : A number can not be written as form a/b where a,b∈Z is called irrational number sush as $\sqrt{2}$, $\sqrt{5}$,e, π . The set of all irrational numbers denoted by I_{rr} .

5- R=Q∪ I_{rr} is called a set of Real numbers and denoted by R .

6- C={a+b i; a,b∈R and $i^2 = -1$} is called a a set of Complex numbers and denoted by C .

Order properties: The order properties allow us to compare the size to any real numbers . The order properties are:

(1) For any a and b , either a≤b or b≤a

(2) If a≤b and b≤a then a=b

(3) If a≤b and b≤c then a≤c

(4) If a≤b then a+c≤b+c

(5) If a≤b and 0≤c then ac≤bc

(6) If a≤b and c<0 then ac ≥ bc

(7) a<b ⇒ a+c<b+c and a-c<b-c for every real number c

(8) a<b and c >0⇒ ac<bc

(10) If a<b and c<0 then ac > bc

(11) If a and b are both positive or both negative , then

(i) if a<b , then 1/b<1/a (ii) a<x<b , then 1/b<1/x< 1/a

We next see that our new notion of exponentiation satisfies certain familiar rules.

Theorem : If a,d>0 and a,b,c,d $\in$R then

(1) $a^{b+c} = a^b a^c$ **(2)** $a^{b-c} = a^b/a^c$ **(3)** $(a^b)^c = a^{bc}$

(4) a^b=d if and only if a=$d^{1/b}$ (provided b≠0) **(5)** $(ad)^c=a^c d^c$

Notation	Set description
(a,b)	{x: a<x<b}
[a,b]	{x: a≤x≤b}
[a,b)	{x: a≤x<b}
(a,b]	{x: a<x≤b}
(a,∞)	{x: a<x }
(−∞,b)	{x: x<b}
[a,∞)	{x: a≤x }
(−∞,b]	{x: x≤b}

Absolute value : The absolute value of a number x, denoted by $|x|$, is defined by the formula

$|x|$=x if x≥0

$|x|$=-x if x<0

Remark: If a any positive real number, then :

(1) $|x|$=a if and only if x=a or x=-a

(2) $|x|$<a if and only if –a<x<a

(3) $|x|$≤a if and only if –a≤x≤a

(4) |x|>a if and only if x>a or –x>a

(5) $|x|$≥a if and only if x≥a or –x≥a

(6) $\sqrt{a^2}$ =$|a|$, for every real number a

Theorem : if x,y$\in$R , then show that

(i) $|x|^2=x^2$

(ii) $|xy|=|x||y|$

(iii) $|x/y|=|x|/|y|$

Proof : (i) If $x\geq 0$ then $|x|=x$ thus $|x|^2=x^2$.

If $x<0$ then $|x|=-x$ thus $|x|^2=x^2$. In both case we get $|x|^2=x^2$

(ii)$|xy|^2=(xy)^2=x^2\ y^2=|x|^2\ |y|^2=(|x|\ |y|)^2$ we take square root of both sides we get $|xy|=|x||y|$

(iii) $|x/y|^2=(x/y)^2=x^2/\ y^2=|x|^2\ /|y|^2=(|x|/\ |y|)^2$ we take square root of both sides we get $|x/y|=|x|/|y|$

Theorem : For all real numbers x,y show that

(i) $|x+y|\leq|x|+|y|$ (ii) $|x-y|\ \geq\big||x|-|y|\big|$

Proof: (i) $|x+y|^2=(x+y)^2=x^2+y^2+2xy\leq|x|^2+|x|^2+2|x||y|=(|x|+|y|)^2$.

Since $|x+y|$ and $|x|+|y|$ are both non negative numbers, so we take square root of booth sides we get $|x+y|\leq|x|+|y|$

(ii) $|x-y|^2=(x-y)^2=x^2+y^2-2xy\geq|x|^2+|x|^2-2|x||y|=(|x|-|y|)^2$

we take square root of both sides we get $|x-y|\geq\big||x|-|y|\big|$.

Example(1): Solve the equalion $|x-3|=2$

Solution: $|x-3|=2 \Leftrightarrow x-3=2$ or $-(x-3)=2 \Leftrightarrow x=5$ or $x=1 \Rightarrow S=\{5,1\}$.

Example(2): Solve the equation $|9x|-11=x$

Solution: $|9x|-11=x \Leftrightarrow 9x-11=x$ or $-9x-11=x \Leftrightarrow 8x=11$ or $-10x=11$

$x=11/8$ or $x=-11/10 \Rightarrow S=\{11/8\ ,\ -11/10\}$.

Example(3): Solve the inequality $|x-3|>4$

Solution: $|x-3|>4 \Leftrightarrow x-3>4$ or $-(x-3)>4 \Leftrightarrow x>7$ or $x<-1 \Rightarrow S=(-\infty,-1)\cup(7,\infty)$

Example(4): Solve the inequality $-|x-5|<-7$

Solution: $-|x-5|<-7 \Leftrightarrow |x-5|>7 \Leftrightarrow x-5>7$ or $-(x-5)>7$

$x>12$ or $-x+5>7 \Leftrightarrow x>12$ or $-x>2 \Leftrightarrow x>12$ or $x<-2 \Rightarrow S=(-\infty,-2)\cup(12,\infty)$.

Example(5): Solve the inequality $|3x+1|\leq 4$

Solution: $|3x+1|\leq 4 \Leftrightarrow -4\leq 3x+1\leq 4 \Leftrightarrow -5/3\leq x\leq 1 \Rightarrow S=[-5/3,1]$.

Example(6) : Solve the inequality $\left|\frac{2}{x}-10\right|<2$

Solution: $\left|\frac{2}{x} - 10\right| <2 \Leftrightarrow -2 < \frac{2}{x} - 10 < 2$

$8 < \frac{2}{x} < 12 \Leftrightarrow 1/8>x/2>1/12 \Leftrightarrow ¼ > x> 1/6 \Rightarrow S=(1/6,1/4)$

Example(7) : Solve the inequality $|2x-5| < |x+4|$

Solution: $|2x-5| < |x+4|$, since $x\neq-4$ then divided both side by $|x+4|$, then weget

$|2x-5|/|x+4| < 1 \Rightarrow |(2x-5)/(x+4)| < 1$

$-1< \frac{2x-5}{x+4} < 1$, since $x\neq-4$ then we have only two cases x<-4 and x>-4

Case 1 if x>-4 then $-1(x+4) < 2x-5 < 1(x+4)$

$-x-4 < 2x-5 <(x+4) \Leftrightarrow -x-4 < 2x-5$ *and* $2x-5 <(x+4)$

$x>1/3$ and $x<9 \Rightarrow 1/3 < x < 9 \Rightarrow S_1=(1/3,9)$

Case 2 if x<-4 then $-1(x+4) > 2x-5 > 1(x+4)$

$-x-4 > 2x-5 >(x+4) \Leftrightarrow -x-4 > 2x-5$ *and* $2x-5 > (x+4)$

$x<1/3$ and $x>9$, this is imposible , then S=(1/3,9)

Example(8) : Solve the inequality $|3x+1| < 2|x-6|$

Solution: $|3x+1| < 2|x-6|$, since $x\neq6$ then divided both side by $|x-6|$, then weget

$|3x+1|/|x-6| < 2 \Rightarrow |(3x+1)/(x-6)| < 2$

$-2< \frac{3x+1}{x-6} < 2$, since $x\neq6$ then we have only two cases x<6 and x>6

Case 1: if x>6 , then $-2(x-6) < 3x+1 < 2(x-6)$

$-2x+12 < 3x+1 < 2x-12 \Leftrightarrow -2x+12 < 3x+1$ *and* $3x+1 < 2x-12$

$x>11/5$ and $x<-13$ this is imposible

Case 2: if x<6 , then $-2(x-6) > 3x+1 > 2(x-6) \Leftrightarrow -2x+12 > 3x+1 > 2x-12$

$-2x+12 > 3x+1$ *and* $3x+1 > 2x-12 \Leftrightarrow x<11/5$ and $x>-13 \Rightarrow S=(-13,11/5)$

Example (9): Solve the inequalities use the result $\sqrt{a^2} =|a|$ as a property :

(1) $4<x^2<9$ (2) $(x+3)^2< 2$

Solution:(1) $4<x^2<9 \Rightarrow 2<|x|<3 \Rightarrow 2<x<3$ or $2<-x<3 \Leftrightarrow 2<x<3$ or $-3<x<-2 \Rightarrow S=(-3,-2)\cup(2, 3)$

(2) $(x+3)^2< 2 \Rightarrow |x+3| < \sqrt{2} \Rightarrow -\sqrt{2} < x+3 <\sqrt{2} \Rightarrow S=(-3-\sqrt{2}, -3+\sqrt{2})$

Example(10): Solve the inequality (x-2) (5-x)>0

Solution: Since (x-2) (5-x)>0 , then ((x-2)>0 and (5-x) >0) or((x-2)<0 and (5-x) <0)

If (x-2)>0 and (5-x) >0 ⇔ x>2 and x<5⇔ 2<x<5 ⇔ S_1=(2,5) .

If (x-2)<0 and (5-x) <0 ⇔ x<2 and x>5 which is a contraduction .

Hence S=(2,5)

Example(11): Solve the inequality (x-2) (5-x)<0

Solution: Since (x-2) (5-x)<0 , then ((x-2)>0 and (5-x) <0) or((x-2)<0 and (5-x) >0)

If ((x-2)>0 and (5-x) <0) ⇔ x>2 and x>5⇔ 5<x ⇒ S_1=(5,∞)

If ((x-2)<0 and (5-x) >0) ⇔ x<2 and x<5 ⇔ x<2 ⇒ S_2=(-∞,2)

Hence S=(-∞,2) ∪ (5,∞)

Example(12): Solve the inequality (x-2) (5-x)≥0

Solution: We have only two cases

Case1 (x-2) (5-x)=0 , then x=2 or x=5 then S_1={2,5}

Case2 (x-2) (5-x)>0 , then ((x-2)>0 and (5-x) >0) or((x-2)<0 and (5-x) <0)

If (x-2)>0 and (5-x) >0 ⇔ x>2 and x<5⇔ 2<x<5 → S_2=(2,5)

If (x-2)<0 and (5-x) <0 ⇔ x<2 and x>5 which is a contraduction

S=(2,5)∪ {2,5} =[2,5]

Example(15): Solve the inequality (x-2) (5-x) ≤ 0

Solution: We have only two cases

Case1 (x-2) (5-x)=0 then x=2 or x=5 then S_1={2,5}

Case2 (x-2) (5-x)<0 , then ((x-2)>0 and (5-x) <0) or((x-2)<0 and (5-x) >0)

If ((x-2)>0 and (5-x) <0) ⇔ x>2 and x>5⇔ 5<x → S_2=(5,∞)

If ((x-2)<0 and (5-x) >0) ⇔ x<2 and x<5 ⇔ x<2 → S_3=(-∞,2)

Hence S==(-∞,2) ∪ (5,∞)∪ {2,5}=(-∞,2] ∪ [5,∞)

Exercise H.w: Solve the inequalities in excersize 1-5

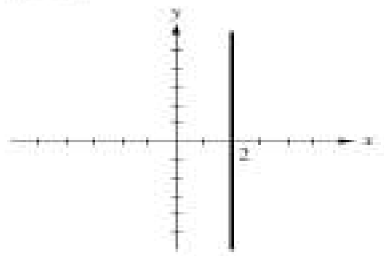
(1) $|6x-3|<5$ (2) $|3-1/x|<$ ½ (3) $|(x+1)/2|>1$ (4) $3<|x+1|<6$

(5) $|x+4|<|2x-6|$ (6) x^2-x = 0 (7) (x-2) (x-6) ≥0 (8) (x-1) (4-x) ≤ 0

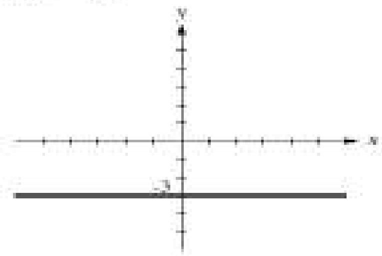
Example: graph each of the following inequality

(a) **x=2** (b) **y= -3** (c) **x ≥ 0** (d) **y=x** (e) **y ≥ x** (f) $|x|$**≥1**

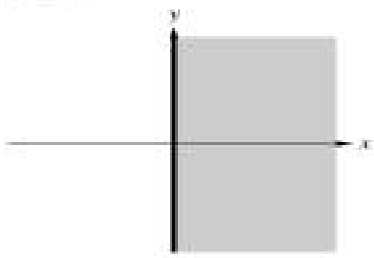
(a) $x = 2$



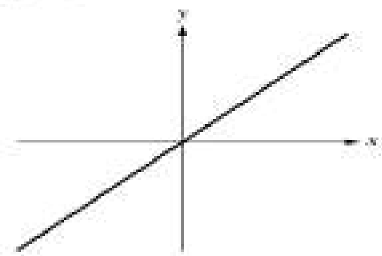
(b) $y = -3$



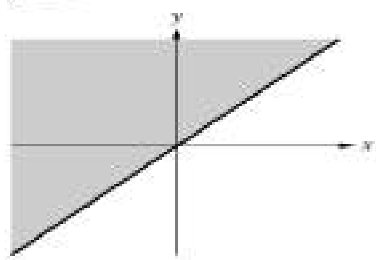
(c) $x \geq 0$



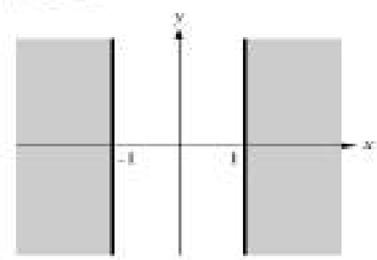
(d) $y = x$



(e) $y \geq x$

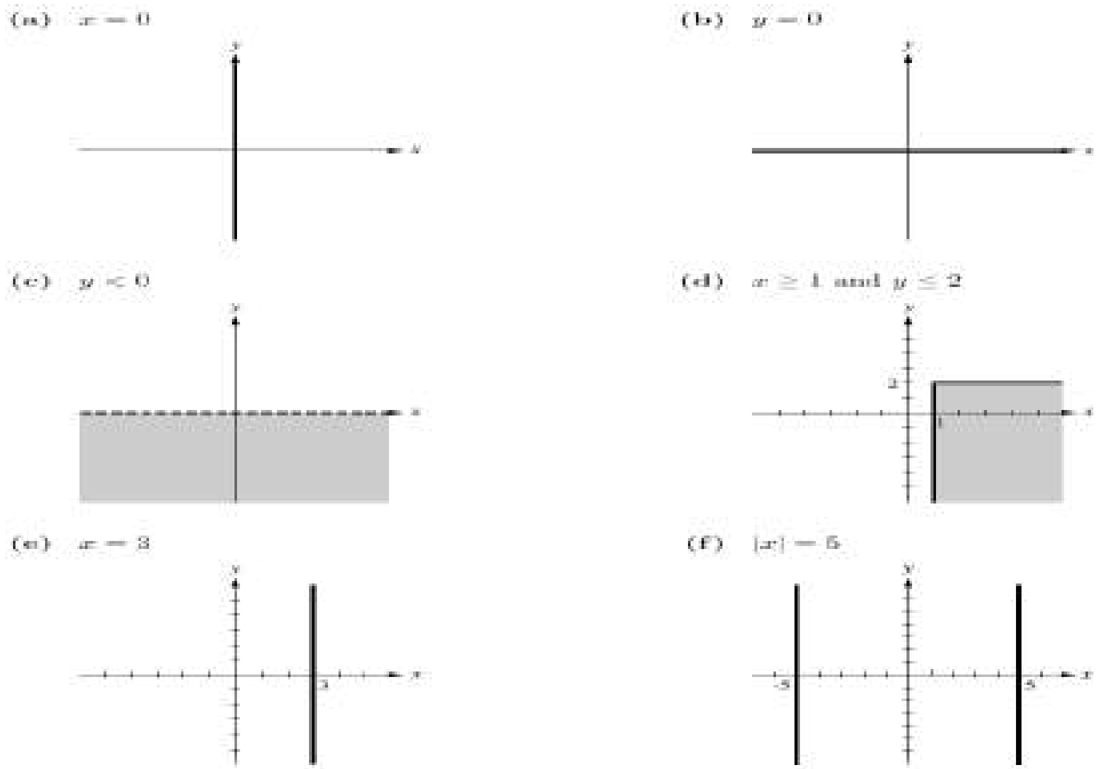


(f) $|x| \geq 1$



Example: graph each of the following inequality

(a) **x=0** (b) **y= 0** (c) **y<0** (d) **x ≥ 1** and y≤2 (e) $x = 3$ **(f)** $|x|$ **=5**



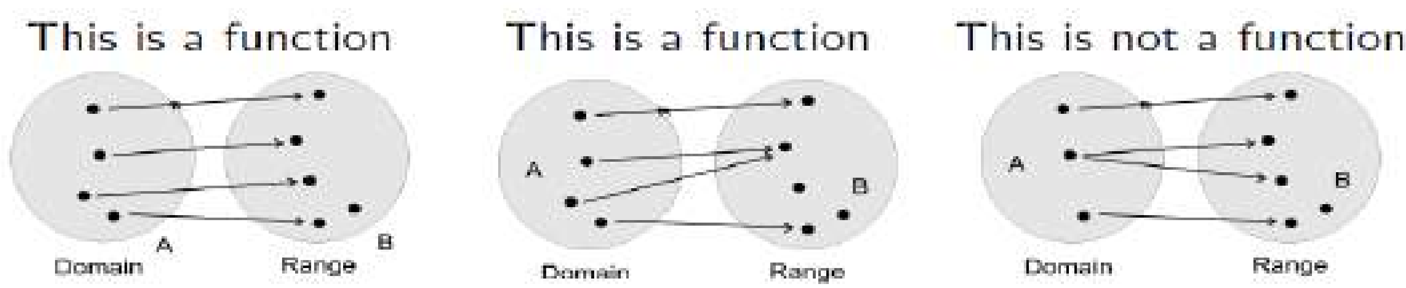
Section 2 : Function and Graphs of Second-Degree Equations and Trigonometric function

Function: Let A and B be two non empty sets.A correspondence which associates each element $x \in A$ with unique element $y \in B$, is called a function or mapping and generally denoted by $f:A \rightarrow B$.Moreover Ais called domain and Bis called co-domain.

$f(A) = \{ y \in B \text{ ; there exist } x \in A \text{ such that } f(x)=y \} = \{ y \in B \text{ ; } f(x)=y \text{ , for some } x \in A \}$ will be called range or image of f ,denoted by f(A).

Example: Let A={1,2,3,4} , B ={ 1,2,3,4,5} and $f:A \rightarrow B$ defined by f(x) =x+1 , then

f(A) = { 2,3,4,5}



Some important notes to find the domain of the function

1- If a function is a polynomial then $D_f=R$

2- If f(x)= h(x)/g(x) where each of h(x)and g(x) are polynomial then $D_f=\{x\in R;\ g(x)\neq 0\}$

For example if $f(x)=(3x^2+5)/(x^2-1)$, then $(x^2-1)=0 \Rightarrow x=-1$ or $x=1$, then $D_f=R-\{-1,1\}$

3- If a function is of the form $f(x)=h(x)/\sqrt{g(x)}$, then $D_f=\{x\in R;\ g(x)>0\}$

4- If a function is of the form $f(x)=\sqrt{f(x)/g(x)}$, then $D_f=\{x\in R;\ f(x)/g(x)\geq 0\}$

Example: Verify the domains and ranges of these functions.

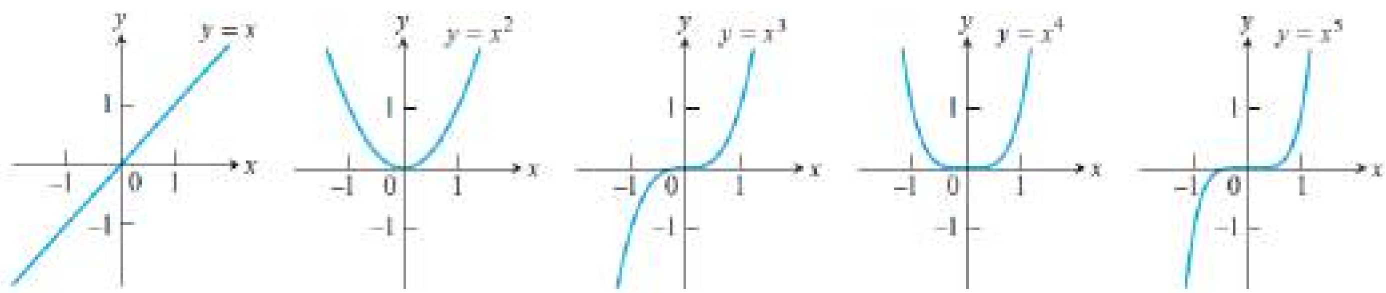
(1) $f(x)=1+x^2$ (2) $f(x)=1-\sqrt{x}$

Solution: (1) domain =R ; range=$[1,\infty)$ **(2)** domain =$[0,\infty)$; range = $(-\infty, 1]$

A- **Identifying Functions;** There are a number of important types of functions frequently encountered in calculus. We identify and briefly summarize them here.

1- Linear Functions: A function of the form $f(x)=mx+b$, for constants *m* and *b*, is called a linear function.

2- **Power Functions :** A function $f(x)=x^n$ where *a* is a constant, is called a power function.



3- **Polynomials :**A function *p* is a polynomial if $p(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1x^1+a_0$ where *n* is a nonnegative integer and the numbers $a_0, a_1,\ldots, a_n$ are real constants.

4- Algebraic Functions: An algebraic function is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)

5- Trigonometric Functions: such as sine and cosine functions.

6- Exponential Functions: Functions of the form $f(x)=a^x$, where the base $a>0$ is a positive constant and $a\neq 1$, are called exponential functions.

7- Logarithmic Functions: These are the functions $f(x)=\log_a^x$, where the base $a\neq 1$is a positive constant.

8- Rational Functions : A rational function is a quotient or ratio of two polynomials: $f(x)=p(x)/q(x)$ where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x)\neq 0$.

9-**Transcendental Functions:**

These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions.

Example: Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category.

(1) $f(x)=1+x-(1/2)\,x^5$ (2) $g(x)=7^x$

(3) $h(z)=z^7$ (4) $y(t)=\sin(t-5)$

Solution:

(1) $f(x)=1+x-(1/2)\,x^5$ is a polynomial of degree 5

(2) $g(x)=7^x$ is an exponential function with base 7

(3) $h(z)=z^7$ is a power function

(4) $y(t)=\sin(t-5)$ is a trigonometric function.

Example(2) H.w: Identify each function given here as one of the types of functions .

(a) $f(x)=7-3x$ (b) $g(x)=\sqrt[3]{x}$ (c) $h(x)=(x+1)/(x+2)$

(d) $r(x)=8^x$ (e) $\sqrt{z^7}$ (f) $y=\log_{12} x$ (g) $y=\sin x$

Step function : A function $f:X\to R$ is called step function if $f(x)=[\![x]\!]$ where we defined $[\![x]\!]$ by greatest integer less than or equal to x , so that $D_f=R$ and $R_f=Z$.

For example

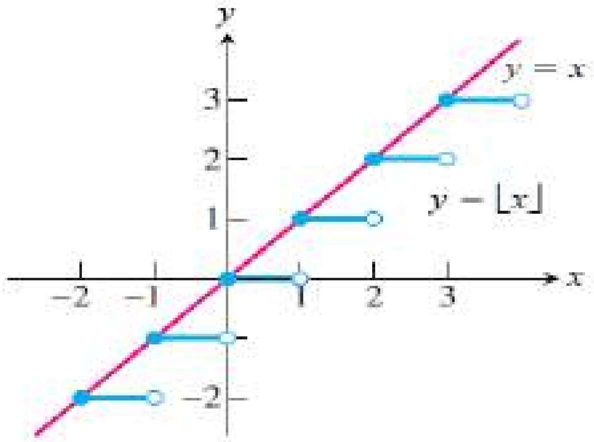
if $0\leq x\leq 1$ then $[\![x]\!]=0$

if $1\leq x<2$ then $[\![x]\!]=1$

if $2\leq x<3$ then $[\![x]\!]=2$

if $-1\leq x<0$ then $[\![x]\!]=-1$

if $-2\leq x<1$ then $[\![x]\!]=-2$



Example(1): find the set of solutionof $\llbracket x \rrbracket^2=\llbracket x \rrbracket$

Solution: $\llbracket x \rrbracket^2=\llbracket x \rrbracket \Rightarrow \llbracket x \rrbracket^2-\llbracket x \rrbracket=0 \Rightarrow \llbracket x \rrbracket(\llbracket x \rrbracket-1)=0 \Leftrightarrow \llbracket x \rrbracket = 0$ or $(\llbracket x \rrbracket-1)=0$

$\llbracket x \rrbracket = 0$ or $\llbracket x \rrbracket = 1 \Leftrightarrow 0\le x<1$ or $1\le x<2 \Leftrightarrow S=[0,1)\cup[1,2)=[0,2)$

Example(2): find the set of solution of $\llbracket x \rrbracket^2-3\llbracket x \rrbracket=-2$.

Solution : $\llbracket x \rrbracket^2-3\llbracket x \rrbracket=-2 \Rightarrow \llbracket x \rrbracket^2-3\llbracket x \rrbracket+2=0 \Rightarrow (\llbracket x \rrbracket-1)(\llbracket x \rrbracket-2)=0$

$(\llbracket x \rrbracket-2)=0$ or $(\llbracket x \rrbracket-1)=0 \Leftrightarrow \llbracket x \rrbracket = 2$ or $\llbracket x \rrbracket = 1$

$2\le x<3$ or $1\le x<2 \Leftrightarrow S=[2,3)\cup[1,2)=[1,3)$

Example(3): find the set of solutionof $\llbracket 1/(2x+1) \rrbracket=1$

Solution: $\llbracket 1/(2x+1) \rrbracket = 1 \Leftrightarrow 1 \le \frac{1}{2x+1} < 2 \Leftrightarrow 1/2 < 2x+1 \le 1$

$-1/2<2x\le 0 \Leftrightarrow -1/4<x\le 0 \Leftrightarrow S=(-1/4,0]$

Example(4): find the set of solutionof $|\llbracket x \rrbracket|<2$

Solution: $|\llbracket x \rrbracket| < 2 \Rightarrow -2 < \llbracket x \rrbracket < -2 \Rightarrow \llbracket x \rrbracket = -1$ or $\llbracket x \rrbracket = 0$ or $\llbracket x \rrbracket = 1$

$-1\le x<0$ or $0\le x<1$ or $1\le x<2 \Leftrightarrow S=[-1,0)\cup[0,1)\cup[1,2)=[-1,2)$

Definition: A function $f:X\rightarrow R$ is called an

even function if $f(-x)=f(x)$ for all $x\in X$

odd function if $f(-x)=-f(x)$ for all $x\in X$

Example (1): In 1–4, say whether the function is even, odd, or neither. Give reasons for your answer.

1. $f(x)=x^2+1$ (2) $f(x)=x^2+x$

(3) $g(x)= x^3+x$ (4) $g(x)=x^4+3x^2-1$

Solution:

1: Since $f(x)=x^2+1= f(x)=(-x)^2+1=f(-x)$, then the function is even.

2:Since $[f(x)=x^2+x]\neq[\ f(-x)=(-x)^2-x]$ and $[[f(x)=x^2+x]\neq[-f(x)=-x^2-x]$, then the function is neither even nor odd

3:Since $g(-x)= -x^3-x=- g(x)$, so the function is odd

4:Since $g(x)=x^4+3x^2-1=(-x)^4+3(-x)^2-1=g(-x)$, thus the function is even.

Example(2):

Can a function be both even and odd? Give reasons for your answer.

Solution: Yes, $f(x)=0$ is both even and odd.

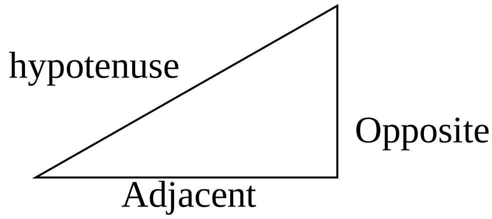
Example (3) H.w:

Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line Which of the following(where defined) are even? odd?

(a)fg (b) f/b (c) g/f (d) f^2=f f (e) g^2=g g (f) $f \circ g$ (g) $g \circ f$ (h) $f \circ f$ (i) $g \circ f$

Definition : A function $f(x)$ is called periodic function if there is a positive number p such that f(x+p) = f(x) for every value of x. The smallest such value of p is the period of f.

This section reviews the basic trigonometric functions. The trigonometric functions are important because they are periodic. For an angle θ the six trigonometric function are defined as ratio of length of sides of right as follows:



Sin x = opposite / hypotenuse,   cos x= adjacent / hypotenuse   and   tan x= opposite / adjacent

csc x= 1 / sin x ,   sec x = 1 / cos x   and   cot x = 1/ tan x . Moreover

Sin x = x – x^3 / 3! + x^5 / 5! - x^7 / 7! + x^9 / 9! + …

Cos x = 1 – x^2 / 2! + x^4 / 4! - x^6 / 6! + x^8 / 8! + …

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \quad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \frac{\pi}{2}\right) = \sin A$$

$$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$$

$$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

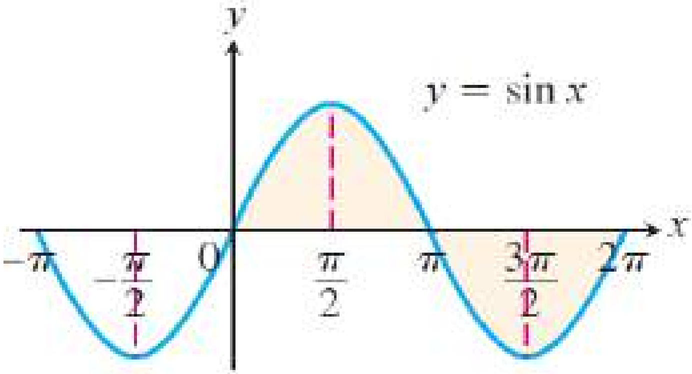
$$\sin A\cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

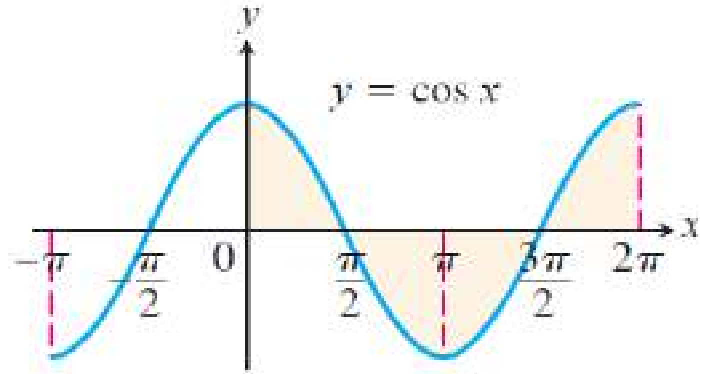
$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

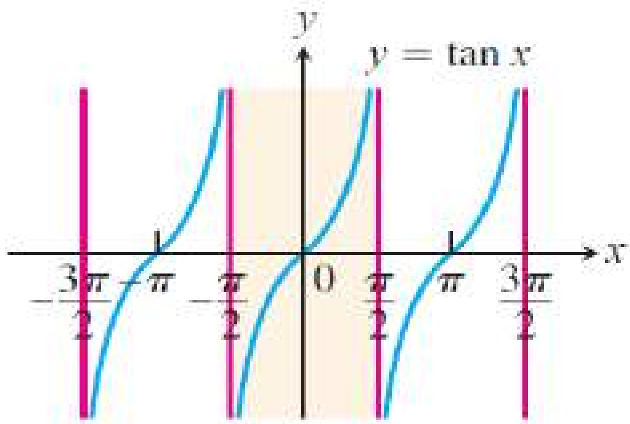
$$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$



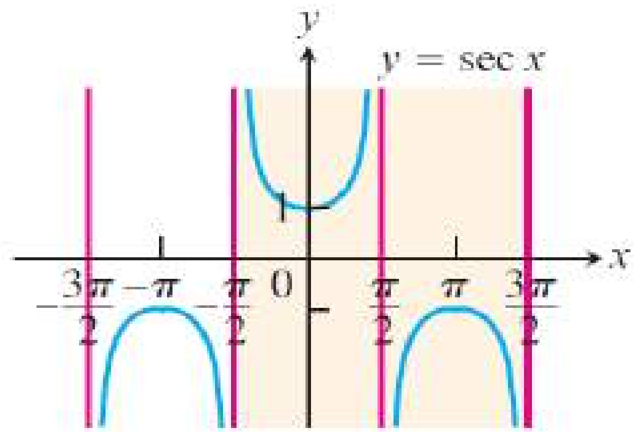
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



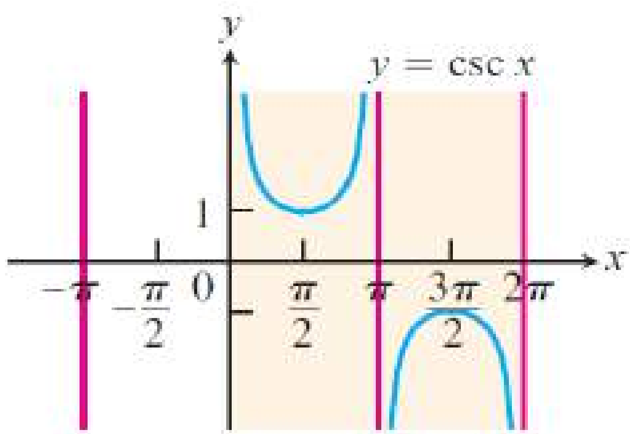
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



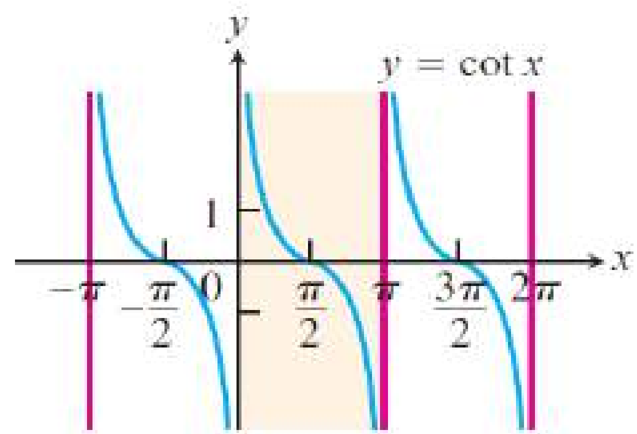
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



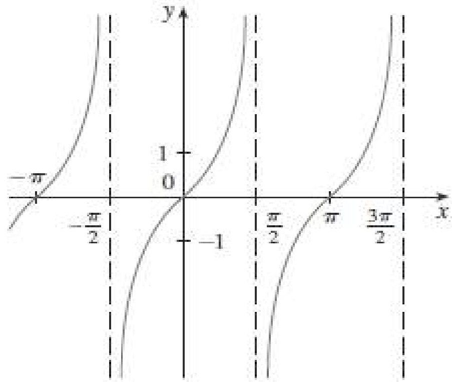
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



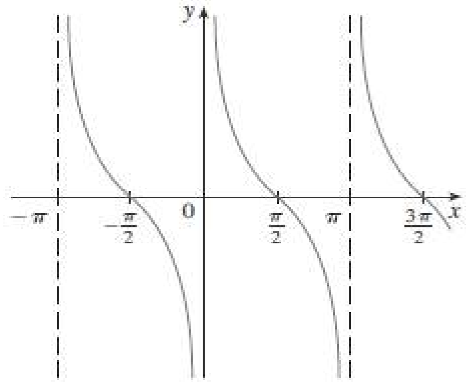
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



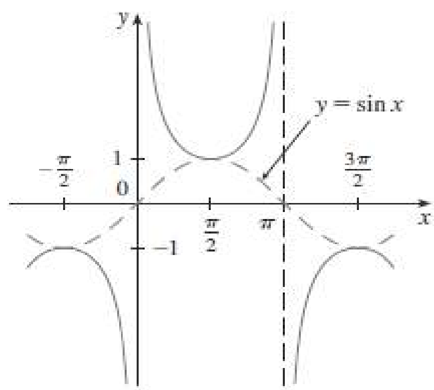
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, \infty)$



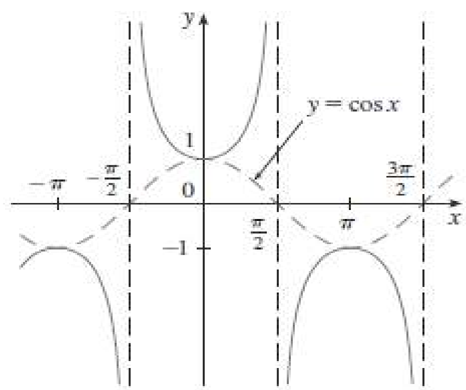
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$

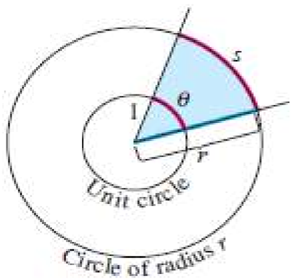


(d) $y = \sec x$

Angle	**Sin**	**Cos**	**Tan**	**Cot**	**Sec**	**Csc**
0	0	1	0	undef	1	undef
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
$\pi/2$	1	0	undef	0	undef	1
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	-2	$2/\sqrt{3}$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	$-2/\sqrt{3}$	2
π	0	-1	0	undef	-1	undef
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$-2/\sqrt{3}$	-2
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$1/\sqrt{3}$	-2	$-2/\sqrt{3}$
$3\pi/2$	-1	0	undef	0	undef	-1
$5\pi/3$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	2	$-2/\sqrt{3}$
$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	$2/\sqrt{3}$	-2

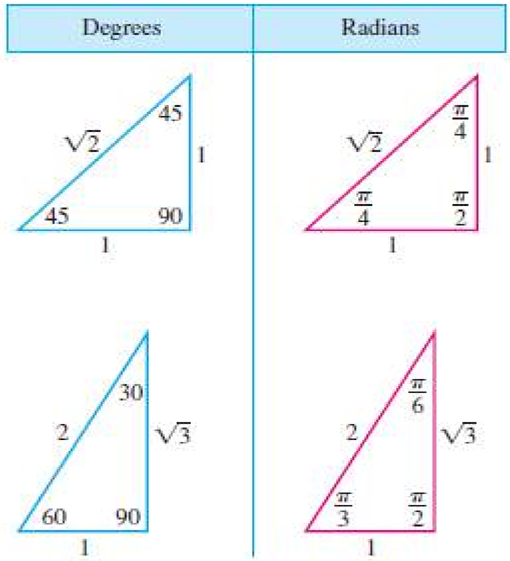
Trigonometric Functions

Radian Measure



$\frac{s}{r} = \frac{\theta}{1} = \theta$ or $\theta = \frac{s}{r}$,

$180° = \pi$ radians.



The angles of two common triangles, in degrees and radians.

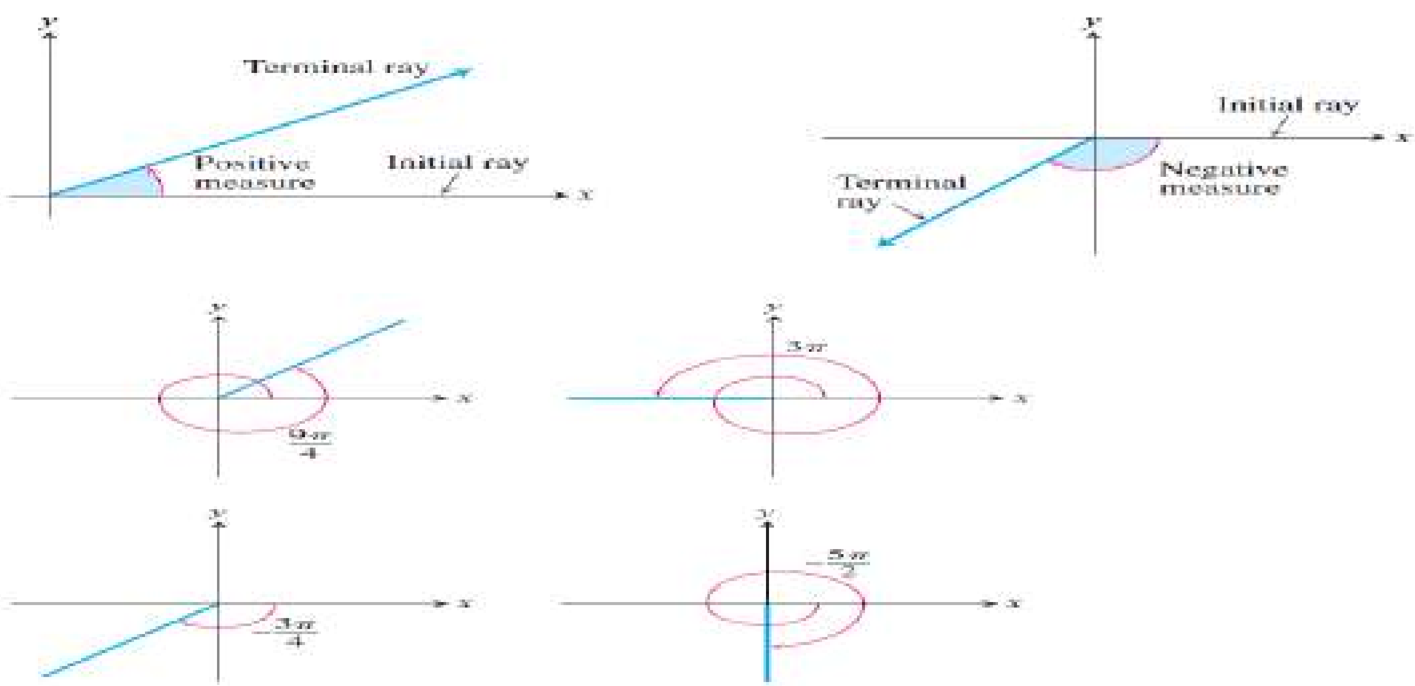
$\pi/180° = S/A$, where S radians measure and A degree measure, is the relation between radians measure and A degree measure

Example: Write 65° as radians measure

Solution: $\pi/180° = S/A \Rightarrow \pi/180° = S/65° \Rightarrow S = 65°\ \pi/180° = 13\ \pi/36$

Example: write 13 π/36 as degree measure

Solution: $\pi/180° = S/A \Rightarrow \pi/180° = (13\ \pi/36)/A \Rightarrow A = (180°/\pi)\ (13\ \pi/36) = 65°$



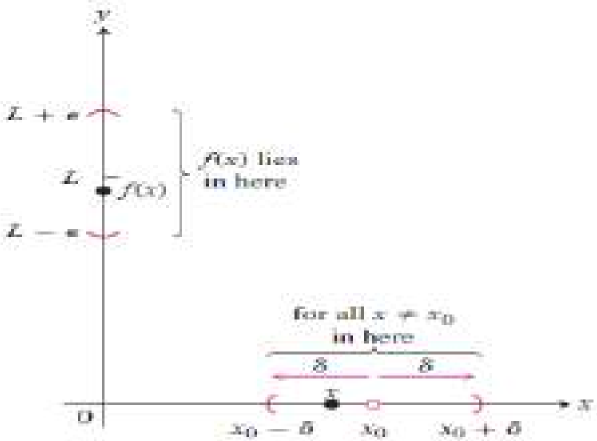
Definition:

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$



Example: prove that $\lim_{x \to 1} f(x) = 5$, where f(x)=2x+3

Solution: Step(1) $|x - a| < \delta \to |x - 1| < \delta \to -\delta < x - 1 < \delta \to -\delta + 1 < x < \delta + 1$

Step (2)$|f(x) - L| < \varepsilon \to |(2x + 3) - 5| < \zeta \to = |2x - 2| < \zeta \to 2|x - 1| < \zeta \to |x - 1| < \zeta/2 \to$

satiesfies $-\zeta/2 < x - 1 < \zeta/2 \to -\frac{\zeta}{2} + 1 < x < \zeta/2 + 1$

$|f(x) - L| = |(2x + 3) - 5| = |2x - 2| = 2|x - 1| < 2\,\frac{\varepsilon}{2} = \varepsilon$, we get $\lim_{x \to 1} f(x) = 5$

Example: prove that $\lim_{x \to 5} f(x) = 5$ where f(x) = x

Solution: $|f(x) - L| = |x - 5|$, take $\delta = \varepsilon$ then $\forall\, x \in D_f$ such that $0 < |x - 5| < \delta$ satiesfies

$|f(x) - L| = |x - 5| < \varepsilon$, we get $\lim_{x \to 1} f(x) = 5$

Example: Let f(x)=bx+c where $b \neq 0$ prove that $\lim_{x \to a} f(x) = b\,a + c$

Solution: $|f(x) - L| = |bx + c - (b\,a + c)| = |bx - ba| = b|x - a|$

Take $\delta = \varepsilon/b$ then $\forall\, x \in D_f$ such that $0<|x-a|<\delta$ satiesfies

$|f(x) - L| = |(bx+c) - (b\,a + c)| = |bx - ba| = b|x-a| < b\,\frac{\varepsilon}{b} = \varepsilon$, we get $\lim_{x\to a} f(x) = ba + c$

Theorem: If the limit of $f: D \to R$ exist, then its unique (If $\lim_{x\to a} f(x)=L_1$ and $\lim_{x\to a} f(x)=L_2$, then $L_1=L_2$)

Proof: If $L_1 \neq L_2$, so $|L_1 - L_2| > 0$ take $\varepsilon=(1/2)\,|L_1 - L_2|$.

Since $\lim_{x\to a} f(x)=L_1$, then there exist $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow |f(x)-L_1|< \varepsilon$ …(1)

Since $\lim_{x\to a} f(x)=L_2$, then there exist $\delta_2>0$ such that $0<|x-a|<\delta_1 \Rightarrow |f(x)-L_2|< \varepsilon$ … (2)

$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| < |f(x) - L_1| + |f(x) - L_2| < \varepsilon + \varepsilon = 2\,\varepsilon = 2\,(1/2)|L_1 - L_2|$

So $|L_1 - L_2| < |L_1 - L_2|$ which is contradiction. Hence $L_1=L_2$.

Theorem : Let $f: D_1 \to R$ and $g: D_2 \to R$ be a function and $D_1 \cap D_2 \neq \emptyset$.

If $\lim_{x\to a} f(x)=L_1$ and $\lim_{x\to a} g(x)=L_2$, then :

(1) $\lim_{x\to a} (f(x)+ g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(2) $\lim_{x\to a} (f(x)\, g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = L_1\ L_2$

(3) $\lim_{x\to a} (f(x)/ g(x)) = \lim_{x\to a} f(x) / \lim_{x\to a} g(x) = L_1 / L_2$ where $L_2 \neq 0$ and $g(x) \neq 0$

Proof(1): Since $\lim_{x\to a} f(x)=L_1$,then for every $\varepsilon/2>0$ there exist $\delta_1>0$ such that

$0<|x-a|<\delta_1 \Rightarrow |f(x)-L_1|< \varepsilon/2$ …..(1)

Since $\lim_{x\to a} g(x)=L_1$,then for every $\varepsilon/2>0$ there exist $\delta_2>0$ such that

$0<|x-a|<\delta_2 \Rightarrow |g(x)-L_2|< \varepsilon/2$ ……(2) take $\delta=\min\{\delta_1 , \delta_1\}$and $x \in D_1 \cap D_2$:

$0<|x-a|<\delta \Rightarrow |(f(x)+g(x))-(L_1+L_2)| < |f(x)-L_1| + |g(x)-L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Hence $\lim_{x\to a} (f(x)+ g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$.

Example: (1) $\lim_{x\to 2} 5x+3 = 5\times 2+3 = 15$

(2) $\lim_{x\to 2} (x^2-4)/(x-2) = \lim_{x\to 2} (x-2)(x+2)/(x-2) = \lim_{x\to 2} x+2 = 4$

(3) $\lim_{x\to -3} (x+3)/(x^2+4x+3) = \lim_{x\to -3} (x+3)/(x+3)(x+1) = \lim_{x\to -3} 1/(x+1) = 1/(-3+1) = -\frac{1}{2}$

(4) $\lim_{x\to 9} (\sqrt{x}-3)/(x-9) = \lim_{x\to 9} (\sqrt{x}-3)/((\sqrt{x}-3)(\sqrt{x}+3)) = \lim_{x\to 9} 1/(\sqrt{x}+3) = 1/(\sqrt{9}+3) = 1/6$

(5) $\lim_{x\to a} (x^n-a^n)/(x-a) = \lim_{x\to a} (x^{-}-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+x^{n-4}a^3+\ldots+x^1a^{n-2}+a^{n-1})/(x-a)$

$= \lim_{x\to a} (x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+x^1a^{n-2}+a^{n-1}) = (a^{n-1}+a^{n-2}a+a^{n-3}a^2+\ldots+a^1a^{n-2}+a^{n-1}) = na^{n-1}$

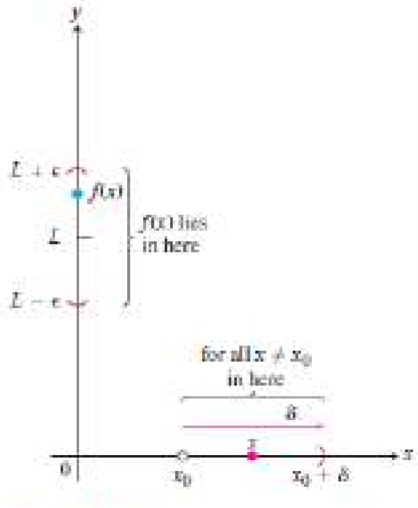


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit** L at x_0, and write

$$\lim_{x\to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit** L at x_0, and write

$$\lim_{x\to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

EXAMPLE 3 Applying the Definition to Find Delta

Prove that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon.$$

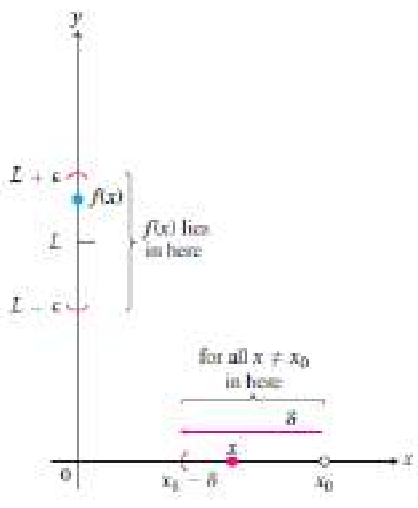


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Example: $\lim_{x\to 1} [\![x]\!]$ does not exist because $\lim_{x\to 1^-} [\![x]\!] = 0$ and $\lim_{x\to 1^+} [\![x]\!] = 1$ which is not equal .

(1) f(x) $= \begin{cases} 3 & x < 1 \\ 5 & x \geq 1 \end{cases}$ has no limit at x=1 by $\lim_{x\to 1^-}$ f(x) = 3 and $\lim_{x\to 1^+}$ f(x) = 5

which is not equal .

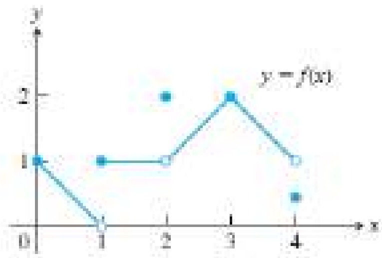


FIGURE 2.24 Graph of the function in Example 2.

EXAMPLE 2 Limits of the Function Graphed in Figure 2.24

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

Example: Find the limits in 16–18.

16. $\lim_{h\to 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x\to -2^+} (x + 3)\frac{|x + 2|}{x + 2}$ **b.** $\lim_{x\to -2^-} (x + 3)\frac{|x + 2|}{x + 2}$

18. a. $\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x - 1)}{|x - 1|}$ **b.** $\lim_{x\to 1^-} \frac{\sqrt{2x}\,(x - 1)}{|x - 1|}$

Solution:

16. $\lim_{h\to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim_{h\to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$= \lim_{h\to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim_{h\to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$

17. (a) $\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = x+2 \text{ for } x > -2)$

$= \lim_{x\to -2^+} (x+3) = (-2)+3 = 1$

(b) $\lim_{x\to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$

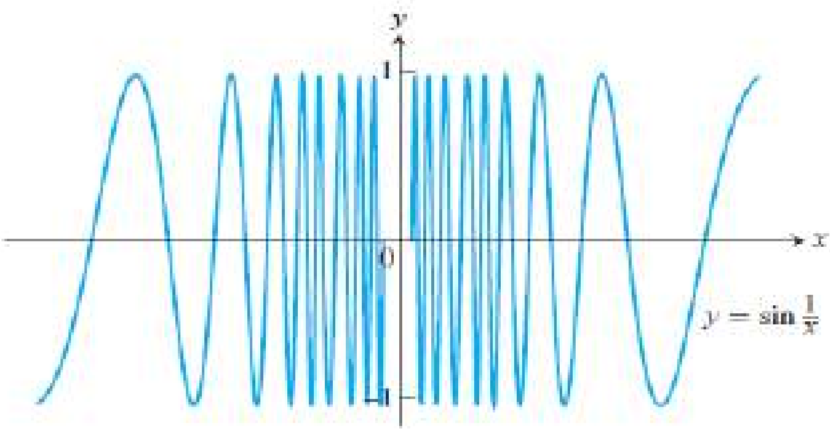
$= \lim_{x\to -2^-} (x+3)(-1) = -(-2+3) = -1$

18. (a) $\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x > 1)$

$= \lim_{x\to 1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x < 1)$

$= \lim_{x\to 1^-} -\sqrt{2x} = -\sqrt{2}$



Theorem : prove that $\lim_{x\to 0} \sin x / x = 1$

Theorem : prove that $\lim_{x\to 0} (\cos x - 1)/x = 0$

Proof: $\lim_{x\to 0} (\cos x - 1)/x = \lim_{x\to 0} -2(\sin(x/2))^2/x = \lim_{x\to 0} \sin(x/2)/(x/2) \;\; \lim_{x\to 0} \sin(x/2) = -1 \times 0 = 0$

Examples:

$$\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad (\text{where } x = \sqrt{2}\theta)$$

$$\lim_{t \to 0} \frac{\sin kt}{t} = \lim_{t \to 0} \frac{k \sin kt}{kt} = \lim_{\theta \to 0} \frac{k \sin \theta}{\theta} = k \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = k \cdot 1 = k \qquad (\text{where } \theta = kt)$$

$$\lim_{y \to 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \to 0} \frac{3 \sin 3y}{3y} = \frac{3}{4} \lim_{y \to 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{3}{4} \qquad (\text{where } \theta = 3y)$$

$$\lim_{h \to 0^-} \frac{h}{\sin 3h} = \lim_{h \to 0^-} \left(\frac{1}{3} \cdot \frac{3h}{\sin 3h}\right) = \frac{1}{3} \lim_{h \to 0^-} \frac{1}{\left(\frac{\sin 3h}{3h}\right)} = \frac{1}{3}\left(\frac{1}{\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta}}\right) = \frac{1}{3} \cdot 1 = \frac{1}{3} \qquad (\text{where } \theta = 3h)$$

$$\lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x} = \lim_{x \to 0} \frac{\sin 2x}{x \cos 2x} = \left(\lim_{x \to 0} \frac{1}{\cos 2x}\right)\left(\lim_{x \to 0} \frac{2 \sin 2x}{2x}\right) = 1 \cdot 2 = 2$$

$$\lim_{t \to 0} \frac{2t}{\tan t} = 2 \lim_{t \to 0} \frac{t}{\left(\frac{\sin t}{\cos t}\right)} = 2 \lim_{t \to 0} \frac{t \cos t}{\sin t} = 2\left(\lim_{t \to 0} \cos t\right)\left(\frac{1}{\lim\limits_{t \to 0} \frac{\sin t}{t}}\right) = 2 \cdot 1 \cdot 1 = 2$$

$$\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \to 0} \left(\frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x}\right) = \left(\frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin 2x}\right)\left(\lim_{x \to 0} \frac{1}{\cos 5x}\right) = \left(\frac{1}{2} \cdot 1\right)(1) = \frac{1}{2}$$

$$\lim_{x \to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x \to 0} \frac{6x^2 \cos x}{\sin x \sin 2x} = \lim_{x \to 0} \left(3 \cos x \cdot \frac{x}{\sin x} \cdot \frac{2x}{\sin 2x}\right) = 3 \cdot 1 \cdot 1 = 3$$

$$\lim_{x \to 0} \frac{x + x \cos x}{\sin x \cos x} = \lim_{x \to 0} \left(\frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x}\right) = \lim_{x \to 0} \left(\frac{x}{\sin x} \cdot \frac{1}{\cos x}\right) + \lim_{x \to 0} \frac{x}{\sin x}$$
$$= \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) \cdot \lim_{x \to 0} \left(\frac{1}{\cos x}\right) + \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) = (1)(1) + 1 = 2$$

$$\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x} = \lim_{x \to 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{\sin x}{x}\right)\right) = 0 - \frac{1}{2} + \frac{1}{2}(1) = 0$$

$$\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = 1 - \cos t \to 0 \text{ as } t \to 0$$

$$\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = \sin h \to 0 \text{ as } h \to 0$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\sin 2\theta} = \lim_{\theta \to 0} \left(\frac{\sin \theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta}\right) = \frac{1}{2} \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x \to 0} \left(\frac{\sin 5x}{\sin 4x} \cdot \frac{4x}{5x} \cdot \frac{5}{4}\right) = \frac{5}{4} \lim_{x \to 0} \left(\frac{\sin 5x}{5x} \cdot \frac{4x}{\sin 4x}\right) = \frac{5}{4} \cdot 1 \cdot 1 = \frac{5}{4}$$

$$\lim_{x \to 0} \frac{\tan 3x}{\sin 8x} = \lim_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x}\right) = \lim_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8}\right)$$
$$= \frac{3}{8} \lim_{x \to 0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$$

$$\lim_{y \to 0} \frac{\sin 3y \cot 5y}{y \cot 4y} = \lim_{y \to 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim_{y \to 0} \left(\frac{\sin 3y}{y}\right)\left(\frac{\sin 4y}{\cos 4y}\right)\left(\frac{\cos 5y}{\sin 5y}\right)\left(\frac{3 \cdot 4 \cdot 5y}{3 \cdot 4 \cdot 5y}\right)$$
$$= \lim_{y \to 0} \left(\frac{\sin 3y}{3y}\right)\left(\frac{\sin 4y}{4y}\right)\left(\frac{5y}{\sin 5y}\right)\left(\frac{\cos 5y}{\cos 4y}\right)\left(\frac{3 \cdot 4}{5}\right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12}{5} = \frac{12}{5}$$

Theorem: (L, Hopital Theorem)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example : (1) $\lim\limits_{x\to a} (x^n-a^n)/(x-a) = \lim\limits_{x\to a} nx^{n-1}=na^{n-1}$ (by using L, Hopital Theorem)

(2) $\lim\limits_{x\to 0} \sin x / x = \lim\limits_{x\to 0} \cos x = \cos 0=1$ (by using L, Hopital Theorem)

(3) $\lim\limits_{x\to 0} (x^2+x)/x = \lim\limits_{x\to 0} 2x+1 =1$ (by using L, Hopital Theorem)

(4) $\lim\limits_{x\to 2} (x^2 -4)/(x-2) = \lim\limits_{x\to 2} 2x = 4$

(5) $\lim\limits_{x\to 2} (x2 -4x)/(x^2-2x) = \lim\limits_{x\to 2} (x^2 -4)/(x-2)$ (by using L, Hospital Theorem) $= \lim\limits_{x\to 2} 2x = 4$

(6) $\lim\limits_{x\to 0} (1/x - 1/\sin x) = \lim\limits_{x\to 0} (\sin x - x)/(x \sin x)$ (by using L, Hopital Theorem)

$= \lim\limits_{x\to 0}(\cos x -1)/(\sin x+ x \cos x)$(by using L,Hopital Theorem)$=\lim\limits_{x\to 0}(-\sin x)/(2 \cos x - x \sin x)=0$

Example: In 1–26, use Hopital's Rule to evaluate the limit:

(1) $\lim\limits_{x\to 2} (x-2)/(x^2-4)$ (2) $\lim\limits_{x\to 0} (\sin 5x) / x$ (3) $\lim\limits_{x\to \infty} (5x^2-3x)/(7x^2+1)$

(4) $\lim\limits_{x\to 1} (x^3-1)/(4x^3-x-3)$ (5) $\lim\limits_{x\to 1} (1- \cos x)/x^2$ (6) $\lim\limits_{x\to \infty} (2x^2+3x)/(x^3+x+1)$

7. $\lim_{t\to 0} \frac{\sin t^2}{t}$

8. $\lim_{x\to \pi/2} \frac{2x-\pi}{\cos x}$

9. $\lim_{\theta\to \pi} \frac{\sin \theta}{\pi-\theta}$

10. $\lim_{x\to \pi/2} \frac{1-\sin x}{1+\cos 2x}$

11. $\lim_{x\to \pi/4} \frac{\sin x-\cos x}{x-\pi/4}$

12. $\lim_{x\to \pi/3} \frac{\cos x-0.5}{x-\pi/3}$

13. $\lim_{x\to(\pi/2)^-} \left(x-\frac{\pi}{2}\right)\tan x$

14. $\lim_{x\to 0} \frac{2x}{x+7\sqrt{x}}$

15. $\lim_{x\to 1} \frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$

16. $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$

17. $\lim_{x\to 0} \frac{\sqrt{a(a+x)}-a}{x}, \quad a>0$

18. $\lim_{t\to 0} \frac{10(\sin t-t)}{t^3}$

19. $\lim_{x\to 0} \frac{x(\cos x-1)}{\sin x-x}$

20. $\lim_{h\to 0} \frac{\sin(a+h)-\sin a}{h}$

21. $\lim_{r\to 1} \frac{a(r^n-1)}{r-1}$, n a positive integer

22. $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

23. $\lim_{x\to\infty} (x-\sqrt{x^2+x})$

24. $\lim_{x\to\infty} x\tan\frac{1}{x}$

25. $\lim_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2}$

26. $\lim_{x\to 0} \frac{\sin 7x}{\tan 11x}$

Solution:

1. l'Hôpital: $\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim_{x\to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim_{x\to 0} \frac{\sin 5x}{x} = 5\lim_{5x\to 0} \frac{\sin 5x}{5x} = 5\cdot 1 = 5$

3. l'Hôpital: $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x\to\infty} \frac{10x-3}{14x} = \lim_{x\to\infty} \frac{10}{14} = \frac{5}{7}$ or $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x\to\infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x^2}} = \frac{5}{7}$

4. l'Hôpital: $\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$
$= \lim_{x\to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$

5. l'Hôpital: $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \left[\frac{(1-\cos x)}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$
$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to\infty} \frac{4}{6x} = 0$ or $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim_{t \to 0} \frac{\sin t^2}{t} = \lim_{t \to 0} \frac{2t \cos t^2}{1} = 0$

8. $\lim_{x \to \pi/2} \frac{2x-\pi}{\cos x} = \lim_{\theta \to \pi/2} \frac{2}{-\sin x} = \frac{2}{-1} = -2$

9. $\lim_{\theta \to \pi} \frac{\sin \theta}{\pi-\theta} = \lim_{\theta \to \pi} \frac{\cos \theta}{-1} = \frac{-1}{-1} = 1$

10. $\lim_{x \to \pi/2} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x \to \pi/2} \frac{-\cos x}{-2 \sin 2x} = \lim_{x \to \pi/2} \frac{\sin x}{-4 \cos 2x} = \frac{1}{-4(-1)} = \frac{1}{4}$

11. $\lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \pi/4} \frac{\cos x + \sin x}{1} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

12. $\lim_{x \to \pi/3} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}} = \lim_{x \to \pi/3} \frac{-\sin x}{1} = -\frac{\sqrt{3}}{2}$

13. $\lim\limits_{x\to\pi/2} -(x-\frac{\pi}{2})\tan x = \lim\limits_{x\to\pi/2} \frac{-(x-\frac{\pi}{2})\sin x}{\cos x} = \lim\limits_{x\to\pi/2} \frac{(\frac{\pi}{2}-x)\cos x+\sin x(-1)}{-\sin x} = \frac{-1}{-1} = 1$

14. $\lim\limits_{x\to0} \frac{2x}{x+7\sqrt{x}} = \lim\limits_{x\to0} \frac{2}{1+\frac{7}{2\sqrt{x}}} = \lim\limits_{x\to0} \frac{4\sqrt{x}}{2\sqrt{x}+7} = \frac{4\cdot0}{2\cdot0+7} = 0$

15. $\lim\limits_{x\to1} \frac{2x^2-(3x+1)\sqrt{x}+2}{x-1} = \lim\limits_{x\to1} \frac{2x^2-3x^{3/2}-x^{1/2}+2}{x-1} = \lim\limits_{x\to1} \frac{4x-\frac{9}{2}x^{1/2}-\frac{1}{2\sqrt{x}}}{1} = -1$

16. $\lim\limits_{x\to2} \frac{\sqrt{x^2+5}-3}{x^2-4} = \lim\limits_{x\to2} \frac{\frac{1}{2}(x^2+5)^{-1/2}(2x)}{2x} = \lim\limits_{x\to2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{6}$

17. $\lim\limits_{x\to0} \frac{\sqrt{a(a+x)}-a}{x} = \lim\limits_{x\to0} \frac{a}{2\sqrt{a^2+ax}} = \frac{a}{2\sqrt{a^2}} = \frac{1}{2}$, where $a > 0$.

18. $\lim\limits_{t\to0} \frac{10(\sin t-t)}{t^3} = \lim\limits_{t\to0} \frac{10(\cos t-1)}{3t^2} = \lim\limits_{t\to0} \frac{10(-\sin t)}{6t} = \lim\limits_{t\to0} \frac{-10\cos t}{6} = \frac{-10\cdot1}{6} = -\frac{5}{3}$

19. $\lim\limits_{x\to0} \frac{x(\cos x-1)}{\sin x-x} = \lim\limits_{x\to0} \frac{-x\sin x+\cos x-1}{\cos x-1} = \lim\limits_{x\to0} \frac{-x\cos x-2\sin x}{-\sin x} = \lim\limits_{x\to0} \frac{x\cos x+2\sin x}{\sin x} = \lim\limits_{x\to0} \frac{-x\sin x+3\cos x}{\cos x} = \frac{3}{1} = 3$

20. $\lim\limits_{h\to0} \frac{\sin(a+h)-\sin a}{h} = \lim\limits_{h\to0} \frac{\cos(a+h)-\cos a}{1} = 0$

21. $\lim\limits_{r\to1} \frac{a(r^n-1)}{r-1} = \lim\limits_{r\to1} \frac{a(nr^{n-1})}{1} = an\lim\limits_{r\to1} r^{n-1} = an$, where n is a positive integer.

22. $\lim\limits_{x\to0^+} \left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right) = \lim\limits_{x\to0^+} \left(\frac{1-\sqrt{x}}{x}\right) = \left(\begin{array}{c}\text{l'Hopital's rule}\\ \text{does not apply}\end{array}\right) = \lim\limits_{x\to0^+} (1-\sqrt{x})\cdot\frac{1}{x} = \infty$

23. $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2+x}\right) = \lim\limits_{x\to\infty} \left(x-\sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right) = \lim\limits_{x\to\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim\limits_{x\to\infty} \frac{-\frac{x}{x}}{\frac{x}{x}+\frac{\sqrt{x^2+x}}{\sqrt{x^2}}}$

$= \lim\limits_{x\to\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x}}} = -\frac{1}{2} \left(\begin{array}{c}\text{l'Hopital's rule}\\ \text{is unnecessary}\end{array}\right)$

24. $\lim\limits_{x\to\infty} x\tan\left(\frac{1}{x}\right) = \lim\limits_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \sec^2\left(\frac{1}{x}\right) = \sec^2 0 = 1$

25. $\lim\limits_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2} = \lim\limits_{x\to\pm\infty} \frac{3}{4x-1} = 0$

26. $\lim\limits_{x\to0} \frac{\sin 7x}{\tan 11x} = \lim\limits_{x\to0} \frac{7\cos(7x)}{11\sec^2(11x)} = \frac{7\cdot1}{11\cdot1} = \frac{7}{11}$

Definition:

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \implies |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \implies |f(x) - L| < \epsilon.$$

Example: Show that

(a) $\lim_{x \to \infty} \frac{1}{x} = 0$ (b) $\lim_{x \to -\infty} \frac{1}{x} = 0$.

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

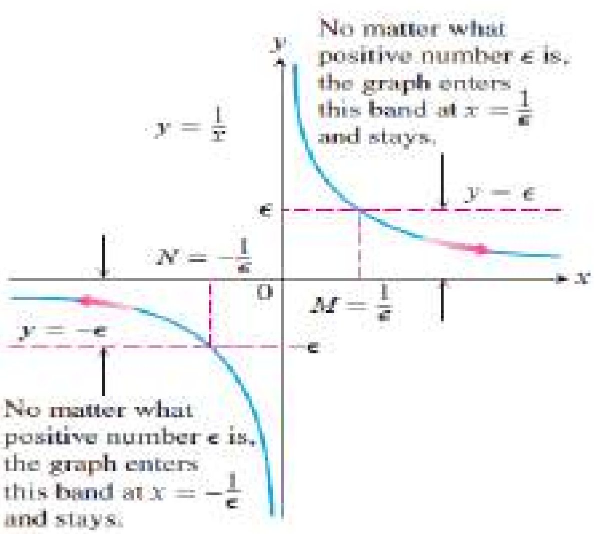
$$x > M \quad \Longrightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.32). This proves $\lim_{x \to \infty}(1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Longrightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.32). This proves $\lim_{x \to -\infty}(1/x) = 0$. ■



Theorem: If f : D → R is a rational function such that

$f(x) = (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) / (b_m x^m + b_{m-1} x^{m-1} + \dots + b_0)$ where $n, m \in Z^+ \cup \{0\}$, then

$$\lim_{n \to \infty} f(x) = \begin{cases} an/bm & if\ n = m \\ 0 & if\ n < m \\ \infty\ or - \infty & if\ n > m \end{cases}$$

Proof: $f(x) = (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) / (b_m x^m + b_{m-1} x^{m-1} + \dots + b_0) =$

$x^n (a_n + a_{n-1} x^{-1} + \dots + a_0 / x^n) / (x^m (b_m + b_{m-1} x^{-1} + \dots + b_0 / x^m))$, we have only three cases case 1: $if\ n = m$, then $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} x^n(a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (x^m(b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m)) = \lim_{n\to\infty} (a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m) = a_n / b_n$.

case 2: $if\ n > m$, then

$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} x^n(a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (x^m(b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m)) = \lim_{n\to\infty} x^{n-m} \lim_{n\to\infty} (a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m) = \infty\ or - \infty$.

case 3: $if\ n < m$, then

$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} x^n(a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (x^m(b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m)) = \lim_{n\to\infty} 1/x^{m-n} \lim_{n\to\infty} (a_n + a_{n-1}x^{-1} + \ldots + a_0 / x^n) / (b_m + b_{m-1}x^{-1} + \ldots + b_0 / x^m) = 0 \times (a_n / b_m) = 0$

Hence $\lim_{n\to\infty} f(x) = \begin{cases} an/bm & if\ n = m \\ 0 & if\ n < m \\ \infty\ or - \infty & if\ n > m \end{cases}$

Example: In 47–56, find the limit of each rational function (a) as x→∞ (b) as x→ -∞:

47. $f(x) = \dfrac{2x + 3}{5x + 7}$

48. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

49. $f(x) = \dfrac{x + 1}{x^2 + 3}$

50. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

51. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

52. $g(x) = \dfrac{1}{x^3 - 4x + 1}$

53. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

54. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

55. $h(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

56. $h(x) = \dfrac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

Solution:

47. (a) $\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$ (b) $\frac{2}{5}$ (same process as part (a))

48. (a) $\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+\left(\frac{7}{x^3}\right)}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$

(b) 2 (same process as part (a))

49. (a) $\lim_{x \to \infty} \frac{x+1}{x^2+3} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ (b) 0 (same process as part (a))

50. (a) $\lim_{x \to \infty} \frac{3x+7}{x^2-2} = \lim_{x \to \infty} \frac{\frac{3}{x}+\frac{7}{x^2}}{1-\frac{2}{x^2}} = 0$ (b) 0 (same process as part (a))

51. (a) $\lim_{x \to \infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x \to \infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} = 7$ (b) 7 (same process as part (a))

52. (a) $\lim_{x \to \infty} \frac{1}{x^3-4x+1} = \lim_{x \to \infty} \frac{\frac{1}{x^3}}{1-\frac{4}{x^2}+\frac{1}{x^3}} = 0$ (b) 0 (same process as part (a))

53. (a) $\lim_{x \to \infty} \frac{10x^5+x^4+31}{x^6} = \lim_{x \to \infty} \frac{\frac{10}{x}+\frac{1}{x^2}+\frac{31}{x^6}}{1} = 0$
(b) 0 (same process as part (a))

54. (a) $\lim_{x \to \infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x \to \infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} = \frac{9}{2}$
(b) $\frac{9}{2}$ (same process as part (a))

55. (a) $\lim_{x \to \infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x \to \infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}} = -\frac{2}{3}$
(b) $-\frac{2}{3}$ (same process as part (a))

56. (a) $\lim_{x \to \infty} \frac{-x^4}{x^4-7x^3+7x^2+9} = \lim_{x \to \infty} \frac{-1}{1-\frac{7}{x}+\frac{7}{x^2}+\frac{9}{x^4}} = -1$
(b) -1 (same process as part (a))

Example:

57. $\lim_{x \to \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$ **58.** $\lim_{x \to \infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

59. $\lim_{x \to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$ **60.** $\lim_{x \to \infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

61. $\lim_{x \to \infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$ **62.** $\lim_{x \to -\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

Solution:

57. $\lim_{x \to \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x \to \infty} \frac{\left(\frac{2}{x^{1/2}}\right) + \left(\frac{1}{x^2}\right)}{3 - \frac{7}{x}} = 0$

58. $\lim_{x \to \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} = \lim_{x \to \infty} \frac{\left(\frac{2}{x^{1/2}}\right) + 1}{\left(\frac{2}{x^{1/2}}\right) - 1} = -1$

59. $\lim_{x \to -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} = \lim_{x \to -\infty} \frac{1 - x^{(1/5) - (1/3)}}{1 + x^{(1/5) - (1/3)}} = \lim_{x \to -\infty} \frac{1 - \left(\frac{1}{x^{2/15}}\right)}{1 + \left(\frac{1}{x^{2/15}}\right)} = 1$

60. $\lim_{x \to \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}} = \lim_{x \to \infty} \frac{x + \frac{1}{x^2}}{1 - \frac{1}{x}} = \infty$

61. $\lim_{x \to \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} = \lim_{x \to \infty} \frac{2x^{1/15} - \frac{1}{x^{19/15}} + \frac{7}{x^{8/5}}}{1 + \frac{3}{x^{3/5}} + \frac{1}{x^{11/10}}} = \infty$

62. $\lim_{x \to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} = \lim_{x \to -\infty} \frac{\frac{1}{x^{2/3}} - 5 + \frac{3}{x}}{2 + \frac{1}{x^{1/3}} - \frac{4}{x}} = -\frac{5}{2}$

Theorem:

1. $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
2. $\lim_{n \to \infty} \sqrt[n]{n} = 1$
3. $\lim_{n \to \infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n \to \infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

Example:

$\lim_{n \to \infty} (0.03)^{1/n} = 1$

$\lim_{n \to \infty} \left(1 + \frac{7}{n}\right)^n = e^7$

$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left[1 + \frac{(-1)}{n}\right]^n = e^{-1}$

$\lim_{n \to \infty} \sqrt[n]{10n} = \lim_{n \to \infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1$

$$\lim_{n \to \infty} \sqrt[n]{n^2} = \lim_{n \to \infty} \left(\sqrt[n]{n}\right)^2 = 1^2$$

$$\lim_{n \to \infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n \to \infty} 3^{1/n}}{\lim_{n \to \infty} n^{1/n}} = \frac{1}{1} = 1$$

$$\lim_{n \to \infty} [\ln n - \ln(n+1)] = \lim_{n \to \infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n \to \infty} \frac{n}{n+1}\right) = \ln 1 = 0$$

$$\lim_{n \to \infty} \sqrt[n]{4^n n} = \lim_{n \to \infty} 4\sqrt[n]{n} = 4 \cdot 1 = 4$$

$$\lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim_{n \to \infty} \exp\left(\frac{1}{\ln n} \ln\left(\frac{1}{n}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1}$$

$$\lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln\left(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right) = \ln e = 1$$

$$\lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(\frac{n}{n+1}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln n - \ln(n+1)}{\left(\frac{1}{n}\right)}\right) = \lim_{n \to \infty} \exp\left(\frac{\frac{1}{n} - \frac{1}{n+1}}{\left(-\frac{1}{n^2}\right)}\right)$$

$$= \lim_{n \to \infty} \exp\left(-\frac{n^2}{n(n+1)}\right) = e^{-1}$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n^2}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(1 - \frac{1}{n^2}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln\left(1 - \frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n \to \infty} \exp\left[\frac{\left(\frac{2}{n^3}\right)/\left(1 - \frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$$

$$= \lim_{n \to \infty} \exp\left(\frac{-2n}{n^2 - 1}\right) = e^0 = 1$$

Definition:

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad f(x) < -B.$$

Example:

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if } x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Definition:

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

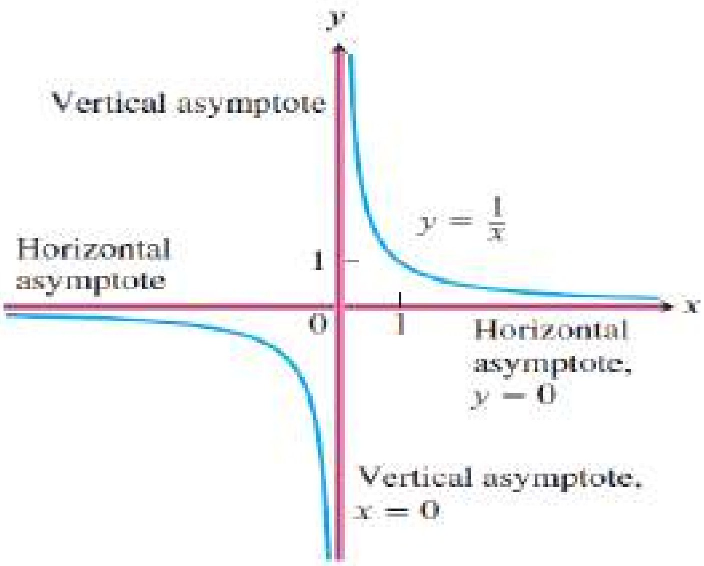
$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty .$$

Definition: A line y=b is a **horizotal asymptote** if $\lim_{x\to \infty} f(x)$ = b or $\lim_{x\to -\infty} f(x)$ = b

Example: find the verticale asymptote and horizotal asymptote of f(x)=1/x if exist .

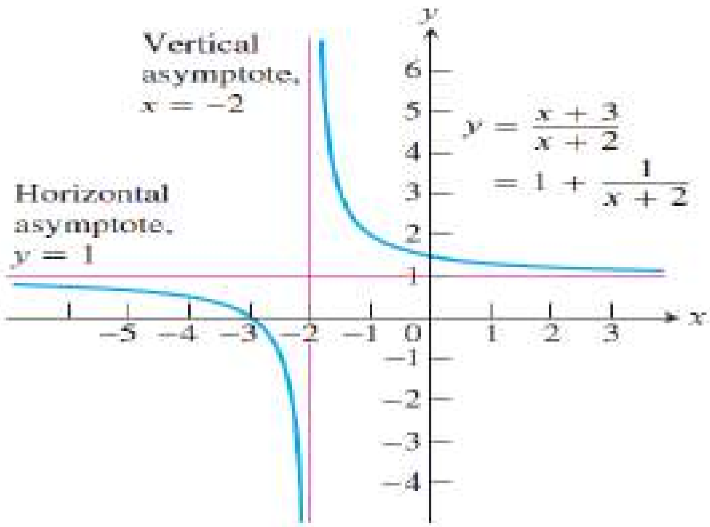
Solution: Since $\lim_{x\to 0^+} 1/x = \infty$ then the verticale asymptote of f(x)=1/x is a line x=0 .

Since $\lim_{x\to \infty} f(x) = \lim_{x\to \infty}$ 1/x =0 , then the horizotal asymptote is a line y=0 .



Example: find the verticale asymptote and horizotal asymptote of f(x)=x+3 / x+2 if exist

Solution: Since $\lim_{x\to -2^+} x + 3 / x + 2 = \infty$ then the verticale asymptote of f(x)=1/x is a line x=-2 . Since $\lim_{x\to \infty} f(x) = \lim_{x\to \infty}$ x+3 / x+2 = 1 , then the horizotal asymptote is a line y=1 .



Example: find the verticale asymptote and horizotal asymptote of f(x)= $(2x^2+3)/x$

Solution: Since $\lim_{x\to 0^+} (2x^2+3)/x = \infty$ then the verticale asymptote of f(x)= $(2x^2+3)/x$ is a line x=0 .Since $\lim_{x\to \infty} f(x) = \lim_{x\to \infty} (2x^2+3)/x = \infty$, then f has not horizotal asymptote.

Example: find the verticale asymptote and horizotal asymptote of f(x)= x/(x-1)

Solution: Since $\lim_{x\to 1^+} x/(x-1) = \infty$ then the verticale asymptote of f(x)= x/(x-1)is a line x=1 .

Since $\lim_{x\to \infty} f(x) = \lim_{x\to \infty} x/(x-1) = 1$, then the horizotal asymptote is a line y=1 .

Definition:

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively.}$$

Continuity test:

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Example: discuss the continuity of f(x) = $\begin{matrix} x^2 & if\ x < 3 \\ x+6 & if\ x \geq 3 \end{matrix}$ at x=3 .

Solution:

(1) f(3) = 3+6 = 9

(2) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+6 = 9$

(3) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} x^2 = 9$

f(3) = $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$ Hence f is continuity at x=3

Example: discuss the continuity of f(x) = $\begin{matrix} (x^2-4)/(x-2) & if\ x \neq 2 \\ 4 & if\ x = 2 \end{matrix}$ at x=2 .

Solution:(1)f(2) = 4

(2) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-4)/(x-2) = \lim_{x\to 2^+}$ x+2 =4

(3) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2-4)/(x-2) = \lim_{x\to 2^-}$ x+2 = 4

f(3) = $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. Hence f is continuity at x=2

Example: discuss the continuity of f(x) = $(x^2-4)/(x-2)\ if\ x \neq 2$, at x=2

Solution:(1)f(2) is undefined

(2) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-4)/(x-2) = \lim_{x\to 2^+}$ x+2 =4

(3) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2-4)/(x-2) = \lim_{x\to 2^-}$ x+2 = 4

f has a limit at x=2 but not continuity at x=2 because f(2) is undefined

Example: discuss the continuity of f(x) = $\begin{matrix} 2/(x-1) & if\ x \neq 1 \\ 0 & if\ x = 1 \end{matrix}$, at x=1

Solution: (1) f(1) = 0

(2) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2/(x-1) = \infty$

(3) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2/(x-1) = -\infty$

Hence f is not continuity at x=1because the limit of f at x=1 does not exist.

Example: If f(x)= $\begin{matrix} ax-b & if\ x>1 \\ 2bx-a+1 & if\ x<1 \\ 1 & if\ x=1 \end{matrix}$ is a continuos funxtion at x=1 then find the value of a and b .

Solution: (1) f(1) = 1

(2) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax-b$ = a-b

(3) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2bx-a+1$ = 2b-a+1

Since f is a continuos funxtion at x=1 then a-b=2b-a+1=1 then

a-b=1 ….(1)

2b-a+1=1 … (2)

Solve equation (1) and (2) we get a=2 and b=1

Example: If f(x)= $\begin{matrix} ax+3 & if\ x\neq 1 \\ x & \text{if } x=1 \end{matrix}$ is a continuos funxtion at x=1 then find the value of a .

Solution: (1)f(1) = 1

(2) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax+3$ = a+3

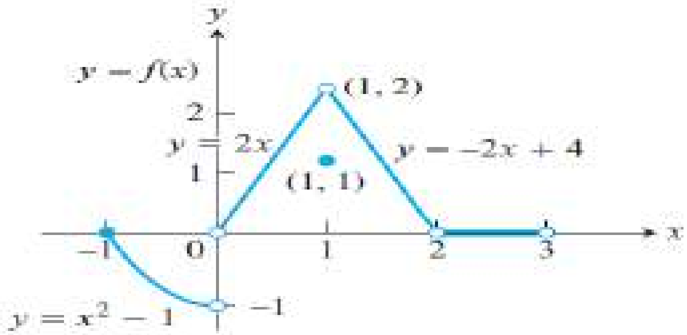
(3) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax+3$ = a+3

Since f is a continuos funxtion at x=1 then a+3=a+3=1 then a+3=1, hence a=-2

Example:

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



5. **a.** Does $f(-1)$ exist?
   b. Does $\lim_{x \to -1^+} f(x)$ exist?
   c. Does $\lim_{x \to -1^+} f(x) = f(-1)$?
   d. Is f continuous at $x = -1$?
6. **a.** Does $f(1)$ exist?
   b. Does $\lim_{x \to 1} f(x)$ exist?
   c. Does $\lim_{x \to 1} f(x) = f(1)$?
   d. Is f continuous at $x = 1$?
7. **a.** Is f defined at $x = 2$? (Look at the definition of f.)
   b. Is f continuous at $x = 2$?
8. At what values of x is f continuous?
9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?
10. To what new value should $f(1)$ be changed to remove the discontinuity?

Solution:

5. (a) Yes (b) Yes, $\lim_{x \to -1^+} f(x) = 0$

(c) Yes (d) Yes

6. (a) Yes, $f(1) = 1$ (b) Yes, $\lim_{x \to 1} f(x) = 2$

(c) No (d) No

7. (a) No (b) No

8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

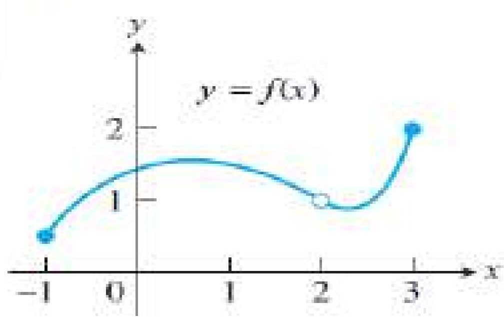
9. $f(2) = 0$, since $\lim_{x \to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x \to 1} f(x)$

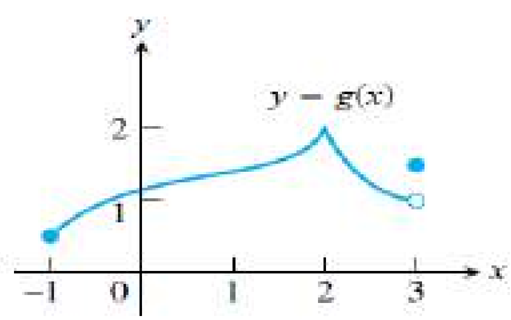
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

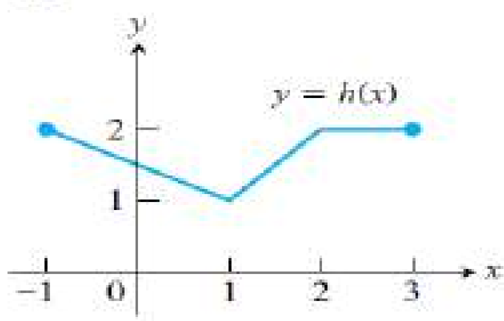
1.



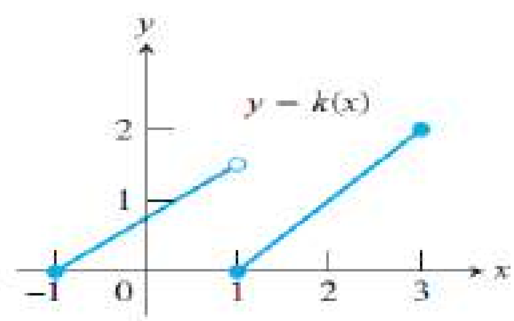
2.



3.



4.



Solution:

1. No, discontinuous at x = 2, not defined at x = 2

2. No, discontinuous at x = 3, $1 = \lim_{x \to 3^-} g(x) \neq g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at x = 1, $1.5 = \lim_{x \to 1^-} k(x) \neq \lim_{x \to 1^+} k(x) = 0$

Theorem:

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*:	$f + g$
2. *Differences*:	$f - g$
3. *Products*:	$f \cdot g$
4. *Constant multiples*:	$k \cdot f$, for any number k
5. *Quotients*:	f/g provided $g(c) \neq 0$
6. *Powers*:	$f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.