***1.1 Definition*** A set is collection of distinct objects. These objects are called the elements, or members.

## *Important Sets of Real Numbers*

The set $\mathbb{N}$ of natural numbers is define by

$\mathbb{N} = \{1, 2, 3, \ldots\}$

The set $\mathbb{Z}$ of integers is define by

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

The set $\mathbb{Q}$ of rational numbers is define by

$\mathbb{Q} = \{a/b: a, b \in \mathbb{Z}, b \neq 0\}$.

Non-periodic decimal fractions are called irrational numbers and denoted by *Irr*.

*For example,* $\sqrt{2}, \sqrt{3}, \pi$

## *Real numbers*

Real Numbers are made up of rational numbers and irrational numbers and denoted by $\mathbb{R}$.

## *The Number Line*

We may use the number line to represent all the real numbers graphically; each real number corresponds to exactly one point on the number line. $\infty$ and $-\infty$ are not real numbers because there is no point on the number line corresponding to either of them.

$\mathbb{C}$, denoting the set of all complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

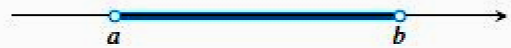
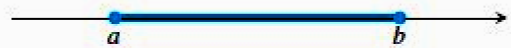
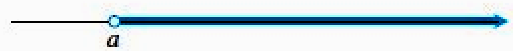
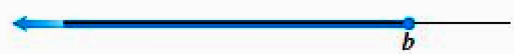
*For example,* $1 + 2i \in \mathbb{C}$.

## *Intervals*

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements.

## *Types of intervals*

**TABLE 1.1** Types of intervals

| | Notation | Set description | Type | Picture |
|---|---|---|---|---|
| **Finite:** | $(a, b)$ | $\{x \mid a < x < b\}$ | Open |  |
| | $[a, b]$ | $\{x \mid a \le x \le b\}$ | Closed |  |
| | $[a, b)$ | $\{x \mid a \le x < b\}$ | Half-open |  |
| | $(a, b]$ | $\{x \mid a < x \le b\}$ | Half-open |  |
| **Infinite:** | $(a, \infty)$ | $\{x \mid x > a\}$ | Open |  |
| | $[a, \infty)$ | $\{x \mid x \ge a\}$ | Closed |  |
| | $(-\infty, b)$ | $\{x \mid x < b\}$ | Open |  |
| | $(-\infty, b]$ | $\{x \mid x \le b\}$ | Closed |  |
| | $(-\infty, \infty)$ | $\mathbb{R}$ (set of all real numbers) | Both open and closed |  |

## *Absolute value*

***1.11 Definition*** The ***absolute value*** of a number $x$, denoted by is defined by the formula

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}.$$

***1.12 Example*** $|2| = 2$, $|-5| = -(-5) = 5$.

## *Some properties of the absolute value*

Let $a, b$ and $x$ be any real numbers then:

1) $|x| = \sqrt{x^2}$
2) $|ab| = |a||b|$
3) $|a+b| \leq |a| + |b|$
4) $|a-b| \geq ||a| - |b||$
5) $|x| \leq a$ if and only if $x \leq a$ and $x \geq -a$ $(\,or\, -a \leq x \leq a)$
6) $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

## *Cartesian product*

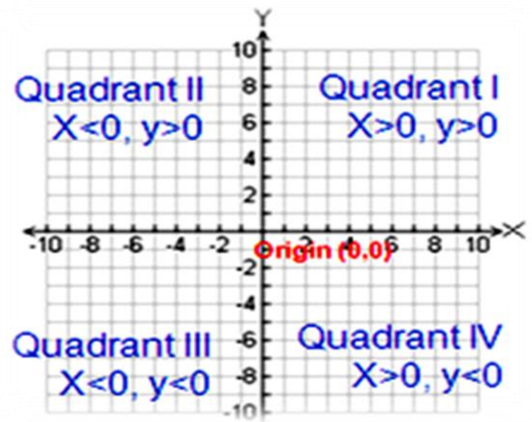
***Definition*** Let $A$ and $B$ be any two non empty sets, the Cartesian product of $A$ with $B$ denoted by $A \times B$ is defined by

$A \times B = \{(a, b): a \in \mathrm{A} \;\; and \; b \in \mathrm{B}\}$

$B \times A = \{(a, b): a \in \mathrm{B} \;\; and \; b \in \mathrm{A}\}$

***Remark***

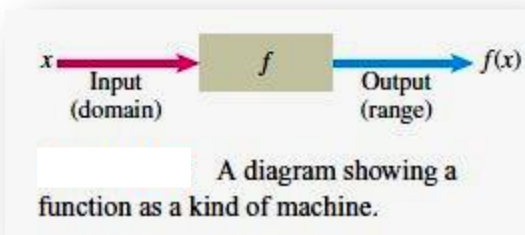
1) $A \times B \neq B \times A$
2) $A \times B = B \times A$ iff $A = B$
3) If $A$ contains $m$ elements and $B$ contains $n$ elements, then $A \times B$ contains $m \times n$ elements.
4) $A \times B = \phi$ iff $A = \phi$ or $B = \phi$
5) The Cartesian product of $\mathbb{R}$ with itself is $\mathbb{R} \times \mathbb{R}$ denoted by $\mathbb{R}^2$, $\mathbb{R}^2 = \{(x, y): x, y \in \mathbb{R}\}$, $\mathbb{R}^2$ denotes the Cartesian plane.




## ***The Function***

***Definition*** Let $A$ and $B$ be any two non empty sets then a function (denoted by $f$) from $A$ to $B$ is a relation from $A$ to $B$ provided that for each $x \in \mathrm{A}$ there exist only a unique $y \in \mathrm{B}$ such that $(x, y) \in f$ and $f$ can be written as:

$$f: A \to B,\ y = f(x) \text{ or } y \overset{f}{\to} x.$$




A diagram showing a function as a kind of machine.

***Example*** Let $f: \mathbb{R} \to \mathbb{R}$. Does the following $f$ are functions or not?

**1)** $f(x) = \sqrt{x}$ **2)** $f(x) = x^2$ **3)** $f(x) = 3$

***Solution:***

**1.** Is not a function because $\sqrt{-1}$ is undefined**.**

**2.** Is a function since for all $x$ there exist $y$ such that $(x, y) \in f$.

**3.** is a function since for all $x$ there exist $y$ such that $(x, y) \in f$.

***Example*** Let $f: \mathbb{R}^{+} \to \mathbb{R},\ x = y^2$. Is not a function because

$$(4, -2) \in f \text{ and } (4, 2) \in f$$

***Example*** Let $f: \mathbb{R}^{+} \to \mathbb{R}^{+},\ x = y^2$. Is a function

***Definitions***

**1)** The set $A$ of all possible input values is called the ***domain*** of the function.

That means $D_f = \{x: x \in A \text{ and } y = f(x) \text{ for a unique } y \in B\}$

**2)** The set of all values of $f(x)$ as $x$ varies throughout $A$ is called the ***range*** of the function, i.e. $R_f = \{y: y \in B \text{ and } y = f(x) \text{ for at least one } x \in A\}$

The range may not include every element in the set $Y$.

***Examples*** Find the domains and ranges of these functions.

| Function | Domain ($x$) | Range ($y$) |
|---|---|---|
| $y = x^2$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $y = 1/x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $y = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $y = \sqrt{4 - x}$ | $(-\infty, 4]$ | $[0, \infty)$ |
| $y = \sqrt{1 - x^2}$ | $[-1, 1]$ | $[0, 1]$ |

**Solution** The formula $y = x^2$ gives a real $y$-value for any real number $x$, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number $y$ is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real $y$-value for every $x$ except $x = 0$. *We cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$.

The formula $y = \sqrt{x}$ gives a real $y$-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real $y$-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real $y$-value for every $x$ in the closed interval from $-1$ to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

***H.W.*** Find the domains and ranges of these functions.

**1)** $y = f(x) = \frac{x+3}{x^3+1}$ **2)** $y = f(x) = \sqrt{\frac{x+1}{x}}$

***Definition (The Graph of Function)***

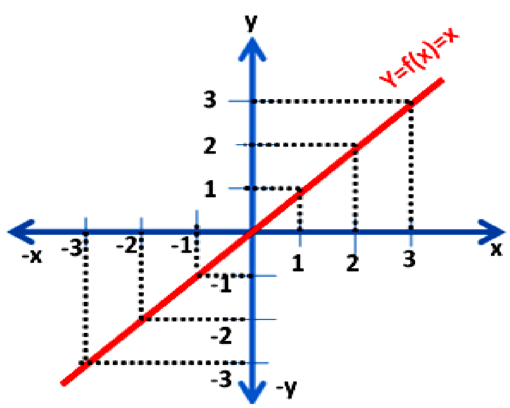
The graph of the function $y = f(x)$ is the set of all points $(x, y)$ in the Cartesian plane $X \times Y$ such that $(x, y)$ satisfies the function $y = f(x)$.

That means the graph is $\{(x, y): y = f(x)\}$.

***Example*** Find the graph of this function $y = f(x) = x$

***Solution:***

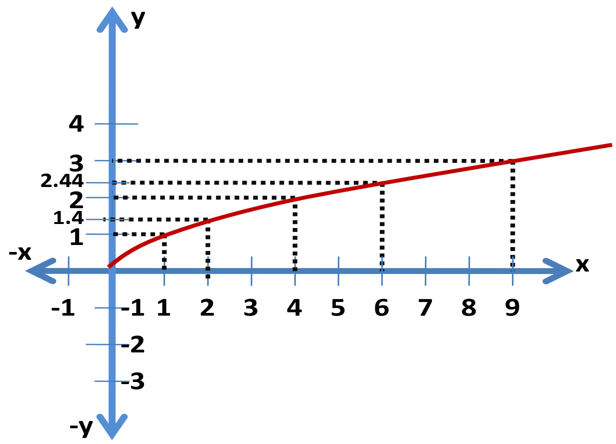
| $x$ | 1 | 2 | 3 | 0 | -1 | -2 | -3 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | 1 | 2 | 3 | 0 | -1 | -2 | -3 |
| $(x, y)$ | $(1, 1)$ | $(2, 2)$ | $(3, 3)$ | $(0, 0)$ | $(-1, -1)$ | $(-2, -2)$ | $(-3, -3)$ |



***1.34 Example :*** Graph this function $y = f(x) = \sqrt{x}$

***Solution:***

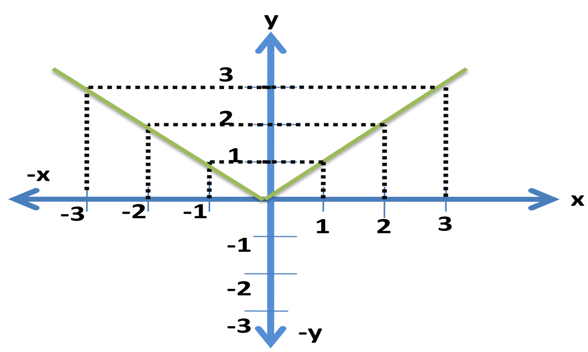
| x | y=f(x) | (x, y) |
|---|---|---|
| **0** | **0** | **(0,0)** |
| **1** | **1** | **(1,1)** |
| **2** | **1.4** | **(2,1.4)** |
| **4** | **2** | **(4,2)** |
| **6** | **2.44** | **(6,2.44)** |
| **9** | **3** | **(9,3)** |
| **0** | **0** | **(0,0)** |




***1.35 Example :*** graph of this function $y = f(x) = |x|$

***Solution:***

| x | y=f(x) | (x, y) |
|---|---|---|
| 1 | 1 | (1,1) |
| 2 | 2 | (2,2) |
| 3 | 3 | (3,3) |
| 0 | 0 | (0,0) |
| -1 | 1 | (-1,1) |
| -2 | 2 | (-2,2) |
| -3 | 3 | (-3,3) |



***Note*** Let $f(x)$ and $g(x)$ be two functions having $D_f$ and $D_g$ as a domain respectively. Then

**1)** $D_{f+g} = D_{f-g} = D_{f\times g} = D_f \cap D_g$

**2)** $D_{f/g} = D_f \cap D_g - \{x: g(x) = 0\}$

***Example*** Find the domain for the function $K(x) = \frac{x+1}{[\![x]\!]-1}$

***Solution:*** Let $f(x) = x + 1$ and $g(x) = [\![x]\!] - 1$

$D_f = \mathbb{R}$ and $D_g = \mathbb{R}$

$\Longrightarrow D_K = \mathbb{R} \cap \mathbb{R} - \{x: [\![x]\!] - 1 = 0\} = \mathbb{R} - \{x: [\![x]\!] = 1\} = \mathbb{R} - [1,2)$

***Definition Type of Functions***

**1)** ***Constant function*** $y = f(x) = c$ where $c \in \mathbb{R}$ is called the constant function.

**2)** ***Identity function*** $y = f(x) = x$ is called the identity function.

**3)** ***Polynomial function*** $y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_i \in \mathbb{R}$, $i = 0,1,\dots,n$. For $a_0 \neq 0$ and $n \geq 0$ an integer is called a polynomial of degree $n$.

$$f(x) = x^6 + x - 6.$$

**4)** ***Even function*** The function $y = f(x)$ is called an even function if

$f(-x) = f(x) \ \forall \ x \in D_f$.

For example $y = f(x) = x^2 + 1$ is an even function.

**5)** ***Odd function*** The function $y = f(x)$ is called an even function if

$f(-x) = -f(x) \ \forall \ x \in D_f$.

For example $y = f(x) = x^3$ is an odd function.