Trigonometric Function

Consider the circle (see Figure 2.1)

$$x^2 + y^2 = r^2, \quad r > 0$$

We define

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

$$\Rightarrow y = r\sin\theta \quad \text{and} \quad x = r\cos\theta$$

Now, since $x^2 + y^2 = r^2$

$$\Rightarrow r^2\sin^2\theta + r^2\cos^2\theta = r^2, \quad r \neq 0$$

$$\Rightarrow \quad \boxed{\sin^2\theta + \cos^2\theta = 1} \quad \dots (1)$$

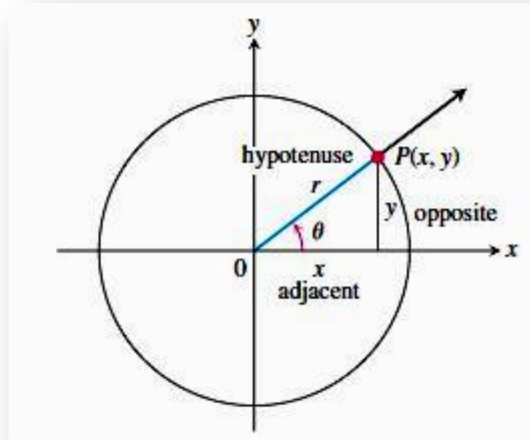


Figure 2.1

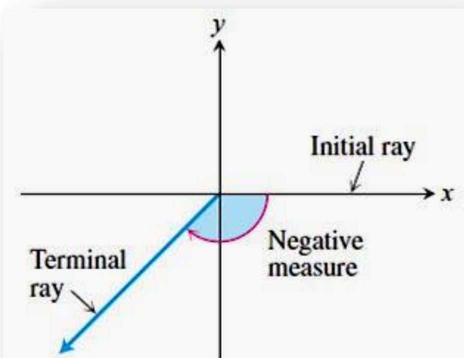


Figure 2.2

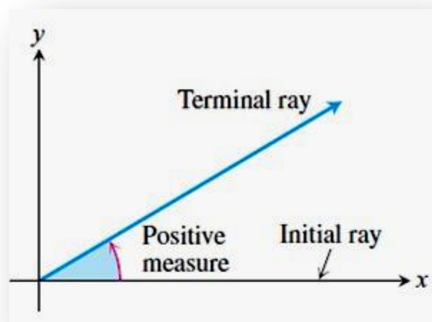


Figure 2.3

Some Important Identities

1. $\sin(\alpha \mp \beta) = \sin\alpha\cos\beta \mp \sin\beta\cos\alpha$

2. $\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

3. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

4. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

5. $\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$

6. $\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$

7. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

8. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

9. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

10. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$

11. $\sin\alpha - \sin\beta = 2\ \cos\frac{\alpha+\beta}{2}\ \sin\frac{\alpha-\beta}{2}$

12. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$

13. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\ \sin\frac{\alpha-\beta}{2}$

14. $\sin 2\alpha = 2\sin\alpha\ \cos\beta$

15. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

16. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

1. Sine Function

It is a function $f\colon\mathbb{R}\to[-1,1]$ defined by $f(x) = \sin x$.

$\sin x = 0$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots = n\pi, n\in\mathbb{Z}$.

$\sin x = \mp 1$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots = \left(n+\frac{1}{2}\right)\pi, n\in\mathbb{Z}$.

Since $\sin(-x) = -\sin x$, therefore the sine is an odd function.

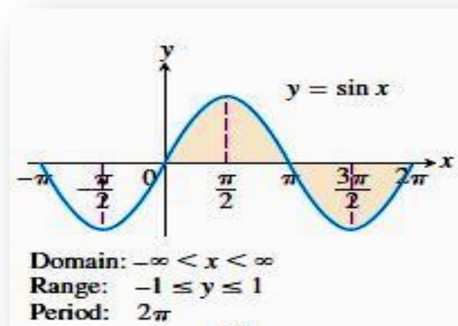


Figure 2.5

2. Cosine Function

It is a function $f\colon\mathbb{R}\to[-1,1]$ defined by $f(x) = \cos x$.

$\cos x = 0$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots = \left(n+\frac{1}{2}\right)\pi, n\in\mathbb{Z}$.

$\cos x = \mp 1$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots = n\pi, n\in\mathbb{Z}$

Since $\cos(-x) = \cos x$, therefore the cosine is an even function.

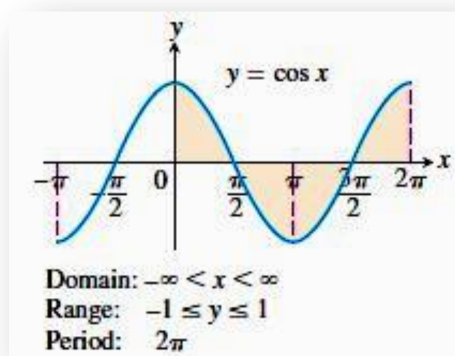


Figure 2.6

It's a one to one function if $0 \le x \le \pi$ or $-\pi \le x \le 0$. Therefore $\cos\colon[0,\pi]\to[-1,1]$ is a bijection function.

3. Tangent Function

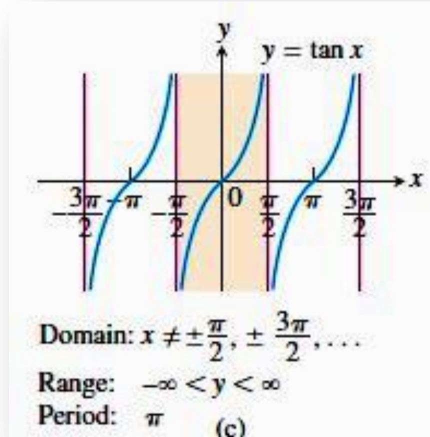


Figure 2.7

It is defined by $\tan: D_{\tan} \to \mathbb{R}$,

$f(x) = \tan x = \frac{\sin x}{\cos x}, \cos x \neq 0.$

$\cos x = 0$ if $x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$,

$$D_{\tan} = \mathbb{R}/\{x: x = \left(n + \tfrac{1}{2}\right)\pi, n \in \mathbb{Z}\}$$

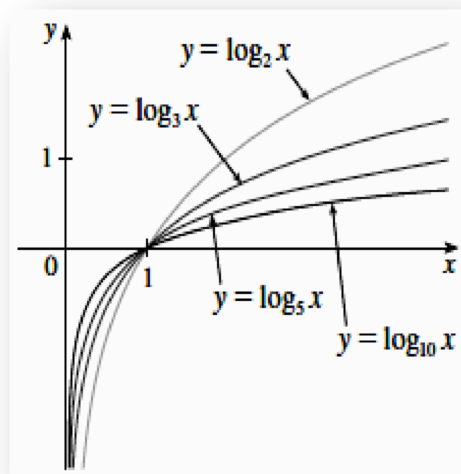
$$= \{x \in \mathbb{R}: x \neq \left(n + \tfrac{1}{2}\right)\pi, n \in \mathbb{Z}\}.$$

$\tan x = 0$ if $\sin x = 0 \implies x = n\pi, n \in \mathbb{Z}$.

Logarithmic Functions

1. The General Logarithmic Functions

Let $a > 0$, $a \neq 1$ be any real number, the general logarithmic from $\text{Log}_a: \mathbb{R}^+ \to \mathbb{R}$ where a is the base of logarithmic function.



Some Properteis

If x and y are positive numbers, then

1. $\text{Log}_a(xy) = \text{Log}_a x + \text{Log}_a y$

2. $\text{Log}_a\left(\frac{x}{y}\right) = \text{Log}_a x - \text{Log}_a y$

3. $\text{Log}_a x^r = r\text{Log}_a x$ (where r is any real number)

4. $\text{Log}_a 1 = 0$

5. $\text{Log}_a a = 1$

6. For $0 < x < 1$, $\text{Log}_a x < 0$

7. For $x \geq 1$, $\text{Log}_a x \geq 0$

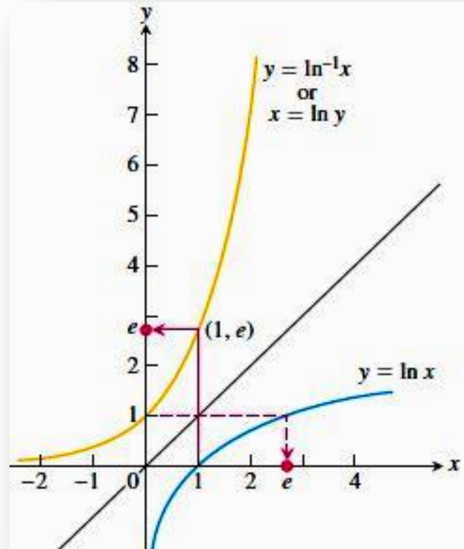
8. $\lim_{x \to 0^+} \text{Log}_a x = -\infty, \lim_{x \to \infty} \text{Log}_a x = \infty$

-If $a = 10$, then we denote this function by $f(x) = \text{Log}\, x$.

-If $a = e$ (e is the Euler's number and $e = 2.718281828 \ldots$),

we denote this function by $f(x) = \ln x$ and it is called the natural logarithmic function.

2. The Natural Logarithmic Functions



It is the logarithmic function with the base $a = e$.

$i.e.$ $f(x) = \text{Log}_a x = \ln x$, $\ln: \mathbb{R}^+ \to \mathbb{R}$.

Some Properteis

If x and y are positive numbers, then

1. $\ln(xy) = \ln x + \ln y$

2. $\frac{\ln x}{\ln y} = \ln x - \ln y$

3. $\ln 1 = 0, \ln e = 1$

4. $\ln x^r = r \ln x$

5. $\ln \frac{1}{a} = \ln a^{-1} = -\ln a$

6. For $0 < x < 1$, $\ln x < 0$

7. For $x \geq 1, \ln x \geq 0$

8. $\lim_{x \to 0^+} \ln x = -\infty, \lim_{x \to \infty} \ln x = \infty$

9. It is one to one and onto function, so it is bijective function.

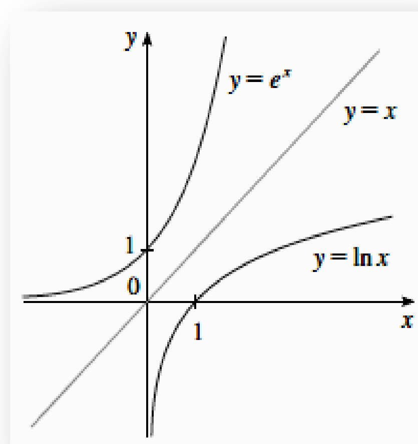
Exponential Functions

1. The Natural Exponential Functions

Since the natural logarithmic function is a bijective function, so it has an inverse, which's the natural exponential, hence exp: $\mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x, \forall x \in \mathbb{R}$.

Some Properteis

The natural exponential e^x obeys the following laws:



1. $e^x \, e^y = e^{x+y} \;\; \forall \, x, y \in \mathbb{R}$

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^x}{e^y} = e^{x-y}$

4. $(e^x)^y = e^{xy} = (e^y)^x$

5. $e^0 = 1$

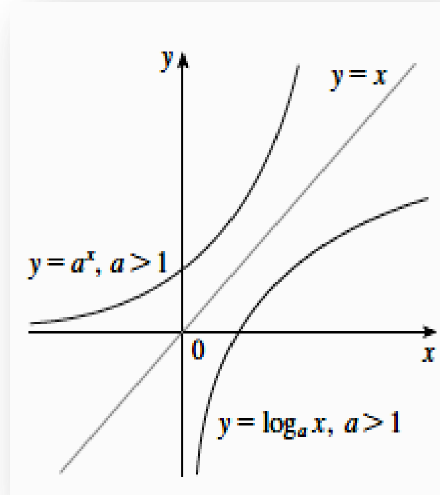
6. $e^{\ln x} = x$

7. $\ln e^x = x$

2. The General Exponential Functions

It is defined $f: \mathbb{R} \to \mathbb{R}^+$ by

$$f(x) = a^x, \forall x \in \mathbb{R}, a > 0, a \neq 1$$



It is the inverse function of the logarithmic function.

Since $e^{\ln x} = x \implies e^{\ln a} = a$

$$\implies \left(e^{\ln a}\right)^x = a^x$$

$$\implies \boxed{a^x = e^{x \ln a}}$$

$$\boxed{\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a}}$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$