Introduction to 3D Elasticity

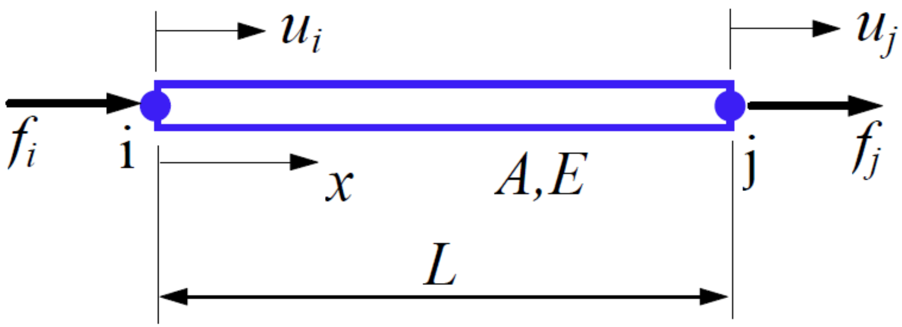
Summary:

- 1D elasticity (Bar Element)
- 3D elasticity problem
  - Governing differential equation
  - Strain-displacement relationship
  - Stress-strain relationship
- Special cases

  2D (plane stress, plane strain)

  Axisymmetric body with axisymmetric loading

1D Elasticity (Bar Element)



L = length
A = cross-sectional area
E = elastic modulus
u = u(x) displacement
ε = ε(x) strain
σ = σ(x) stress

Strain-displacement relation:

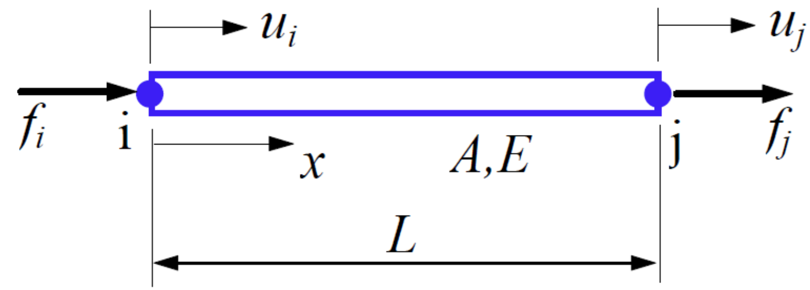
$$\varepsilon = \frac{du}{dx}$$

Stress-strain relation:

$$\sigma = E\varepsilon$$

Assuming that the displacement u is varying linearly along the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j$$



we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \; (\Delta = \text{ elongation }), \qquad \sigma = E\varepsilon = \frac{E\Delta}{L}$$

We also have

$$\sigma = \frac{F}{A} \; (F = \text{ force in bar })$$

Thus

$$F = \frac{EA}{L} \Delta = k\Delta$$

The bar is acting like a spring in this case, and we conclude that element stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix}$$

or

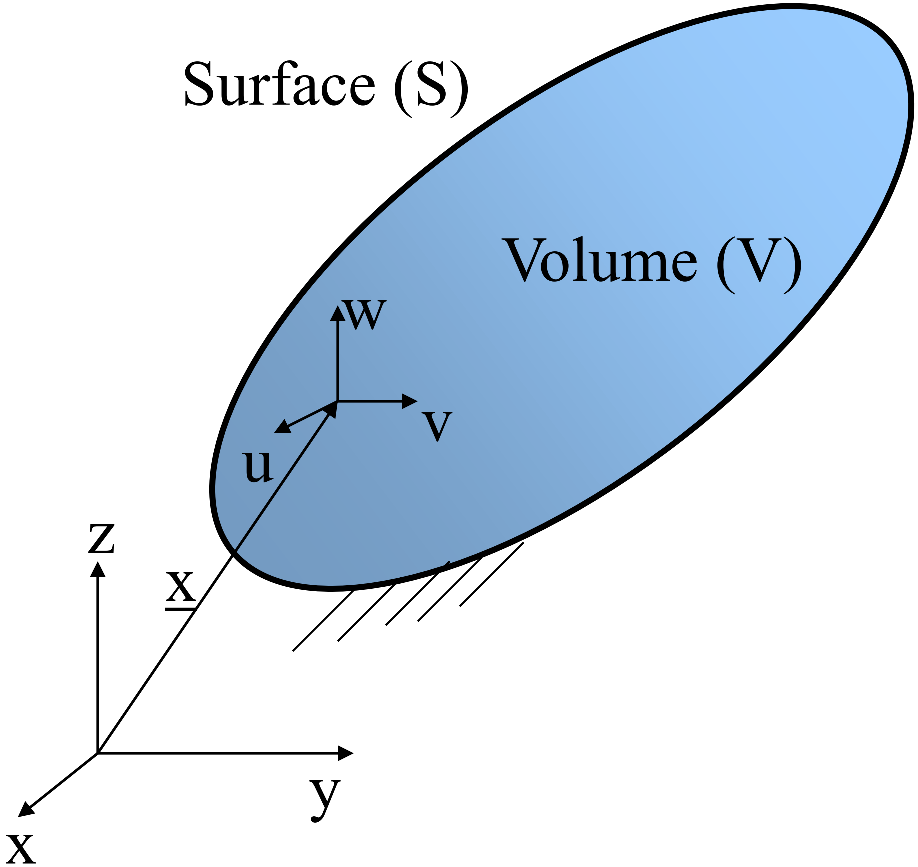
$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This can be verified by considering the equilibrium of the forces at the two nodes.

Element equilibrium equation is $$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

3D Elasticity

Problem definition



V: Volume of body

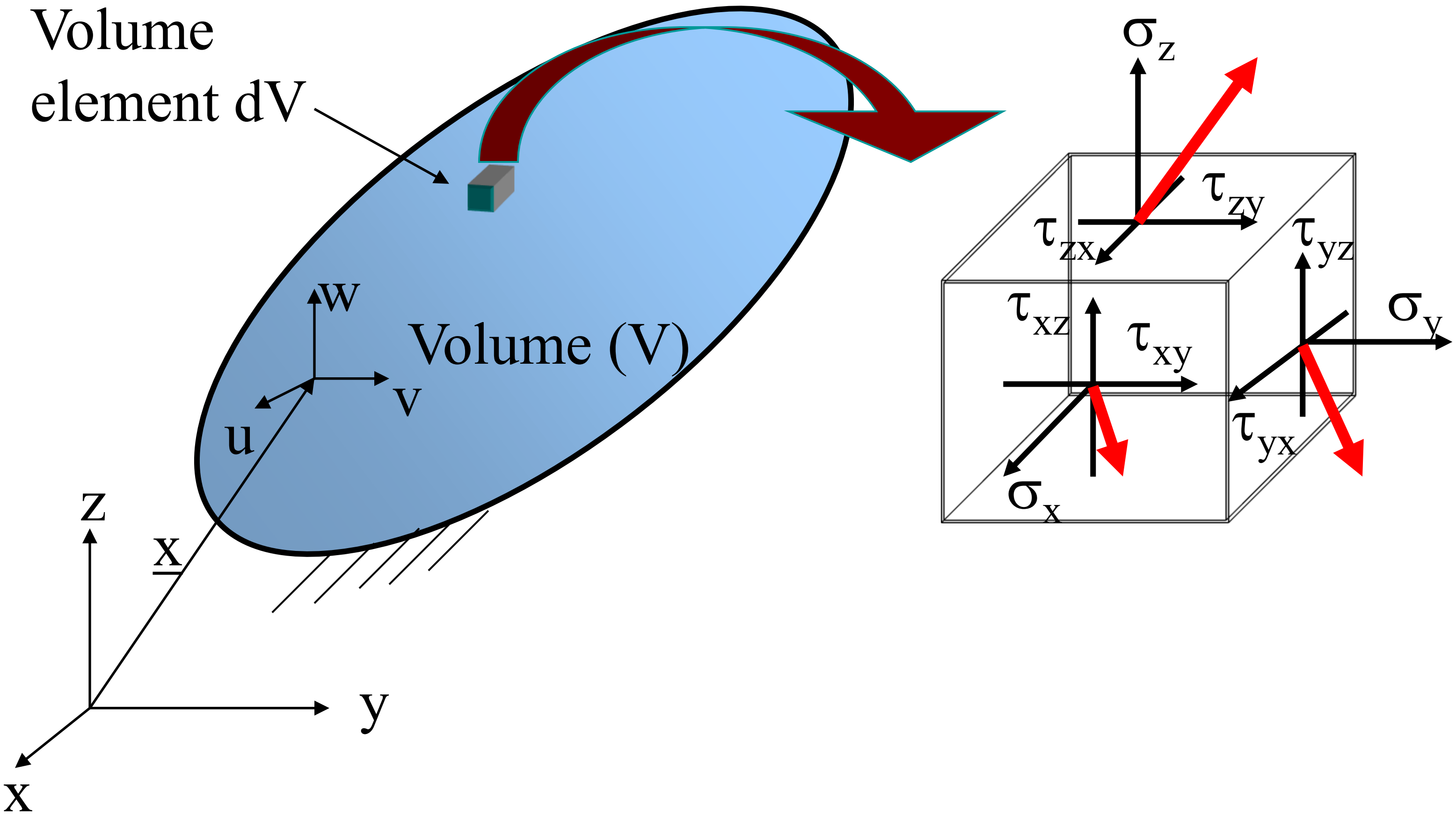
S: Total surface of the body

The deformation at point $\underline{x} = [x,y,z]^T$ is given by the 3 components of its displacement

$$\underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

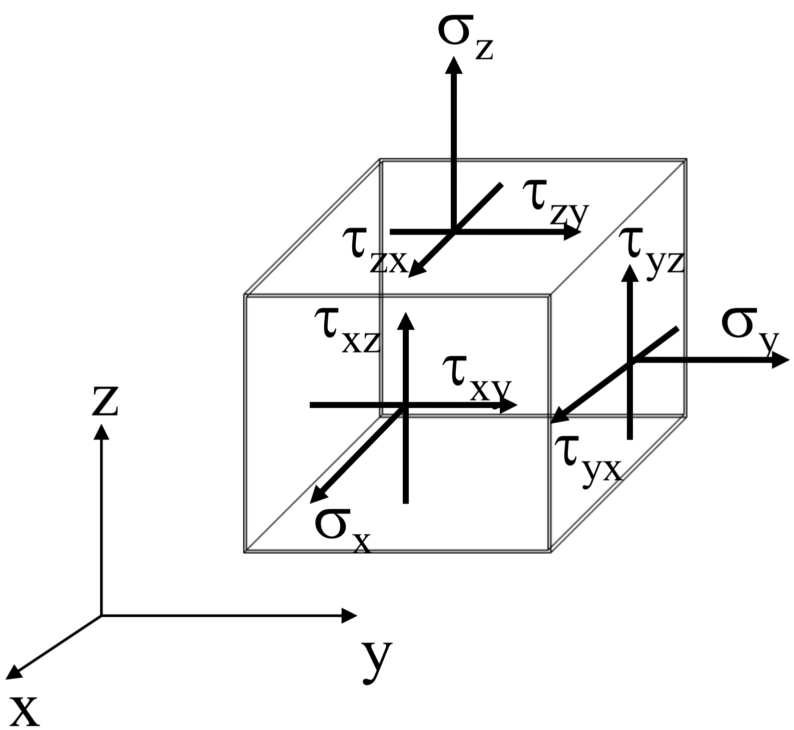
NOTE: $\underline{u} = \underline{u}(x,y,z)$, i.e., each displacement component is a function of position

3D Elasticity: INTERNAL FORCES



If we take out a piece of material from the body, we will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the **internal reaction forces** **per unit area** (**red arrows**) , on each surface, may be decomposed into three orthogonal components.

3D Elasticity



σ_x, σ_y and σ_z are **normal stresses**.

The rest 6 are the **shear stresses**

Convention

τ_{xy} is the stress on the face perpendicular to the x-axis and points in the +ve y direction

Total of 9 stress components of which only 6 are independent since

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{zx} = \tau_{xz}$$

The **stress vector** is therefore

$$\underline{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

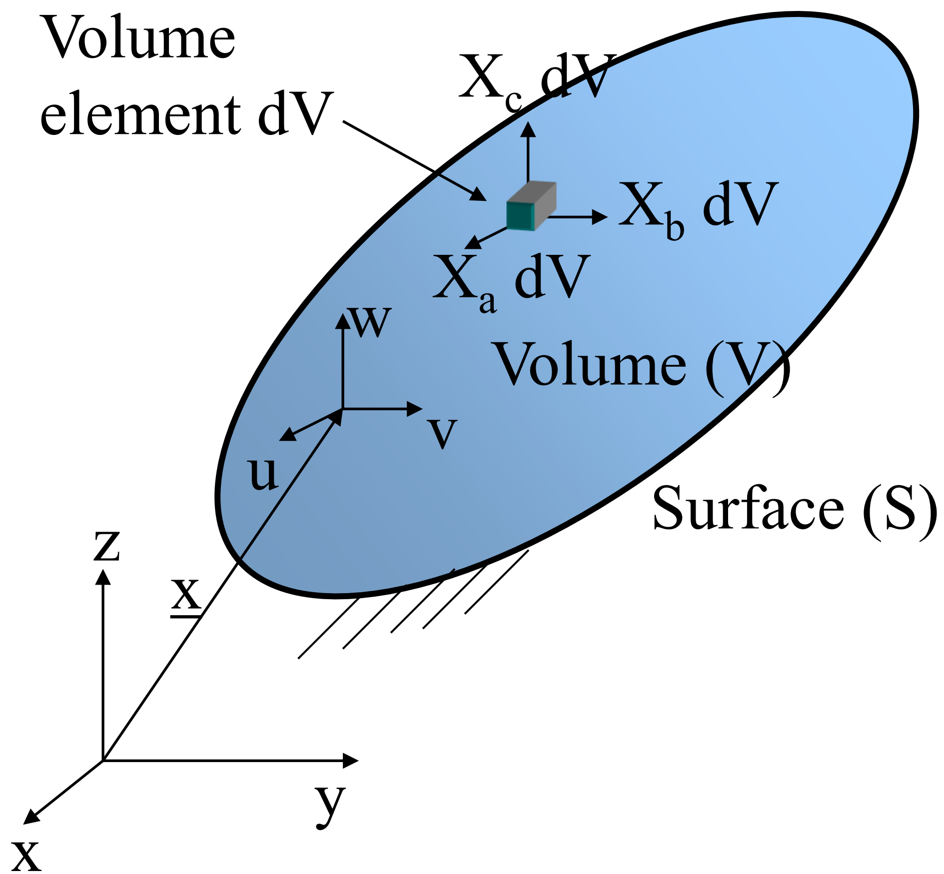
6 independent **strain components**

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

BODY FORCE

EXTERNAL FORCES ACTING ON THE BODY

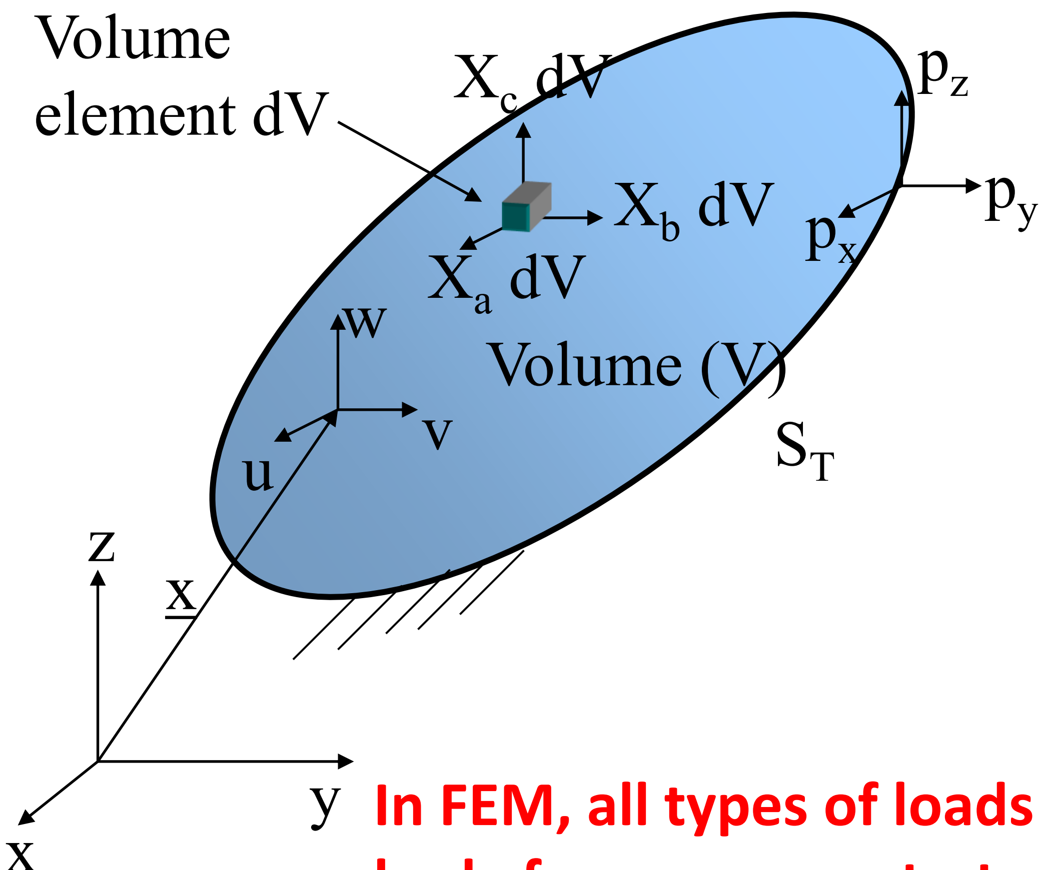
1. Body force: distributed force per unit volume (e.g., weight, inertia, etc)



$$\underline{X} = \begin{Bmatrix} X_a \\ X_b \\ X_c \end{Bmatrix}$$

SURFACE TRACTION

2. Surface traction (force per unit **surface area**) e.g., friction



$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

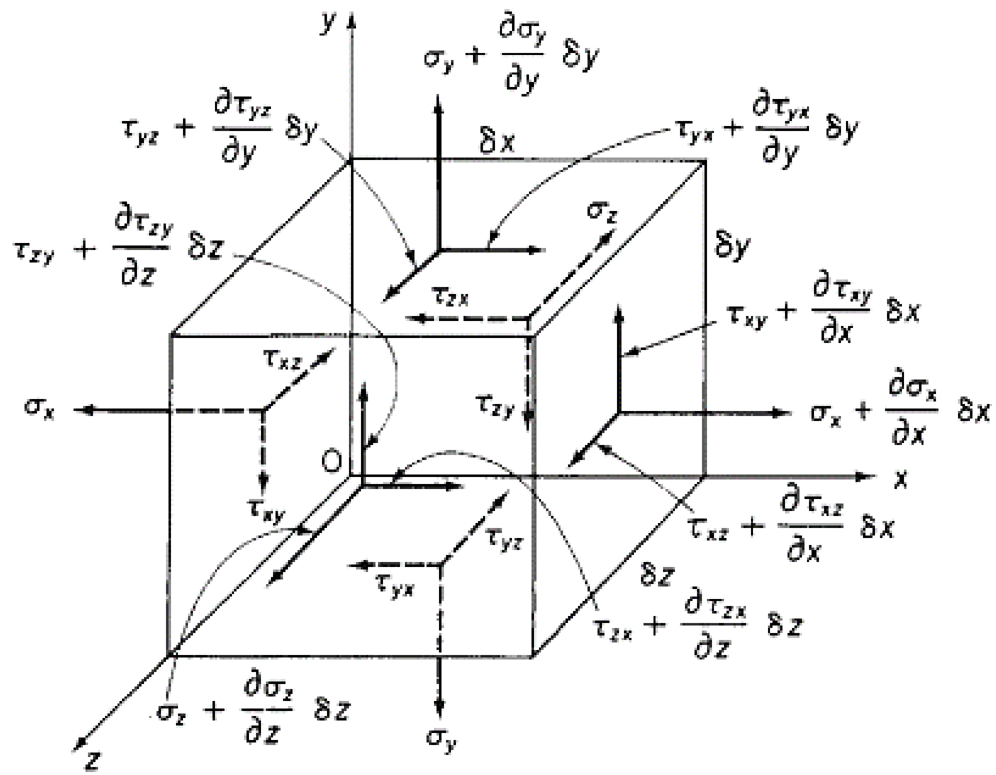
In FEM, all types of loads (distributed surface loads, body forces, concentrated forces and moments, etc.) are converted to point forces acting at the nodes.

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$



Compactly;

EQUILIBRIUM EQUATIONS

$$\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0} \qquad (1)$$

where

$$\underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

Strain-displacement relationships in 3D elasticity problem :

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

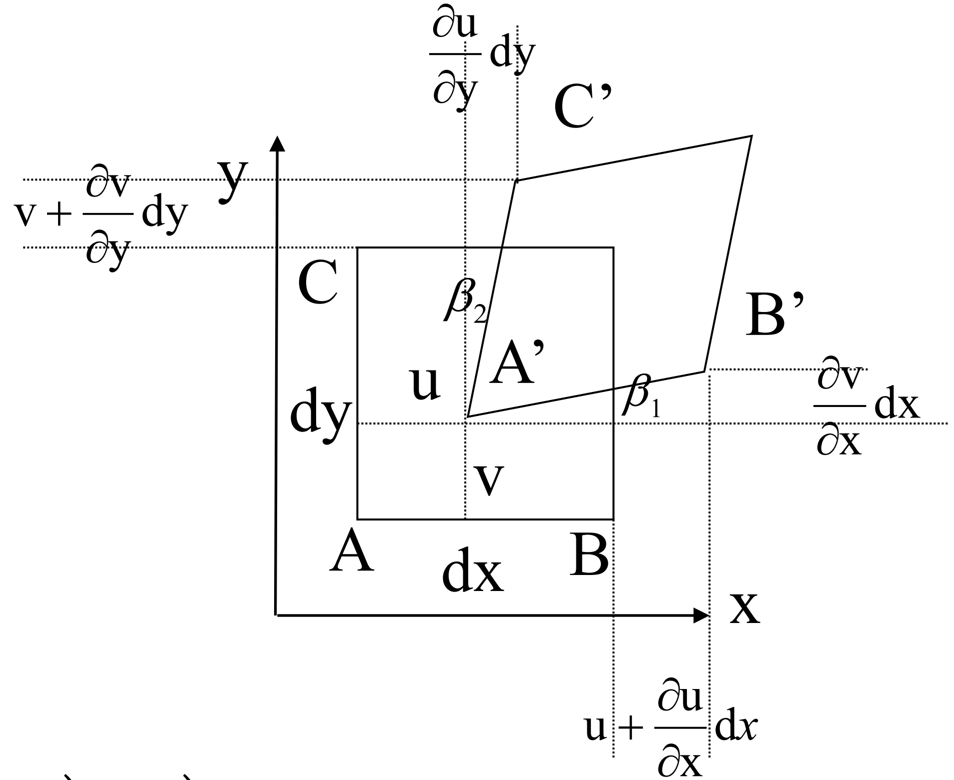
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Compactly; $\boxed{\underline{\varepsilon} = \underline{\partial}\ \underline{u}}$ (2)

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{\mathrm{u}} = \begin{Bmatrix} \mathrm{u} \\ \mathrm{v} \\ \mathrm{w} \end{Bmatrix}$$

<u>In 2D</u>



$$\boldsymbol{\varepsilon_x} = \frac{\mathrm{A'B'} - \mathrm{AB}}{\mathrm{AB}} = \frac{\left(\mathrm{dx} + \left(\mathrm{u} + \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\mathrm{dx}\right) - \mathrm{u}\right) - \mathrm{dx}}{\mathrm{dx}} = \frac{\boldsymbol{\partial \mathrm{u}}}{\boldsymbol{\partial \mathrm{x}}}$$

$$\boldsymbol{\varepsilon_y} = \frac{\mathrm{A'C'} - \mathrm{AC}}{\mathrm{AC}} = \frac{\left(\mathrm{dy} + \left(\mathrm{v} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\mathrm{dy}\right) - \mathrm{v}\right) - \mathrm{dy}}{\mathrm{dy}} = \frac{\boldsymbol{\partial \mathrm{v}}}{\boldsymbol{\partial \mathrm{y}}}$$

$$\boldsymbol{\gamma_{xy}} = \frac{\pi}{2} - \text{ angle } (\mathrm{C'A'B'}) = \beta_1 + \beta_2 \approx \tan\beta_1 + \tan\beta_2$$

$$\approx \frac{\boldsymbol{\partial \mathrm{v}}}{\boldsymbol{\partial \mathrm{x}}} + \frac{\boldsymbol{\partial \mathrm{u}}}{\boldsymbol{\partial \mathrm{x}}}$$

Stress-Strain relationship in 3D elasticity problem:

Linear elastic material (Hooke's Law)

$$\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}} \qquad (3)$$

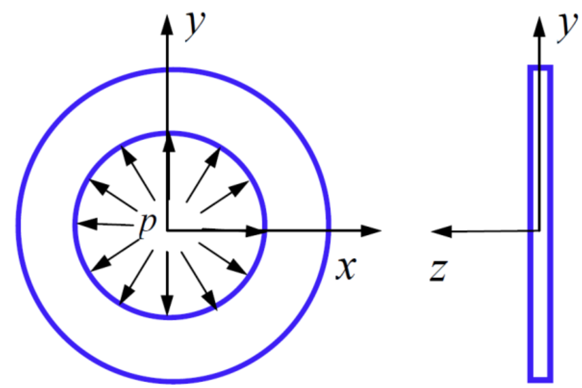
Linear elastic isotropic material

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Plane Stress and Plane Strain

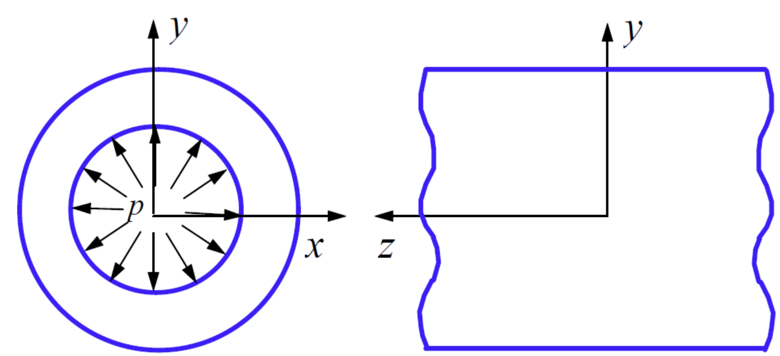
Plane stress is defined to be a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero.

$$\boldsymbol{\sigma_z = \tau_{xz} = \tau_{yz} = 0}$$



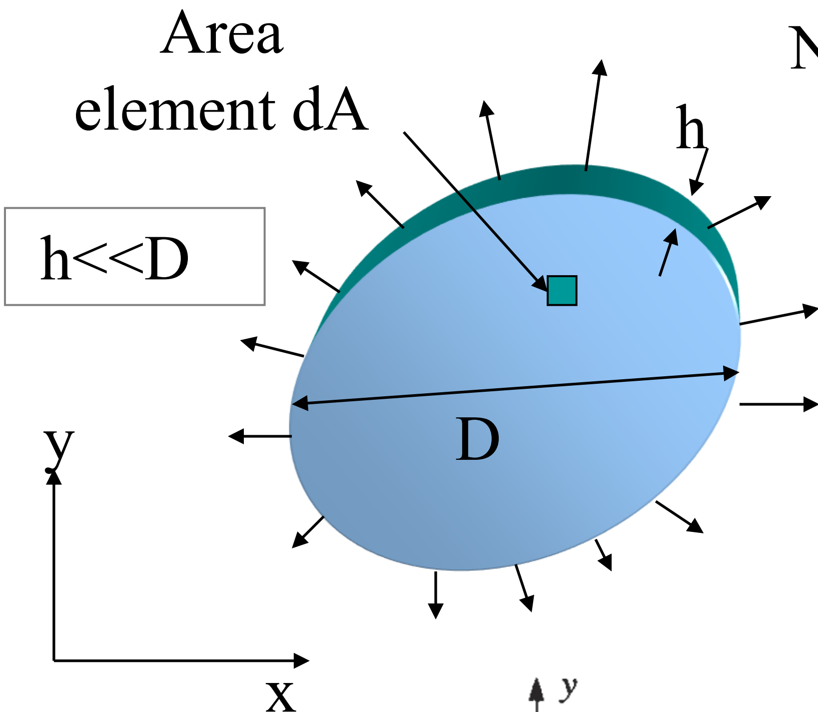
Plane strain is defined to be a state of strain in which the strain normal to the x – y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.

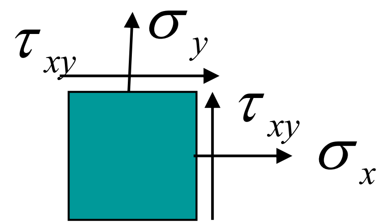
$$\boldsymbol{\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0}$$



<u>PLANE STRESS</u>: Only the in-plane <u>stress</u> components are nonzero

Non-zero <u>stress</u> components $\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}, \boldsymbol{\tau_{xy}}$





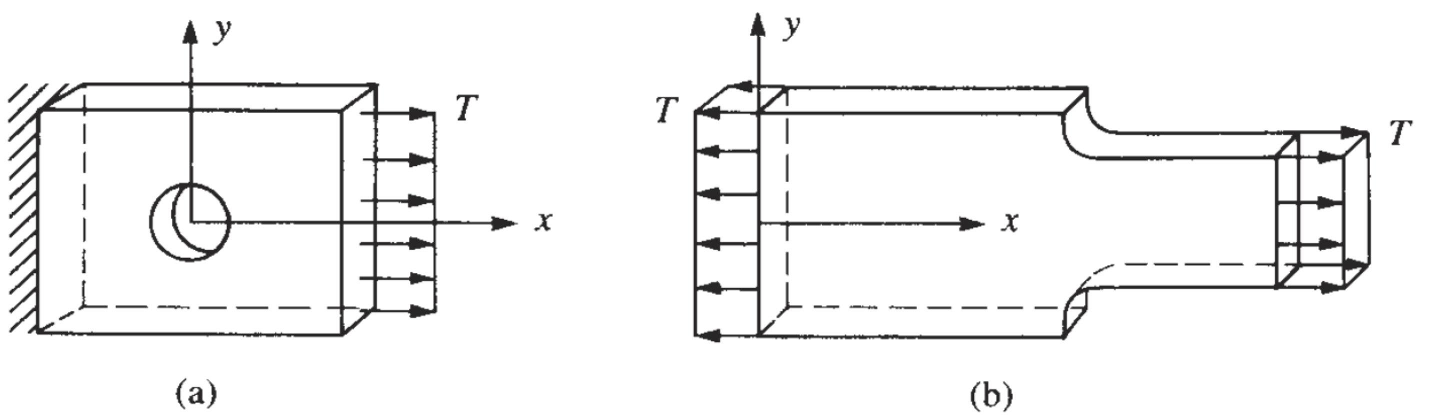
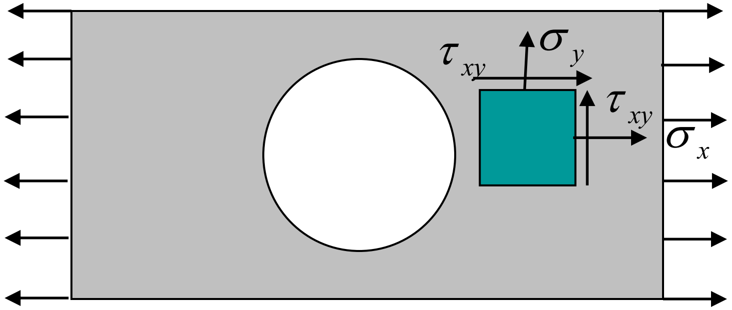


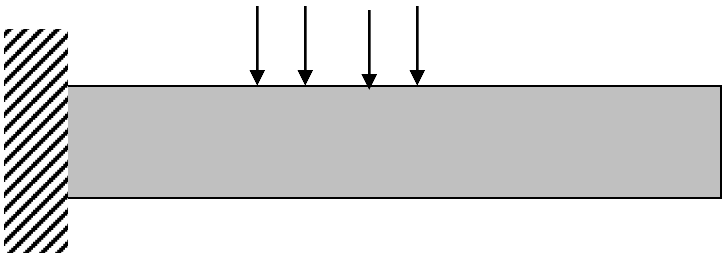
Figure 6–1 Plane stress problems: (a) plate with hole; (b) plate with fillet

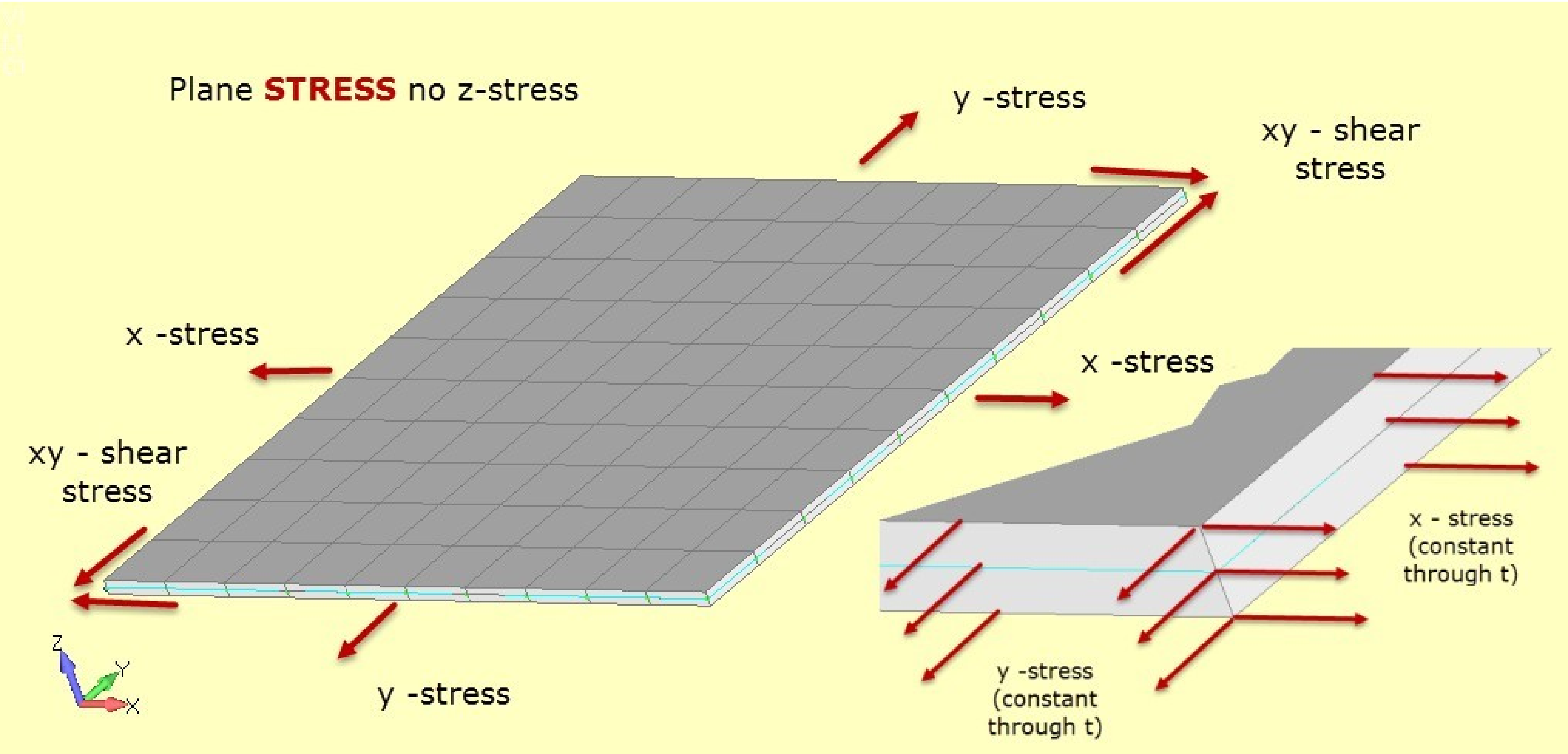
PLANE STRESS Examples:

1. Thin plate with a hole



2. Thin cantilever plate





Plane STRESS no z-stress
y -stress
xy - shear
stress
x -stress
x -stress
xy - shear
stress
x - stress
(constant
through t)
y -stress
y -stress
(constant
through t)
Z
Y
X

PLANE STRESS

Non-zero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$

Non-zero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

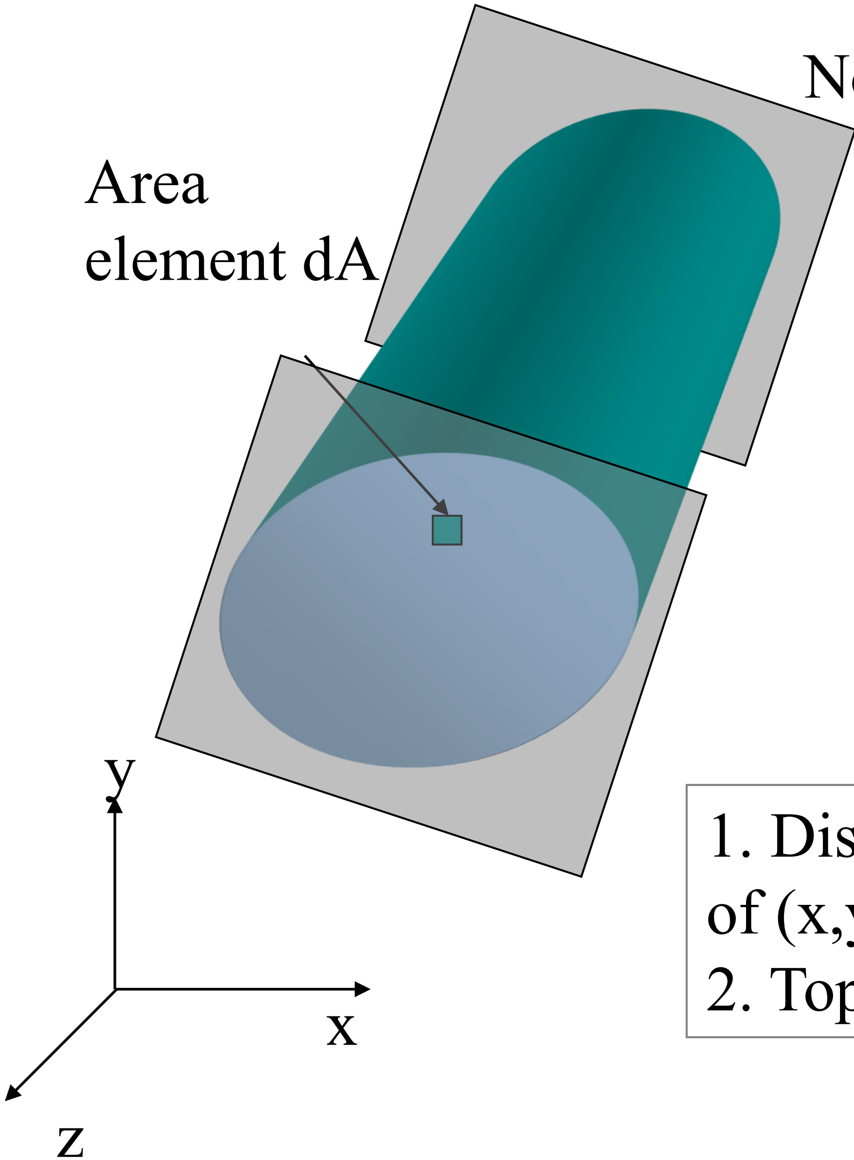
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

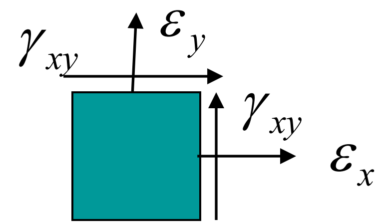
Hence, the <u>D</u> matrix for the **<u>plane stress case</u>** is

$$\underline{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

<u>PLANE STRAIN</u>: Only the in-plane <u>strain</u> components are non-zero

Non-zero **<u>strain</u>** components $\boldsymbol{\varepsilon_x, \varepsilon_y, \gamma_{xy}}$

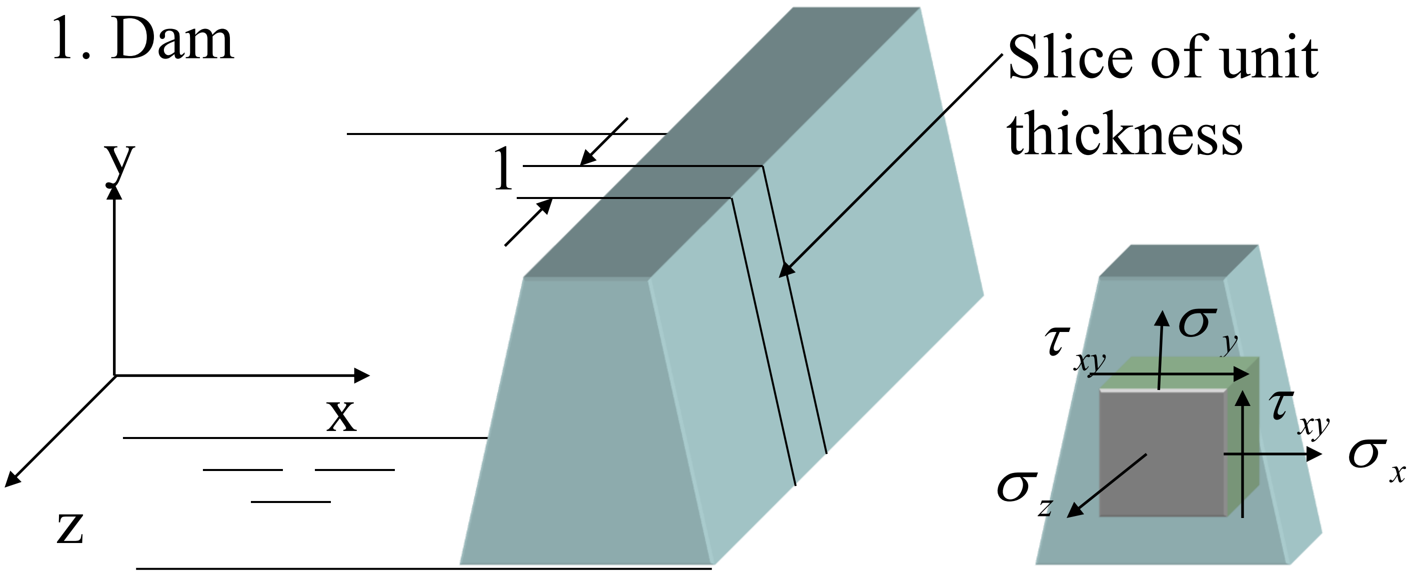




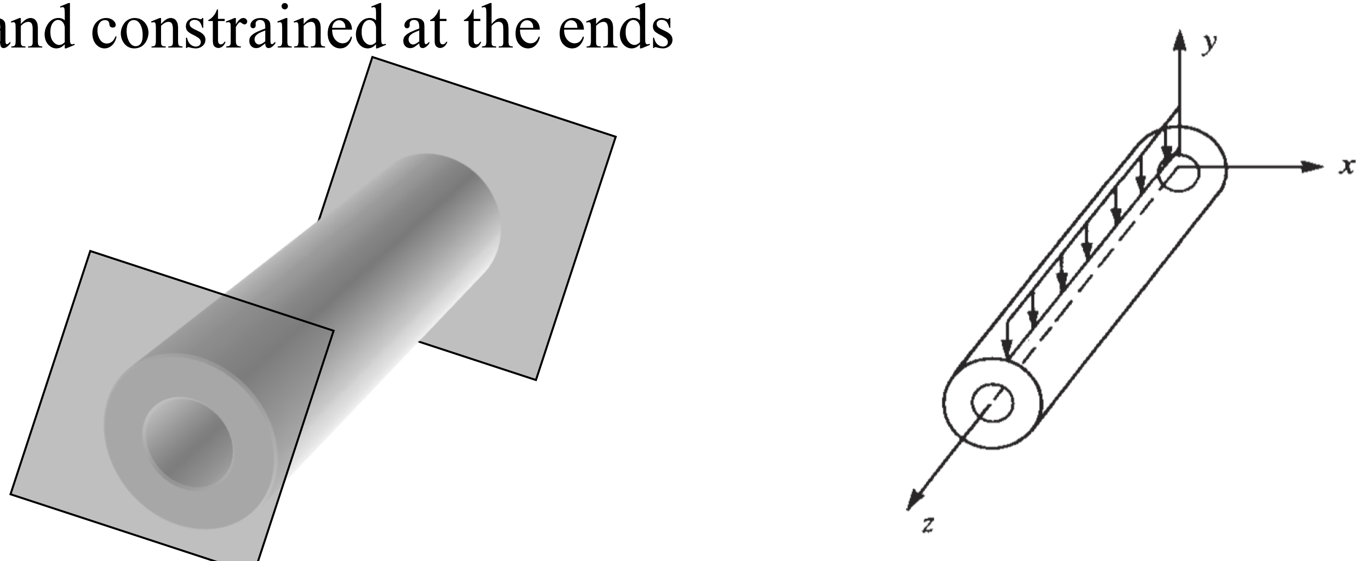
1. Displacement components u,v functions of (x,y) only and w=0
2. Top and bottom surfaces are fixed

PLANE STRAIN Examples:

1. Dam



2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends



PLANE STRAIN

Nonzero **stress**: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$

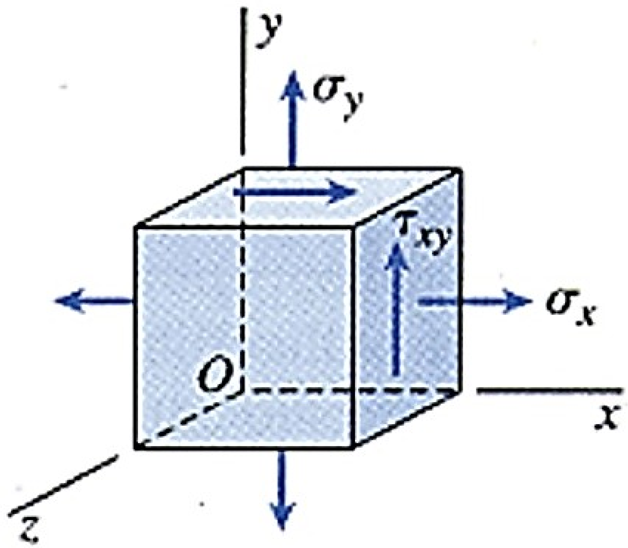
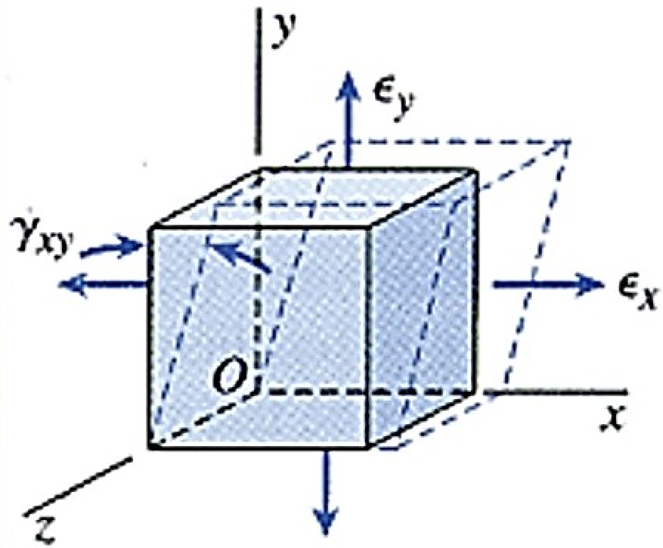
Nonzero **strain** components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

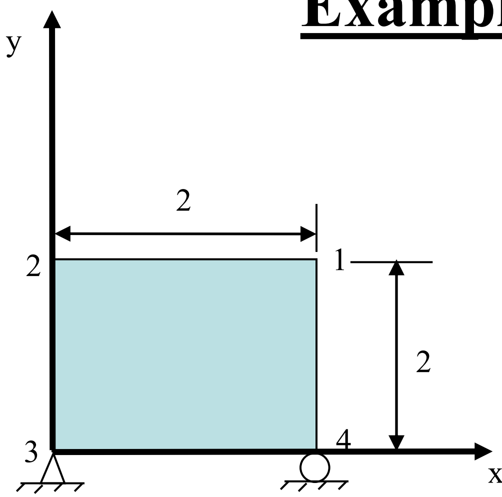
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

Hence, the D matrix for the **plane strain case** is

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values

Example problem



The square block is in **plane strain** and is subjected to the following strains

$$\varepsilon_x = 2xy$$

$$\varepsilon_y = 3xy^2$$

$$\gamma_{xy} = x^2 + y^3$$

Compute the displacement field (i.e., displacement components u(x,y) and v(x,y)) within the block

Solution

Recall from definition

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2xy \qquad (1)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 3xy^2 \qquad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \qquad (3)$$

Arbitrary function of 'x'

Integrating (1) and (2)

$$u(x, y) = x^2 y + C_1(y) \qquad (4)$$

$$v(x, y) = xy^3 + C_2(x) \qquad (5)$$

Arbitrary function of 'y'

Plug expressions in (4) and (5) into equation (3)

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=x^2+y^3 \quad (3)$$

$$\Rightarrow \frac{\partial\left[x^2y+C_1(y)\right]}{\partial y}+\frac{\partial\left[xy^3+C_2(x)\right]}{\partial x}=x^2+y^3$$

$$\Rightarrow x^2+\frac{\partial C_1(y)}{\partial y}+y^3+\frac{\partial C_2(x)}{\partial x}=x^2+y^3$$

$$\Rightarrow \frac{\partial C_1(y)}{\partial y}+\frac{\partial C_2(x)}{\partial x}=0$$

Function of ‘y’

Function of ‘x’

Hence

$$\frac{\partial C_1(y)}{\partial y} = -\frac{\partial C_2(x)}{\partial x} = C \text{ (a constant)}$$

Integrate to obtain

$$C_1(y) = Cy + D_1$$

$$C_2(x) - Cx + D_2$$

D_1 and D_2 are two constants of integration

Plug these back into equations (4) and (5)

(4) $u(x, y) = x^2 y + Cy + D_1$

(5) $v(x, y) = xy^3 - Cx + D_2$

How to find C, D_1 and D_2?

Use the 3 **boundary conditions**

$$u(0,0)=0$$

$$v(0,0)=0$$

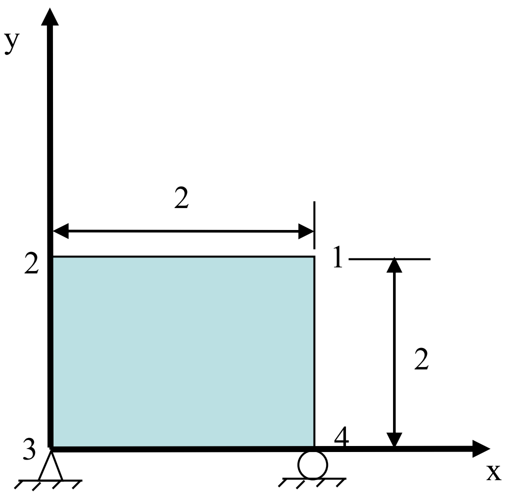
$$v(2,0)=0$$

To obtain

$$C=0$$

$$D_1=0$$

$$D_2=0$$



Hence the solution is

$$u(x,y)=x^2 y$$
$$v(x,y)=xy^3$$