Dr. Feirusha Kakshar, UKH

Lecture - 3

- Inclined Surface
- Stress Invariants
- Maximum Shear Stress

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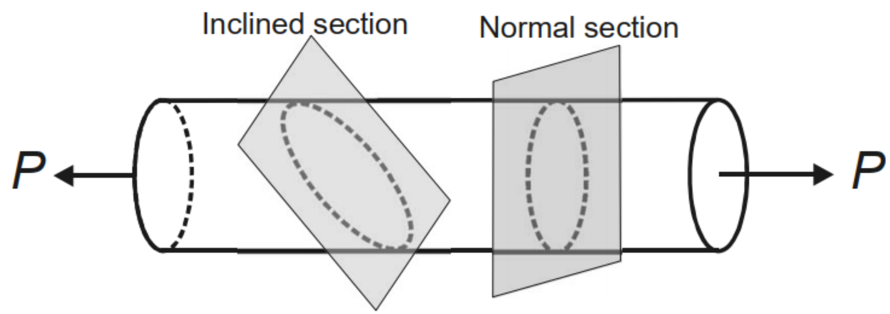
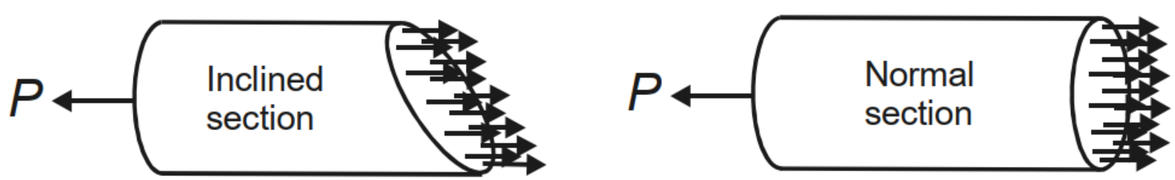
Stresses on Inclined Plane

Normal and shear stresses on inclined sections

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an “inclined” (as opposed to a “normal”) section through the bar.

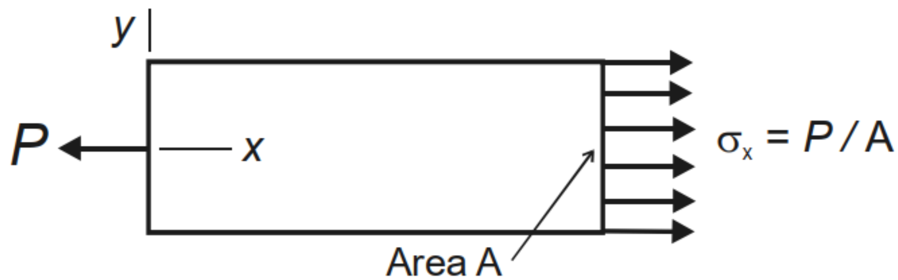


Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.

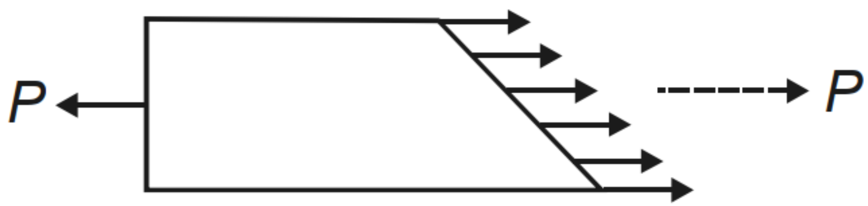


Stresses on Inclined Plane

2D view of the normal section
(but don't forget the thickness perpendicular to the page)

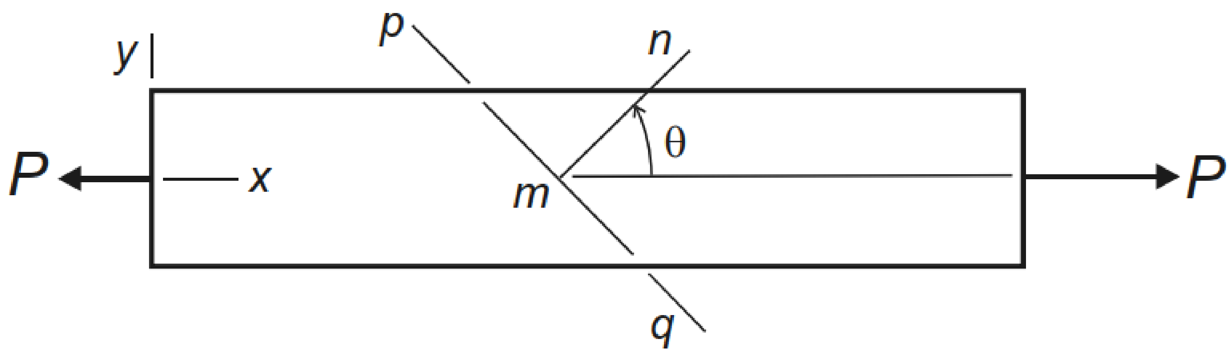


2D view of the inclined section

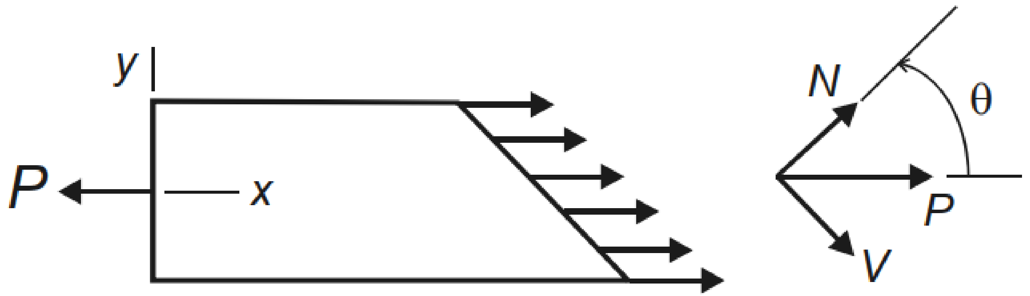


Stresses on Inclined Plane

Specify the orientation of the inclined section *pq* by the angle θ between the *x* axis and the normal to the plane.



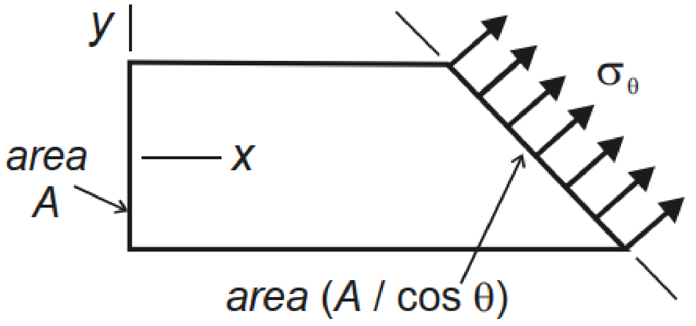
"Normal section" θ = 0°
Top face θ = 90°
Left face θ = 180°
Bottom face θ = 270° or -90°

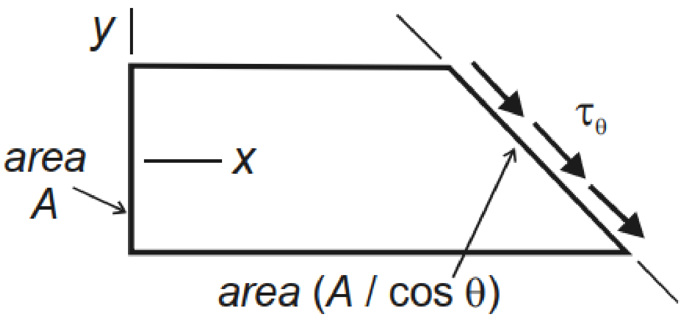


The force *P* can be resolved into components:
Normal force *N* perpendicular to the inclined plane, $N = P \cos\theta$
Shear force *V* tangential to the inclined plane $V = P \sin\theta$

Stresses on Inclined Plane

If we know the areas on which the forces act, we can calculate the associated stresses.





$$\sigma_\theta = \frac{Force}{Area} = \frac{N}{Area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

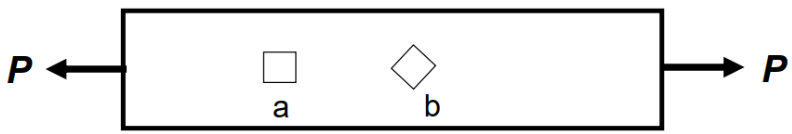
$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta) \quad ☆$$

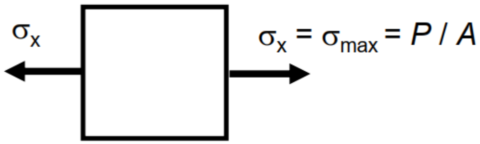
$$\tau_\theta = \frac{Force}{Area} = \frac{-V}{Area} = \frac{-P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta) \quad ☆$$

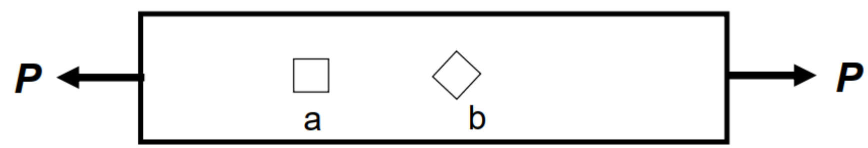
Stresses on Inclined Plane

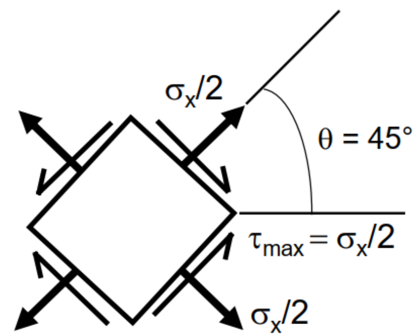
Maximum stresses on a bar in tension





a
No shear stresses





b

Angle	σ_θ	τ_θ
$\theta = 45°$	$\sigma_x/2$	$-\sigma_x/2$
$\theta = 135°$	$\sigma_x/2$	$\sigma_x/2$
$\theta = -45°$	$\sigma_x/2$	$\sigma_x/2$
$\theta = 225°$	$\sigma_x/2$	$-\sigma_x/2$

In case b ($\theta = 45°$), the normal stresses on all four faces are the same, and all four shear stresses have **equal** and **maximum** magnitude.

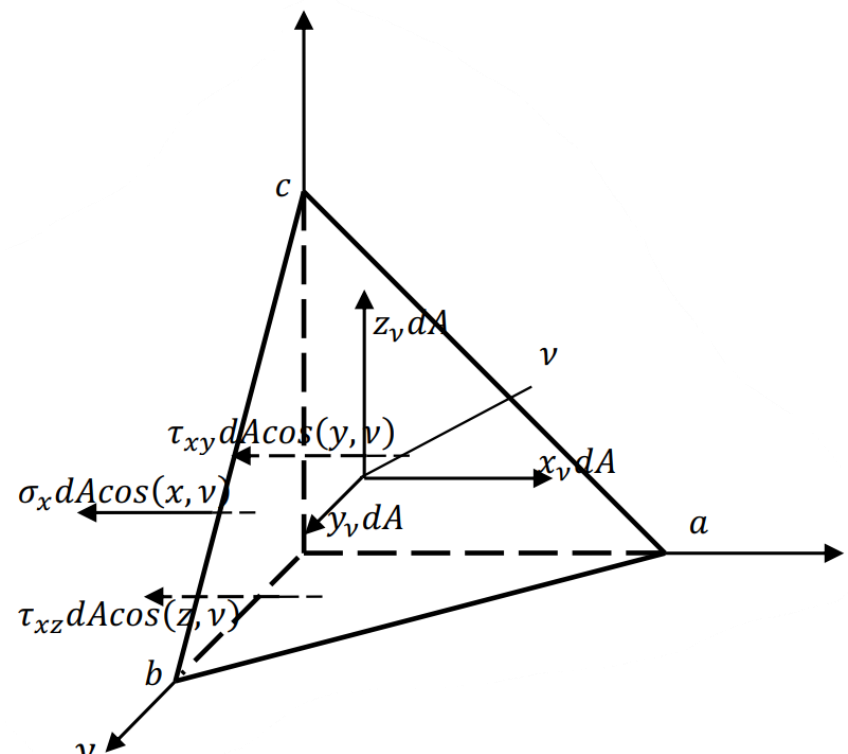
Stresses on Inclined Plane

- As it was noted, if the loaded body is in equilibrium, then any of its cut parts must also be in equilibrium, i.e. the principal vector and the principal moment of all loads applied to this part must be equal to zero. This also applies to the surface of the body. In general, the surface of a body, like any of its elementary plane, is inclined to coordinate axes.

- An external load may be applied to this surface. Therefore, it is necessary to establish a relationship between the projections of the external load (external stress) on a small inclined plane and the stresses arising at the faces parallel to the coordinate Planes.

Stresses on Inclined Plane

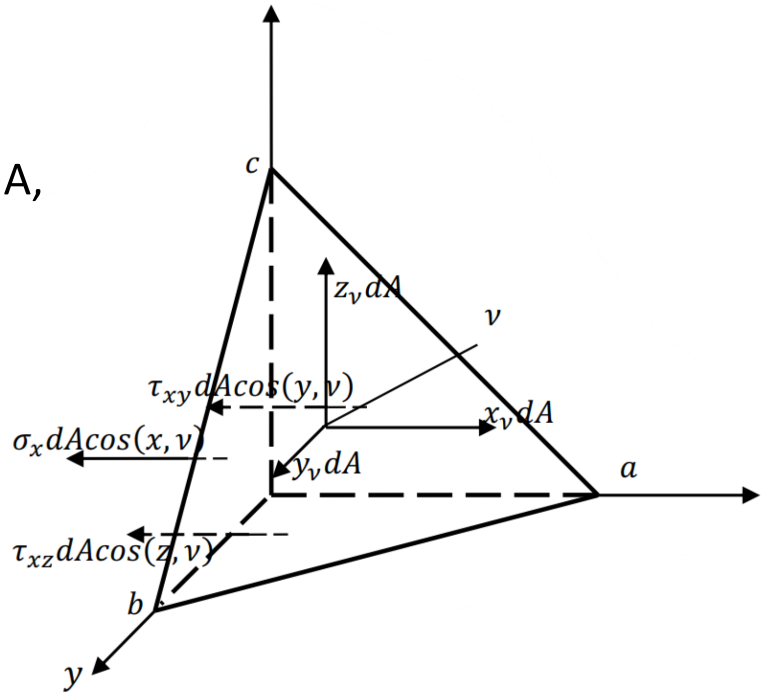
- Extract from the body the elementary tetrahedron Oabc (Fig.) with the planes Oa = dx, Ob = dy, and Oc = dz. For the inclined plane abc, draw the normal vector ν .
- Denote the **cosines** of its inclination angles with the x, y, and z coordinate axes, respectively: **cos(x, ν), cos(y, ν)** and **cos(z, ν)**, i.e. the **cosines** of the angles between the external normal ν and the x, y, and z coordinate axes, respectively, through l, m, and n: cos(x, ν) = l, cos(y, ν) = m and cos(z, ν) = n.



Stresses on Inclined Plane

- Suppose the total external stress is p, then its components will be Xv, Yv, and Zv.
- If we denote the area of the inclined face abc by dA, then the areas of the faces coinciding with the coordinate planes will be, respectively:

$$A_{Oab} = dA \cdot n,$$
$$A_{Obc} = dA \cdot l,$$
$$A_{Oac} = dA \cdot m.$$



The equilibrium equation on the x-axis, i.e. ∑ X = 0:

$$X_v dA - \sigma_x dA \cdot l - \tau_{xy} dA \cdot m - \tau_{xz} dA \cdot n = 0.$$

$$X_v = \sigma_x l + \tau_{xy} m + \tau_{xz} l$$

Stresses on Inclined Plane

- In the same way, we get the other two equations with respect to the y and z axes.

$$\left.\begin{aligned} X_v &= \sigma_x l + \tau_{xy} m + \tau_{xz} n; \\ Y_v &= \tau_{yx} l + \sigma_y m + \tau_{yz} n; \\ Z_v &= \tau_{zx} l + \tau_{zy} m + \sigma_z n. \end{aligned}\right\}$$

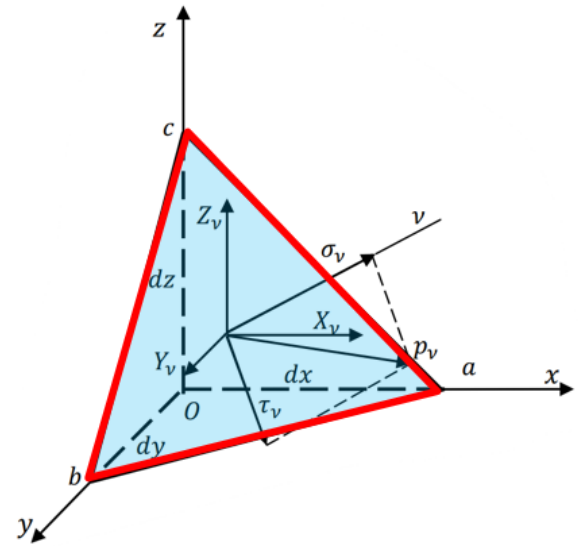
In the case of a plane stress state, we will have:

$$\left.\begin{aligned} X_v &= \sigma_x l + \tau_{xy} m; \\ Y_v &= \tau_{yx} l + \sigma_y m. \end{aligned}\right\}$$

Principal Stresses and Stress Invariants

Investigation of the stress state at the point of the body.
By equation

$$\left.\begin{aligned} X_\nu &= \sigma_x l + \tau_{xy} m + \tau_{xz} n; \\ Y_\nu &= \tau_{yx} l + \sigma_y m + \tau_{yz} n; \\ Z_\nu &= \tau_{zx} l + \tau_{zy} m + \sigma_z n. \end{aligned}\right\}$$



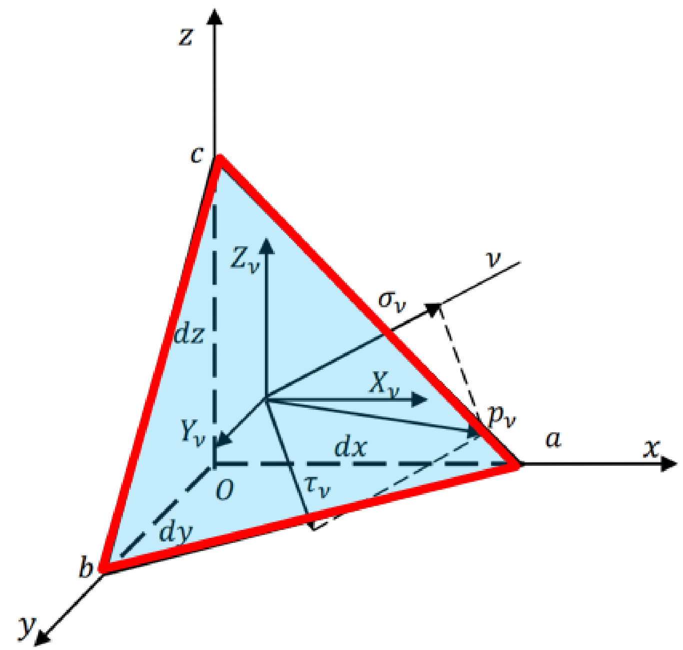
It is possible to compute the stress components Xv, Yv, and Zv of the stress pv at any inclined plane.
The total stress pv is computed as the geometric sum of these components Xv, Yv, and Zv:

$$p_\nu = \sqrt{X_\nu^2 + Y_\nu^2 + Z_\nu^2}$$

Principal Stresses and Stress Invariants

- Decompose the obtained total stress pv into its components along the normal v and along the plane of the face, i.e. into the normal σv and the shear τv stress (Fig.).

The normal stress σv is calculated as the sum of the projections of the components Xv, Yv and Zv on the v axis.



$$\sigma_\nu = X_\nu \cdot l + Y_\nu \cdot m + Z_\nu \cdot n$$

$$\left.\begin{aligned} X_\nu &= \sigma_x l + \tau_{xy} m + \tau_{xz} n; \\ Y_\nu &= \tau_{yx} l + \sigma_y m + \tau_{yz} n; \\ Z_\nu &= \tau_{zx} l + \tau_{zy} m + \sigma_z n. \end{aligned}\right\}$$

$$\sigma_\nu = (\sigma_x l + \tau_{xy} m + \tau_{xz} n) \cdot l + (\tau_{yx} l + \sigma_y m + \tau_{yz} n) \cdot m + (\tau_{zx} l + \tau_{zy} m + \sigma_z n) \cdot n$$

or

$$\boxed{\sigma_\nu = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl.}$$

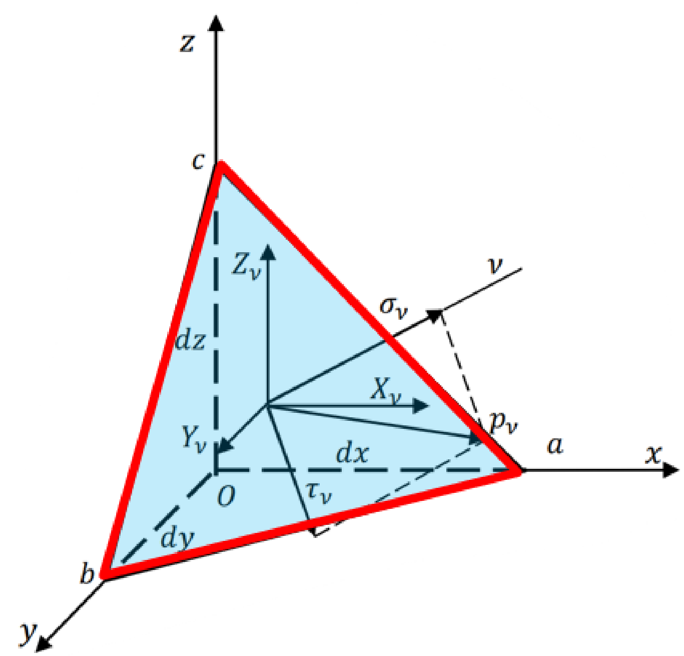
Principal Stresses and Stress Invariants

- The shear stress is calculated from a right-angled triangle according to the Pythagorean theorem: $$\tau_\nu^2 = p_\nu^2 - \sigma_\nu^2$$

This formula only gives the value of the shear stress, but does not specify its direction is in the plane of the site.

Find the component of the shear stress in the plane with the normal $\boldsymbol{\nu}$ in the given direction $\boldsymbol{\eta}$ with direction cosines $\boldsymbol{l1}$, $\boldsymbol{m1}$, $\boldsymbol{n1}$ (Fig.). Since the directions of ν and η are mutually perpendicular, their direction cosines, known from analytic geometry:

$$ll_1 + mm_1 + nn_1 = 0$$



Principal Stresses and Stress Invariants

- The desired shear stress is equal to the sum of the projections of the stress components Xν, Yν, Zν on the direction of η

$$\tau_{\eta\nu} = X_\nu l_1 + Y_\nu m_1 + Z_\nu n_1$$

Substituting here the values of the constituents Xν, Yν, and Zν :

$$\tau_{\eta\nu} = (\sigma_x l + \tau_{xy} m + \tau_{xz} n)\cdot l_1 + (\tau_{yx} l + \sigma_y m + \tau_{yz} n)\cdot m_1 + (\tau_{zx} l + \tau_{zy} m + \sigma_z n)\cdot n =$$
$$\sigma_x l l_1 + \sigma_y m m_1 + \sigma_z n n_1 + \tau_{yx}(l m_1 + l_1 m) + \tau_{yz}(m n_1 + m n_1) + \tau_{zx}(n l_1 + l n_1)$$

If we take τν = 0 in the expression (below), we get that pν = σν, i.e. on the principal plane the total stress pν coincides with the normal σν in magnitude and direction.

$$\tau_\nu^2 = p_\nu^2 - \sigma_\nu^2$$

Using the condition τν = 0, we determine the value of the principal stresses and the position of the principal planes. Let's denote the principal stress with the letter σ. By projecting the σ onto the coordinate axes, we find its components

$$X_\nu = \sigma\cdot l,\;\; Y_\nu = \sigma\cdot m,\;\; Z_\nu = \sigma\cdot n$$

Principal Stresses and Stress Invariants

$$X_\nu = \sigma \cdot l, \quad Y_\nu = \sigma \cdot m, \quad Z_\nu = \sigma \cdot n$$

$$\left.\begin{aligned} X_\nu &= \sigma_x \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n; \\ Y_\nu &= \tau_{yx} \cdot l + \sigma_y \cdot m + \tau_{yz} \cdot n; \\ Z_\nu &= \tau_{zx} \cdot l + \tau_{zy} \cdot m + \sigma_z \cdot n. \end{aligned}\right\}$$

$$\left.\begin{aligned} \sigma \cdot l &= \sigma_x \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n; \\ \sigma \cdot m &= \tau_{yx} \cdot l + \sigma_y \cdot m + \tau_{yz} \cdot n; \\ \sigma \cdot n &= \tau_{zx} \cdot l + \tau_{zy} \cdot m + \sigma_z \cdot n. \end{aligned}\right\}$$

$$\left.\begin{aligned} (\sigma_x - \sigma) \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n &= 0; \\ \tau_{yx} \cdot l + (\sigma_y - \sigma) \cdot m + \tau_{yz} \cdot n &= 0; \\ \tau_{zx} \cdot l + \tau_{zy} \cdot m + (\sigma_z - \sigma) \cdot n &= 0. \end{aligned}\right\}$$

Principal Stresses and Stress Invariants

- Three linear homogeneous equations with respect to *l, m,* and *n* were obtained. In our case, the system can't have a zero *l=m=n=0* solution:

$$l^2 + m^2 + n^2 = 1$$

Therefore, the system can have solutions at zero determinant

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

or

$$(\sigma_x - \sigma)(\sigma_y - \sigma)(\sigma_z - \sigma) + 2\tau_{yx}\tau_{zy}\tau_{xz} - (\sigma_y - \sigma)\tau_{zx}{}^2 - (\sigma_x - \sigma)\tau_{yz}{}^2 - (\sigma_z - \sigma)\tau_{xy}{}^2 = 0$$

Principal Stresses and Stress Invariants

- After multiplying and grouping by powers of σ, we get the cubic equation:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{yx}{}^2 - \tau_{yz}{}^2 - \tau_{zx}{}^2)\sigma - \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = 0$$

or shorter $\sigma^3 - S_1\sigma^2 - S_2\sigma - S_3 = 0$

Where:

$$\left.\begin{aligned} S_1 &= \sigma_x + \sigma_y + \sigma_z \\ S_2 &= -\sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x + \tau_{yx}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2 \\ S_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{yx}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}{}^2 - \sigma_y\tau_{zx}{}^2 - \sigma_z\tau_{xy}{}^2 \end{aligned}\right\}$$

To solve the cubic equation, we use the following substitution $\sigma = x + \frac{1}{3}S_1$

Equation would then take the form

$$\tau_v^2 = p_v^2 - \sigma_v^2 \quad \Rightarrow \quad x^3 + 3px + 3q = 0$$

$$p = \frac{1}{3}\left(S_2 - \frac{1}{3}S_1^2\right)$$

$$q = -\frac{1}{27}S_1^3 + \frac{1}{6}S_1S_2 - \frac{1}{2}S_3$$

Principal Stresses and Stress Invariants

- All three roots, σ1, σ2 and σ3, are valid when the discriminant is negative:

$$\Delta = p^3 + q^2 < 0$$

If substitute the corresponding numerical values of the coefficients S1, S2, S3 into the expressions for p and q, and then compute Δ, you can see that Δ is always negative. This also follows from physical considerations: the principal stresses can only be real quantities.

At Δ<0, the so-called trigonometric method is used to solve the cubic equation

In this case, the roots of the below cubic equation can be represented as follows

$$\sigma^3 - S_1\sigma^2 - S_2\sigma - S_3 = 0$$

$$x_1 = 2\sqrt{|p|}cos\varphi, \;\; x_2 = 2\sqrt{|p|}cos(\varphi + 120°), \;\; x_3 = 2\sqrt{|p|}cos(\varphi - 120°),$$

$$\varphi = \frac{1}{3} arccos \frac{-q}{|p|^{\frac{3}{2}}}.$$

Principal Stresses and Stress Invariants

- For the subsequent determination of the principal stresses σ1, σ2 and σ3, the determined values of the roots x1, x2 and x3 are substituted into the expression below:

$$\sigma = x + \frac{1}{3} S_1$$

Assuming: $\sigma_1 \geq \sigma_2 \geq \sigma_3$

To find the ***cosines*** of any principal stresses σi (i = 1, 2, 3) , we need to insert its value into equations

$$\left.\begin{aligned} (\sigma_x - \sigma) \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n = 0; \\ \tau_{yx} \cdot l + (\sigma_y - \sigma) \cdot m + \tau_{yz} \cdot n = 0; \\ \tau_{zx} \cdot l + \tau_{zy} \cdot m + (\sigma_z - \sigma) \cdot n = 0. \end{aligned}\right\}$$

and then solve together with equation $l^2 + m^2 + n^2 = 1$

any two of them. For example:

$$\left.\begin{aligned} (\sigma_x - \sigma) \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n = 0; \\ \tau_{yx} \cdot l + (\sigma_y - \sigma) \cdot m + \tau_{yz} \cdot n = 0; \\ l^2 + m^2 + n^2 = 1. \end{aligned}\right\}$$

Principal Stresses and Stress Invariants

- **Cosines** are found in the same way for the other two principal stresses. The obtained **cosine** values correspond to the principal stresses, which are mutually perpendicular, so the principal plane will be mutually perpendicular. This proves that at any point of a stressed body it is possible to draw three mutually perpendicular principal planes. In this particular case:

$$\left.\begin{aligned} S_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ S_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ S_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}\right\}$$

Obviously, the roots of the cubic equation cannot depend on the choice of coordinate axes, therefore, its coefficients S1, S2, S3 must remain constant when transforming the axes, i.e. they must essentially be invariants.
The first invariant is called the value

$$S_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

It shows that the sum of the normal stresses at the three mutually perpendicular planes is a constant quantity.

Principal Stresses and Stress Invariants

- The second invariant is:

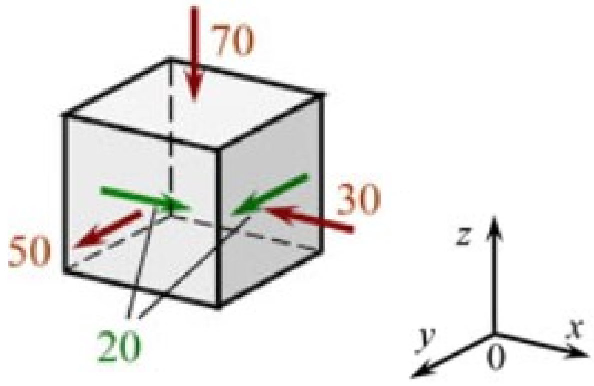
$$S_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - {\tau_{yx}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

It is used in the theory of plasticity.

The third invariant is the quantity that is the determinant composed from the stress tensor elements:

$$S_3 = \begin{vmatrix} \sigma_x \tau_{xy} \tau_{xz} \\ \tau_{yx} \sigma_y \tau_{yz} \\ \tau_{zx} \tau_{zy} \sigma_z \end{vmatrix} = \sigma_x\sigma_y\sigma_z + 2\tau_{yx}\tau_{yz}\tau_{xz} - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 = \sigma_1\sigma_2\sigma_3$$

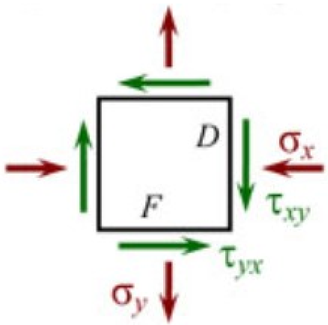
Example



- The shown stresses act on the element at the critical section of the cast-iron member (in MPa). Check the strength of the member.

The plane on which τ=0 is principal plane (perpendicular to z axis

Let show stress state on the other two planes in x0z plane



$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{-30+50}{2} \pm \sqrt{\left(\frac{-30-50}{2}\right)^2 + 20^2}$$

$$\sigma_{max} = 10 + 44{,}7 = 54.7 \qquad \sigma_{min} = 10 - 44{,}7 = -34.7$$

Principal Stresses

$$\sigma_1 = 54.7 \qquad \sigma_2 = -34{,}7 \qquad \sigma_3 = -70$$

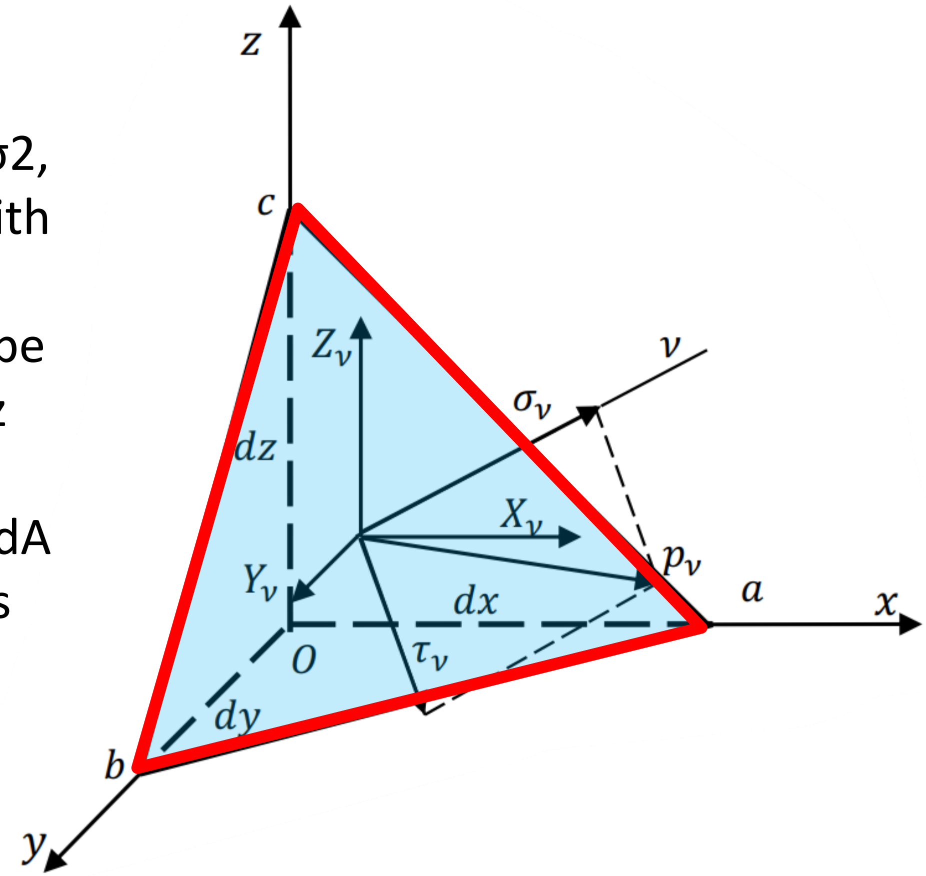
Check the results using sum of normal stresses

$$\sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 = \text{const};$$

$$-30 + 50 - 70 = 54.7 - 34.7 - 70 = -50.$$

Maximum Shear Stress

The x, y, and z coordinate axes are compatible with the directions of the previously found principal stresses σ1, σ2, σ3. Let's draw an arbitrary plane ABC with the area dA and the normal ν.
Let the total stress acting on this plane be equal to pν, its components in the x, y, z axes are equal to Xν, Yν and Zν, and the normal and shear stresses at the plane dA are equal to σν and τν. We have obvious equations.



$$p_\nu^2 = X_\nu^2 + Y_\nu^2 + Z_\nu^2$$

$$p_\nu^2 = \sigma_\nu^2 + \tau_\nu^2$$

Maximum Shear Stress

$$\left.\begin{aligned} X_\nu &= \sigma_x \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n; \\ Y_\nu &= \tau_{yx} \cdot l + \sigma_y \cdot m + \tau_{yz} \cdot n; \\ Z_\nu &= \tau_{zx} \cdot l + \tau_{zy} \cdot m + \sigma_z \cdot n. \end{aligned}\right\}$$

$$p_\nu^2 = X_\nu^2 + Y_\nu^2 + Z_\nu^2$$

$$p_\nu^2 = \sigma_\nu^2 + \tau_\nu^2$$

$$X_\nu = \sigma_1 \cdot l, \quad Y_\nu = \sigma_2 \cdot m, \quad Z_\nu = \sigma_3 \cdot n$$

$$p_\nu^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2$$

By projecting Xv, Yv, and Zv in the direction v, we get the expression for σv

$$\sigma_\nu = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$$

Maximum Shear Stress

$$p_\nu^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2$$

$$\sigma_\nu = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$$

$$p_\nu^2 = \sigma_\nu^2 + \tau_\nu^2$$

$$\tau_\nu^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2$$

Substituting the equation $n^2 = 1 - l^2 - m^2$, obtained from the geometric relation $n^2 + l^2 + m^2 = 1$ we get the following expression for $\tau^2 \nu$.

$$\tau_\nu^2 = (\sigma_1^2 - \sigma_3^2)\, l^2 + (\sigma_2^2 - \sigma_3^2) m^2 + \sigma_3^2 - [(\sigma_1 - \sigma_3) l^2 + (\sigma_2 - \sigma_3) m^2 + \sigma_3]^2$$

Maximum Shear Stress

- Thus, the magnitude of the shear stress τv depends on two independent variables ***l*** and ***m***. To determine the extremum of this magnitude, it is necessary to take the partial derivatives of τv by ***l*** and ***m*** and equate them to zero:

$$\frac{\partial(\tau_v)^2}{\partial l} = 2l\{\sigma_1^2 - \sigma_3^2 - 2(\sigma_1 - \sigma_3)[(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2 + \sigma_3]\} = 0,$$

$$\frac{\partial(\tau_v)^2}{\partial m} = 2m\{\sigma_2^2 - \sigma_3^2 - 2(\sigma_2 - \sigma_3)[(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2 + \sigma_3]\} = 0.$$

After reducing the first equation by (σ1 - σ3) and the second by (σ2 - σ3), we get the following system of equations for finding the values ***l, m, n*** that satisfy the conditions of the extremum of the shear stress τv:

$$\left.\begin{array}{c} l\{\sigma_1 - \sigma_3 - 2[(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2]\} = 0, \\ m\{\sigma_2 - \sigma_3 - 2[(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2]\} = 0, \\ l^2 + m^2 + n^2 - 1 = 0. \end{array}\right\}$$

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Maximum Shear Stress

- The conditions *l=0* and n=1 correspond to the principal planes where the shear stresses are zero.
- If *l≠0* and *m=0*, then the second equation is satisfied at any values of *l*, and the first equation is satisfied at

$$\sigma_1 - \sigma_3 - 2(\sigma_1 - \sigma_3)\, l^2 = 0$$

From where $2l^2 = 1$ and $l = \pm {}^{1}/_{\sqrt{2}}$. Then, from the third

$$l^2 + m^2 + n^2 - 1 = 0.$$

For *n* we get $n = \pm {}^{1}/_{\sqrt{2}}$.

The same can be obtained with *l=0* and *m≠0* by the second equation $m=\pm 1/(\sqrt{2})$ and from the third equation $n=\pm 1/(\sqrt{2})$

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Maximum Shear Stress

- As a result, for angles indicating the direction of extreme plane shear stresses, we get Table.

l	m	n
0	0	± 1
0	± 1	0
± 1	0	0
0	$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{1}{\sqrt{2}}$
$\pm \frac{1}{\sqrt{2}}$	0	$\pm \frac{1}{\sqrt{2}}$
$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{1}{\sqrt{2}}$	0

Maximum Shear Stress

- The first three rows of the table correspond to the directions of the v normal that coincide with the principal axes of coordinates

  (τ = 0).

l	m	n
0	0	± 1
0	± 1	0
± 1	0	0
0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0

The other three rows correspond to the planes that pass through one of the principal axes and divide the corner between the other two in half.

Thus, the planes of extreme shear stresses are at an angle of 45° with the principal planes. Substituting in the expression –

$$\tau_v^2 = (\sigma_1^2 - \sigma_3^2)\, l^2 + (\sigma_2^2 - \sigma_3^2)m^2 + \sigma_3^2 - [(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2 + \sigma_3]^2$$

for τv value l, m, n, turning it into an extremum, we get the following extreme Shear Stress Values:

$$\tau_v^{max} = \pm\frac{\sigma_2 - \sigma_3}{2}, \tau_v^{max} = \pm\frac{\sigma_1 - \sigma_3}{2}, \quad \tau_v^{max} = \pm\frac{\sigma_1 - \sigma_2}{2}$$

Since $\sigma_1 > \sigma_2 > \sigma_3$, TO $\tau_v^{max} = \pm\frac{\sigma_1 - \sigma_3}{2}$

Problems

1-Determine the maximum shear stress for the stress state:

σ_1=120 MPa σ_2 =50 MPa σ_3=-60 MPa

2- Determine the stress invariants in a body under stress state shown by below matrix:

$$\begin{matrix} 10 & -8 & -4 \\ -8 & 8 & 5 \\ -4 & 5 & -5 \end{matrix}$$

All stresses are in MPa

3- A body is under a direct tensile stress of 400 MPa in one plane and a shear stress of 150 MPa on the same plane.
Determine the maximum normal stress on this plane.