زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Nil-Clean-Rings

Research Project
Submitted to the department of mathematics in partial fulfillment of the requirements for the degree of BSc. in mathematics

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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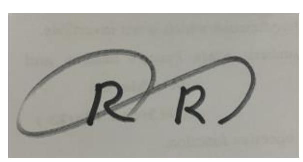
Date: 1 / 4 / 2023

In view of the available recommendations, I forward this report for debate by the examining committee.

Signature: 

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ABSTRACT

In this paper, we introduce the notion of nil-clean ring was defined as a ring for which every element is the sum of a nilpotent and an idempotent. In this short article, we construct a flowchart to show which ring is nil-clean rings. So, we develop a program to illustrate that the ring is nil-clean. Finally, we classify nil-clean rings in Z_n.

INTRODUCTION

Many families of clean rings were investigated in previous decades. In recent years, a particular attention has been paid to the nil-clean rings and its relatives. A nil-clean ring is a ring in which every element is nil-clean, which means that every element can be written as a sum of an idempotent element and a nilpotent one. The study of nil-clean rings was initiated by Alexander J. Diesl in 2013 as a generalization of clean rings.

Chapter One

Background

Definition 1.1 : A nonempty set R is called a ring, if it has two binary operations called addition denoted by $+$ and multiplication denoted by **.** for $a, b \in R$ satisfying the following axioms:
(1) $(R, +)$ is an abelian group.
(2) Multiplication is associative, $i.e.\ a(bc) = (ab)c$ for all $, b, c \in R$.
(3) Distributive laws hold: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in R$.
Definition 1.2:- An element e of a ring $(R, +, .)$ is called idempotent element if

$e^2 = e$.

Definition 1.3:- In mathematics an element x of ring R is called nilpotent if there exists some positive integer n such that $\mathrm{x}^{\mathrm{n}} = 0$.

Definition 1.4:- An element a of a ring R is nil-clean if $a = e + b$ where $e^2 = e \in R$ and b is a nilpotent.

Theorem 1.5 [6]: Let $(Z_n, +, .)$be a ring such that $n = p_1p_2 \cdots p_k$, where the p_i's are distinct primes, then Z_n has no nonzero nilpotent elements.

Proof: We claim that n is a square free integer, $i.e., n = p_1p_2 \cdots p_k$, where the p_i's are distinct primes. Suppose that $n = p_1p_2 \cdots p_k$, p_i's are distinct primes. Let $[a] \in Z_n$ be nilpotent. Then $[a]\, m = [0]$ for some integer m. Hence, n divides a^m, so $p_1p_2 \cdots p_{k1}$ divides am. Then $p_i \mid a^m$ for all $i = 1, 2, \ldots, k$. Because the p_i's are prime, $p_i \mid a$ for all $i = 1, 2, \ldots, k$. Because $p_1p_2 \cdots p_k$ are distinct primes, we must have $p_1p_2 \cdots p_{k1} | a$, i.e., $n \mid a$, so $[a] = [0]$. This implies that Z_n has no nonzero nilpotent elements. Conversely, suppose that Z_n has no nonzero nilpotent elements. Let $n = p_1^{m_1}p_1^{m_2} \cdots p_k^{m_k}$, where the p_i's are distinct primes and $m_i \geq 1$. Let $m = max\{m_1, m_2 \cdots m_k\}$. Now $[p_1p_2 \cdots p_k]^m = [p_1{}^m p_2{}^m \cdots p_k{}^m] = [0]$ because $n \mid (p_1{}^m p_2{}^m \cdots p_k{}^m)$. Also, because Z_n has no nonzero nilpotent elements, $[p_1p_2 \cdots p_k] = [0]$. Hence, $n \mid (p_1p_2 \cdots p_k)$, so $(p_1{}^m p_2{}^m \cdots p_k{}^m) \mid (p_1p_2 \cdots p_k)$. Thus, $m_i \leq 1$ for all $i = 1, 2, \ldots, k$. Hence, $m_i = 1$ for all $i = 1, 2, \ldots, k$, so n is a square free integer.

Theorem 1.6 [6]: Let $(Z_{mn},+,.)$be a ring, where $m > 1, n > 1$, and m and n are relatively prime Show that the number of idempotent elements in Z_{mn}, is at least 4.

Proof: Clearly, [0] and [1] are idempotent elements. Because m and n are relatively prime, there exist integers a and b such that $am + bn = 1$. We now show that n does not divide a and m does not divide b. Suppose that $n \mid a$. Then $a = nr$ for some integer r. Thus, $n(rm + b) = nrm + nb = am + nb = 1$. This implies that $n = 1$, which is a contradiction. Therefore, n does not divide a and similarly m does not divide b. Now $m^2 a = m(1 - nb)$. This implies that $[m^2 a] = [m]$. Hence, $[ma]^2 = [ma]$. If $[ma] = [0]$, then $mn \mid ma$, so $n \mid a$, which is a contradiction. Consequently, $[ma] \neq [0]$. If $[ma] = [1]$, then $mn \mid (ma - 1)$. Hence, $ma + mnt = 1$ for some integer t. Thus, $m(a + nt) = 1$. This implies $m = 1$, which is a contradiction. Hence, $[ma] \neq [1]$. Thus, $[ma]$ is an idempotent such that $[ma] \neq [0]$ and $[ma] \neq [1]$. Similarly, $[nb]$ is an idempotent such that $[nb] \neq [0]$ and $[nb] \neq [1]$. Clearly $[ma] \neq [nb]$. Thus, we find that $[0], [1], [ma]$, and $[nb]$ are idempotent elements of Z_{mn}.

Theorem 1.7 [6]: Let $(Z_n,+,.)$be a ring such that $n = p^r$, for some prime p and some positive integer r , then Z_n has no idempotent elements other than [0] and [1].

Solution: We show that $n = p^r$ for some prime p and some integer $r > 0$. First assume that $n = p^r$ for some prime p and some positive integer r and $[x] \in Z_n$ be an idempotent. Then $[x]^2 = [x]$. Thus, $p^r \mid (x^2 - x)$ or $p^r \mid x(x - 1)$. Because x and $x - 1$ are relatively prime, $p^r \mid x$ or $p^r \mid (x - 1)$. If $p^r \mid x$, then $[x] = [0]$ and if $p^r \mid (x - 1)$, then $[x] = [1]$. Thus, [0] and [1] are the only two idempotent elements. Conversely, suppose that [0] and [1] are the only two idempotent elements. Let $= p_1^{m1} p_1^{m2} \cdots p_k^{mk}$, where the p_i's are distinct primes, $m_i \geq 1$, and $k > 1$. Let $t = p_1^{m1}$ and $n = p_1^{m2} \cdots p_k^{mk}$. Then t and s are relatively prime and $n = ts$. By Worked-Out Exercise 3, $Z_n = Z_{ts}$ must have at least four idempotents, which is a contradiction. Therefore, $k = 1$. Thus, $n = p^r$ for some prime p and some positive integer r.

Chapter Two

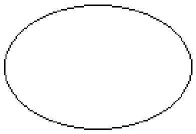
Algorithm and Programs by Using GAP

3.1 Basic Information

A *program flowchart* is an extremely useful tool in program development. First, any error or omission can be more easily detected from a program flow-chart than it can be from a program because a program flowchart is a pictorial representation of the logic of a program. Second, a program flowchart can be followed easily and quickly. Third, it serves as a type of documentation, which may be of great help if the need for program modification arises in future. In addition, the following five rules should be followed while creating program flowcharts:

1. Only the standard symbols should be used in program flowcharts.
2. The program logic should depict the flow from top to bottom and from left to right.
3. Each symbol used in a program flowchart should contain only one entry point and one exit point, with the exception of the decision symbol. This is known as the *single rule*.
4. The operations shown within a symbol of a program flowchart should be expressed independently of any particular programming language.
5. All decision branches should be well-labeled.

The following are the standard symbols used in program flowcharts:

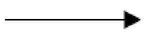


Terminal: used to show the beginning and end of a set of computer-related processes

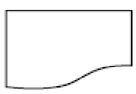
Input/Output: used to show any input/output operation

Computer processing: used to show any processing performed by a computer system

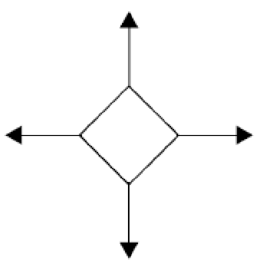
Comment: used to write any explanatory statement required to clarify something



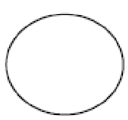
Flow line: used to connect the symbols



Document Input/Output: used when input comes from a document and output goes to a document.



Decision: used to show any point in the process where a decision must be made to determine further action

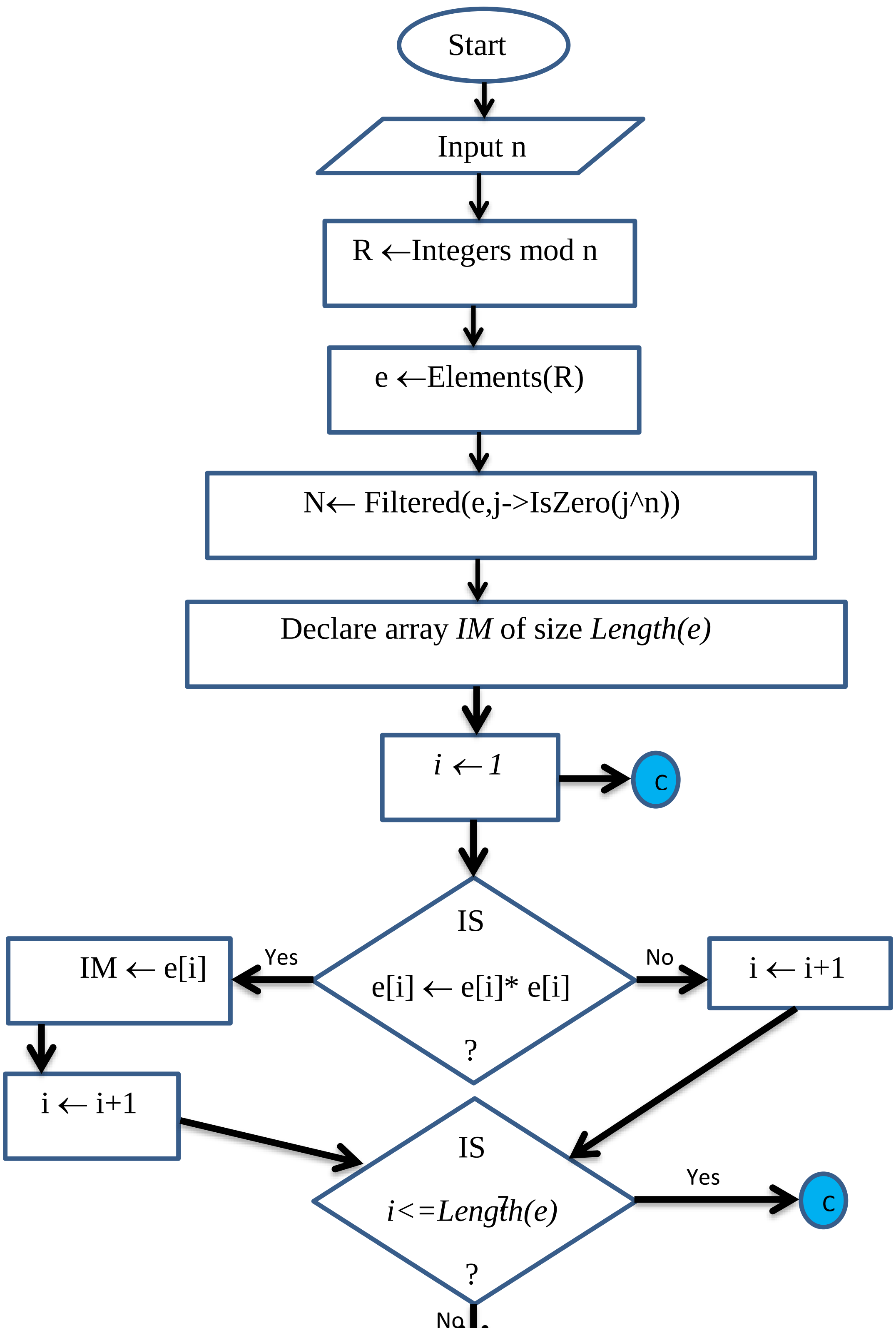


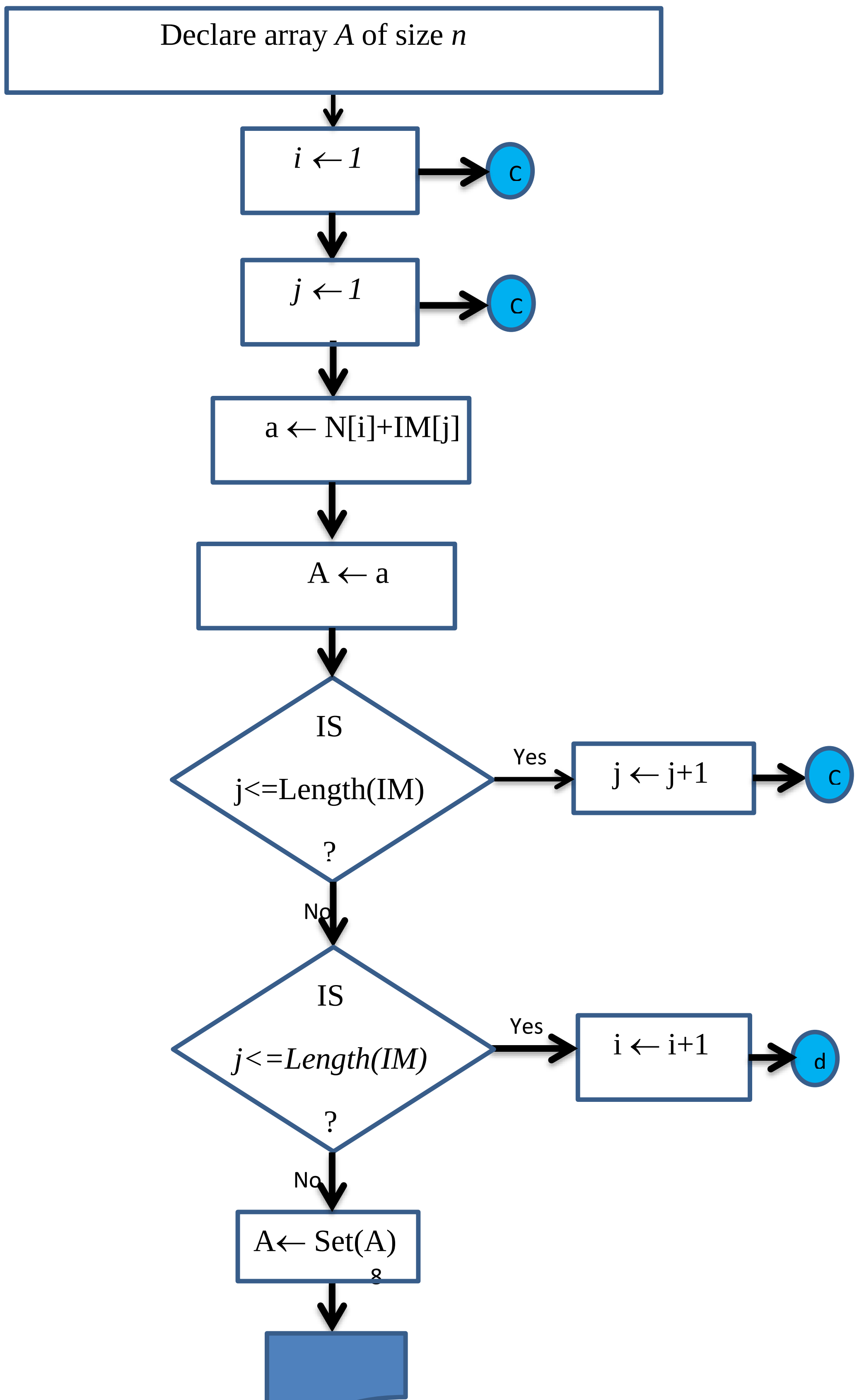
On-page connector: used to connect parts of a flowchart continued on the same page

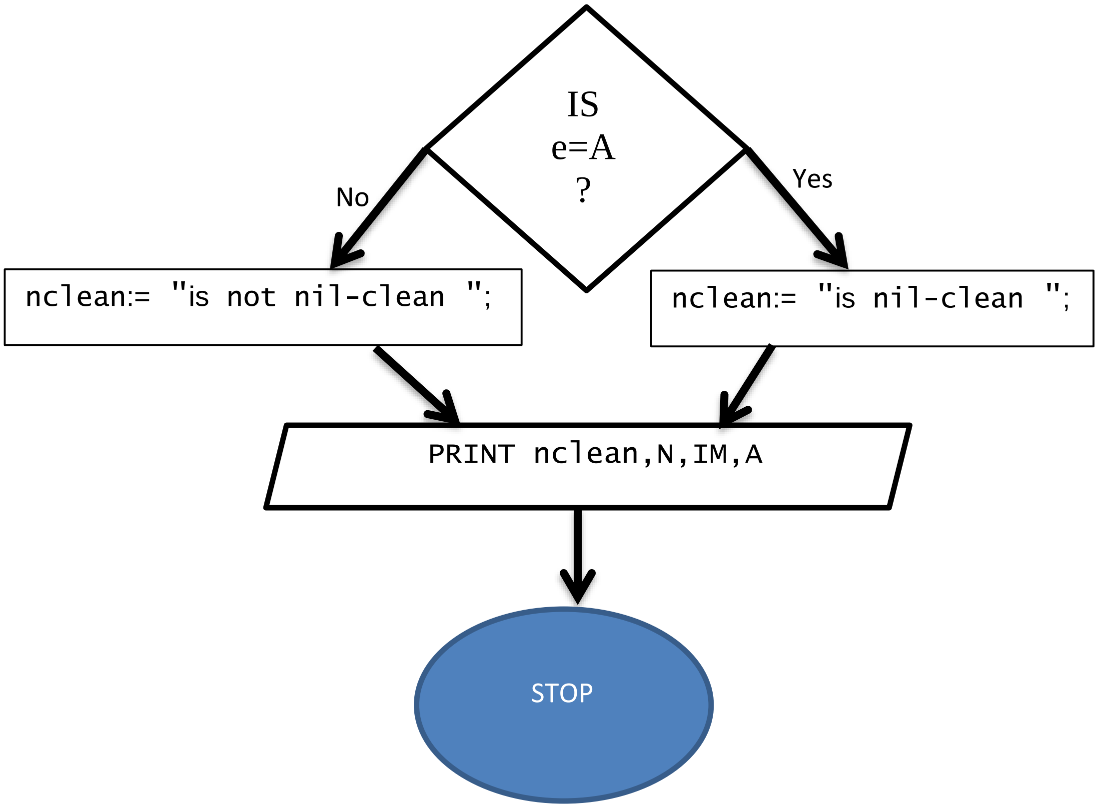
3.2 program flowchart

In this section, we construct a flowchart to show which ring is nil-clean rings. So, we can begin writing simple computer algorithms using GAP commands. Writing an algorithm causes us to solidify a new concept. For example, in order to write an algorithm that finds all the nilpotent elements in a ring. In addition, we develop a program by using GAP to illustrate that the ring is nil-clean.

A flowchart to illustrate which ring is nil-clean ring.




Declare array *A* of size *n*
i ← *1*
c
j ← *1*
c
a ← N[i]+IM[j]
A ← a
IS
j<=Length(IM)
?
Yes
j ← j+1
c
No
IS
j<=Length(IM)
?
Yes
i ← i+1
d
No
A← Set(A)



A program to illustrate which ring is nil-clean ring.

```
% nil-clean rings
nilclean:= function(n)
local R,e,i,j,N,IM,nclean,a,A;
R:=Integers mod n;
e:=Elements(R);
N:=Filtered(e,j->IsZero(j^n));
IM:=[];
for i in [1..Length(e)] do
if e[i]= e[i]*e[i] then
Add(IM, e[i]);
fi;
od;
```

```
A:=[];
for i in [1..Length(N)] do
for j in [1..Length(IM)] do
a:=N[i]+IM[j];
Add(A,a);
od;
od;
A:=Set(A);
if e=A then
nclean:= "is nil-clean ";
fi;
if e<>A then
nclean:= "is not nil-clean ";
fi;
return [nclean,N,IM,A];

end;
```

Chapter Three

Classification of Nil-clean rings in Z_n

1. Let $n = p$ such that $p > 2$,p is prime number. Firstly, we list some prime numbers to check which of them are nil-clean rings.

```
gap> List([3,5,7,11,13,17,19,23,29,31],j-> nilclean(j));
["is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ",
"is not nil-clean ",
  "is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ",
"is not nil-clean " ]
```

On this basis, we try to illustrate the reasons why z_n is not nil-clean ring such that $n = p$ and $p > 2$, p is prime number. Clearly, $(z_n, +, .)$ is a field if and only if n is prime. Thus, z_n has not non zero nilpotent element. On the other hand, let a be an idempotent element in a field. Then, $a^2 = a$, so $a^2 - a = 0$ which implies that $a(a - 1) = 0$. Since a field has no zero divisors either $a = 0$ or $a = 1$. Hence, the idempotent elements of a division ring are exactly 0 and 1. Therefore, we can not construct all elements of z_n by the sum of a nilpotent and an idempotent. Hence, we can establish a new corollary about nil-clean rings:

Corollary 3.1: $(z_p, +, .)$ is not a nil-clean ring where $p > 2$ is prime.

2. If $n = p_1 p_2 \cdots p_k$ where the p_i's are distinct primes. Firstly, we list some numbers which can be expressed as $p_1 p_2 \cdots p_k$ to check which of them are nil-clean rings.

List([6,15,21,22,26,30,38],j-> nilclean(j));

["is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean "]

On this basis, we try to illustrate the reasons why z_n is not nil-clean ring such that $p_1 p_2 \cdots p_k$ where the p_i's are distinct primes. Clearly, by Theorem 1.5, $(Z_n, +, .)$ has no nonzero nilpotent elements. Therefore, we can not construct all elements of z_n by the sum of a nilpotent and an idempotent. Hence, we can establish a new corollary about nil-clean rings:

Corollary 3.2: $(z_p, +, .)$ is not a nil-clean ring where $n = p_1 p_2 \cdots p_k$, where the p_i's are distinct primes.

3. If $n = p^r$ where p is prime for some $r \in \mathbb{Z}^+$. Firstly, we list some numbers which can be expressed as p^r to check which of them are nil-clean rings.

$$n = 2^r$$

List([2,4,8,32,64],j-> nilclean(j));

["is nil-clean ", "is nil-clean ", "is nil-clean ", "is nil-clean ", "is nil-clean "]

$$n = 3^r$$

gap>

List([3,9,27,81,243],j-> nilclean(j));

["is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean "]

gap>

$$n = 5^r$$

gap> List([5,25,125,625],j->nilclean(j));

["is not nil-clean ", "is not nil-clean ", "is not nil-clean ", "is not nil-clean "]

On this basis, we try to illustrate the reasons why z_n is not nil-clean ring such that p^r. Clearly, by Theorem 1.7, $(Z_n, +, .)$ Z_n has no idempotent elements other than [0] and [1]. Therefore, we can not construct all elements of z_n by the sum of a nilpotent and an idempotent. Hence, we can establish a new corollary about nil-clean rings:

Corollary 3.3: $(z_p, +, .)$ is not a nil-clean ring where $n = p^r$, for some prime p and some positive integer r.

References

[1] Barnes, W.E., 1963. Introduction to Abstract Algebra Boston. *Mass.: DC Heath & Co*.

[2] Chen, H. and Sheibani, M., 2017. On strongly nil clean rings. *Communications in Algebra*, *45*(4), pp.1719-1726.

[3] Han, J. and Nicholson, W.K., 2001. Extensions of clean rings. *Communications in Algebra*, *29*(6), pp.2589-2595.

[4] Chen, H. and Sheibani, M., 2017. Strongly 2-nil-clean rings. *Journal of Algebra and its Applications*, *16*(09), p.1750178.

[5] Diesl, A.J., 2013. Nil clean rings. *Journal of Algebra*, *383*, pp.197-211.

[6] Malik, D.S., Mordeson, J.N. and Sen, M.K., 2007. MTH 581-582: Introduction to Abstract Algebra. *United States of America*.

لەم توێژینەوەیەدا، ئێمە چەمکی نیل-کلین ڕینگمان ناساندوە کە بریتییە لەو ڕینگەی کە هەموو دانەکانی بریتییە لە کۆکراوەی نیل پۆتێنتێک و ئایدێنت پۆتێنێتێک.لەم توێژینەوە کورتەدا ئێمە فلۆچارتێکمان نەخشاندوە کە پیشانی دەدات ئەو ڕینگەی هەمانە هەمان نیل-کلین ڕینگە یان نا. هەروەها ئێمە پڕۆگرامێکمان دیزاین کردوە بۆ پیشاندانی ئەو ڕینگەی هەمانە نیل-کلین ڕینگە یان نا لە کۆتایدا پۆلێنی نیل-کلین ڕینگمان کردووە لە Zn دا.