

Chapter one

Differential Equations (DEs.).

Equation: Equations describe the relations between the dependent and independent variables. An equal sign "=" is required in every equation.

Differential Equations: Equations that involve dependent variables and their *derivatives* with respect to the independent variables are called *differential* equations.

Differential Equation:- Is an equation consist algebraic function or non-algebraic function or both of them which contains derivative. Divide DE. in two types

- 1- Ordinary DEs. .
- 2- Partial DEs.

Ordinary Differential Equations: Differential equations that involve only *ONE independent variable* are called *ordinary* differential equations.

Partial Differential Equations: Differential equations that involve *two or more independent variables* are called *partial* differential equations.

Examples:

$$1) \left(\frac{\partial^2 z}{\partial x \partial y} \right) + 6xy \left(\frac{\partial^2 z}{\partial x^2} \right) = 2x \quad P$$

$$2) \frac{dy}{dx} + 2xy = e^x \quad O.$$

$$3) \frac{d^2 y}{dx^2} + y \cos x \frac{dy}{dx} = \tanh x \quad O.$$

$$4) \left(\frac{\partial f}{\partial x} \right)^2 - 3x \frac{\partial f}{\partial x} = \cos x \quad P.$$

$$5) y''' - 3yy'' - 2x(y')^3 = 7 \quad O.$$

Order of ODE:- Is an order of the highest derivative in which occurs.

Degree of ODE:- Is the highest power of the highest derivative in which occurs.

Examples:

$$1) \left(\frac{\partial^2 z}{\partial x \partial y} \right) + 6xy \left(\frac{\partial^2 z}{\partial x^2} \right) = 2x \quad Or.2 \quad D.1$$

$$2) \frac{dy}{dx} + 2xy = e^x \quad Or.1 \quad D.1$$

$$3) \frac{d^2 y}{dx^2} + y \cos x \frac{dy}{dx} = \tanh x \quad Or.2 \quad D.1$$

$$4) \left(\frac{\partial f}{\partial x} \right)^2 - 3x \frac{\partial f}{\partial x} = \cos x \quad Or.1 \quad D.2$$

$$5) y''' - 3yy'' - 2x(y')^3 = 7 \quad Or.3 \quad D.1$$

Note: if the O.D.E containing the roots or rational power then to find the degree of this ODE. can be reduces this roots or rational powers.

Example:1) $y''' + 6\sqrt{(y')^2 + y^2} = 0$

$$\frac{d^3 y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2} = 0$$

$$\frac{d^3 y}{dx^3} = -6\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2}$$

Square both side of the equation

$$\left(\frac{d^3 y}{dx^3} \right)^2 = 36 \left[\left(\frac{dy}{dx} \right)^2 + y^2 \right] \quad \text{Order 3, degree e2}$$

$$2) \sqrt[3]{(y'')^2} = \sqrt{1 + (y')^2} \quad (\text{Ordinary DE.}) \quad \text{Or. 2} \quad \text{D.4}$$

Linear O.D.E

A.D.E. in any order is said to be linear if satisfies:-

- 1) The dep.v .is exist and of the first degree.
- 2) The derivatives y', y'', y''', \dots exist and each of them of the first degree.
- 3) The dep.v. and the derivatives not multiply by each other.

Note: If one of these conditions is not satisfied, then the equation considerate non- Linear.

Examples:

$$1) y'' + 4xy' + 2y = \tan x \quad (\text{L.})$$

$$2) \frac{d^2 y}{dx^2} + 3x \left(\frac{dy}{dx} \right)^2 + 5y = x^2 \quad (\text{Non.L.})$$

$$3) y'' + 4yy' + 2y = \cos x \quad (\text{Non.L.})$$

4) $y'' + 3xy' + 5y^6 = x^2$ (Non.L.)

The Solutions of ODE

Solutions: A functional relation between the dependent variable y and the independent variable x that satisfies the given ODE in some interval J is called a solution of the given ODE on J .

Type of solutions:

- 1) The general solution, denoted by y_G
- 2) The particular solution, denoted by y_p .
- 3) The singular solution, denoted by S .

General Solution: Solutions obtained from integrating the differential equations are called general solutions. The general solution of a n^{th} order ordinary differential equation contains n arbitrary constants resulting from integrating n times.

Example:solve: $\frac{dy}{dx} = 2y$, $\frac{dy}{y} = 2dx$ ($y \neq 0$)

$$\text{Lin } y=2x+c \quad y=e^{2x} \cdot e^c, \quad \text{suppose } e^c=k$$

$y=k e^{2x}$ Is the G. solution.

Particular Solution: Particular solutions are the solutions obtained by assigning(giving) specific values to the arbitrary constants in the general solutions.

Example/pervious example

Choose $(x, y) = (0, 1) \Rightarrow k = 1$

$y=e^{2x}$ is the particular solution.

Note: To find the particular solution of O.D.E in the G. solution by giving the value of arbitrary constant as follows.

By giving the value of dependent variable and the value of independent variable (Represent the integral curve).Obtained the value of arbitrary constant and substituted in the G. solution ,we get the particular solution .(These values of (x, y) is called initial conditions or boundary conditions).

Example: $y'' + y = 0$ $y(0) = 0, y\left(\frac{\pi}{2}\right) = 2.$,

Singular Solutions: Solutions that cannot be expressed by the general solutions are called singular solutions.

Example: $y' = 2y \Rightarrow \frac{dy}{y} = 2dx$ If $y=0$ then $\left(\frac{dy}{dx} = 2y\right)$ is undefined, then $y=0$
is the singular solution.

Elimination the arbitrary constants (Finding the O.D.E if the G. Solution is Exist)

- 1) Differential the G. solution n-times (the number of arbitrary constant =the order of O.D.E)
- 2) We obtain (n+1) of equations.
- 3) we can solve these equations by simultaneous way (or by determinant way).
- 4) Substituted the value of arbitrary constants in the G. solution, we get the O.D.E.

Three kind of method to solve the problems

1- Elimination method

Example: find the D.E if the G. solution is $y=Ax^3+Bx^2+Cx$ ----- (1)

Solution: $y'=3Ax^2+2Bx+C$ ----- (2)

$$y''=6Ax+2B$$

$$y'''=6A$$

$$A=\frac{1}{6}y''', B=\frac{1}{2}y''-\frac{1}{2}xy''', C=y'+\frac{1}{2}x^2y'''-xy''$$

Substituted value of A, B and C in the G. solution , we get the O.D.E of third order

$$Y=\frac{1}{6}y'''x^3+\left(\frac{1}{2}y''-\frac{1}{2}y'''x\right)x^2+\left(y'+\frac{1}{2}x^2y'''-xy''\right)x$$

2- Simultaneous method

Example: find the D.E if the G. solution is $y = Ae^x + B \cos x$

Solution

$$y = Ae^x + B \cos x ----- (1)$$

$$y' = Ae^x - B \sin x ----- (2)$$

$$y'' = Ae^x - B \cos x ----- (3)$$

From 1 & 2 we get

$$y + y'' = 2Ae^x$$

$$A = \frac{y+y''}{2e^x} ----- (*)$$

Put (*) in (2) we get

$$y' = \frac{y+y''}{2e^x}e^x - B \sin x$$

$$B = \frac{y+y''}{2 \sin x} - \frac{y'}{\sin x} ----- (**)$$

Put (*) & (**) in (1) we get ODE

$$y = \frac{y+y''}{2} + \left(\frac{y+y''}{2 \sin x} - \frac{y'}{\sin x} \right) \cos x$$

3- determinant method

Example: find the D.E if the G. solution is $y = Ax^2 + 6Bx$

Solution

$$y = Ax^2 + 6Bx$$

$$y' = 2Ax + 6B$$

$$y'' = 2A$$

$$y - Ax^2 - 6Bx = 0$$

$$y' - 2Ax - 6B = 0$$

$$y'' - 2A = 0$$

$$\begin{vmatrix} y & -x^2 & -6x \\ y' & -2x & -6 \\ y'' & -2A & 0 \end{vmatrix} = 0$$

$$y(-12) + x^2(6y'') - 6x(-2y' + 2xy'') = 0$$

$$y + \frac{1}{2}x^2y'' - xy' = 0 \text{ is DE}$$

Q/ prove that $y = Ax^2 + 6Bx$ is G.S $y + \frac{1}{2}x^2y'' - xy' = 0$ is DE

Example: find the O.D.E if the G. solution is, (H.W)

$$1) y = A e^{-x} + B e^x$$

$$2) y = C_1 \sin x + C_2 \cos x$$

$$3) y = A e^x + B x^2 + C x$$

$$4) y = c_1 e^{-2x} - c_2 \sin x$$

Chapter Two

Methods for Solving the O.D.E in the first order and first degree

- 1) Separation variable (separable).
- 2) Substitution method.
- 3) Homogenous D.E.
- 4) Non-homogenous D.E. of linear coefficient.
- 5) Exact D.E.

1) Separable DEs

Separable Function: A function $F(x, y)$ is called separable if can be written of the form.

$$F(x, y) = g(x).h(y) \text{ or } = \frac{h(y)}{g(x)} ; g(x) \neq 0.$$

Where g is a function of (x) only and h is a function of (y) only.

$$\text{Ex1: } F(x, y) = x^2 y \quad \text{SF}$$

$$\text{Ex2: } F(x, y) = x^2 \pm y \quad \text{Non-SF}$$

(S.D.E.): A D.E. $M(x, y) dx + N(x, y) dy = 0$ is called separable if both M and N are S. functions.

Example 1: solve: $\sin x \cos y dx + \sin y \cos x dy = 0$.

$$\text{Solution: } \frac{\sin x}{\cos x} dx + \frac{\sin y}{\cos y} dy = 0$$

$$\int \sin x \frac{1}{\cos x} dx + \int \sin y \frac{1}{\cos y} dy = 0$$

$$-\ln \cos x - \ln \cos y = c$$

$$\ln \cos x + \ln \cos y = -c$$

$$\ln(\cos x \cos y) = c$$

Take **exp** to both side we get

$$\cos x \cos y = e^{-c} \quad \text{let } e^{-c} = k$$

$$\cos x \cos y = k$$

$$\cos y = \frac{k}{\cos x}$$

$$y = \cos^{-1} \frac{k}{\cos x} \text{ is G.S.}$$

Example 2 : $(x^2 - y) dx + (x+1) dy = 0$ Non-S.D.E

Example 3 : solve: $4xy dy - ydx = x^2 dy$

$$(4x - x^2) dy - ydx = 0 \quad S.D.E \Rightarrow \int \frac{dy}{y} - \int \frac{dx}{(4x - x^2)} = 0$$

You finish it (one mark for any one solve it) $\rightarrow \ln(y) - (1/2)(\tanh^{-1}((x-2)/2)) = k$.

$$\ln y - \int \frac{dx}{x(4-x)} = o \Rightarrow \ln y - \int \left[\frac{A}{x} + \frac{B}{(4-x)} \right] dx = c$$

Example 4): solve :

$$x^3 y^3 dx + (x^2 + x^2 y) dy = 0$$

$$\{x^3 y^3 dx + x^2(1+y) dy = 0\} \quad \frac{1}{y^3 x^2}$$

$$xdx + \frac{(1+y)}{y^3} dy = 0 \Rightarrow \frac{x^2}{2} + y^{-3}(1+y) dy = 0$$

$$\frac{x^2}{2} + \frac{y^{-2}}{-2} + \frac{y^{-1}}{-1} = C_1 \Rightarrow \frac{x^2}{2} - \frac{y^{-2}}{2} - y^{-1} = C_1$$

H.w/ Solve:

$$1) x^3 dx + (y+1)^2 dy = 0$$

$$2) x^2(y+1) dx + y(x-1) dy = 0$$

$$3) xy + \sqrt{1+x^2} y' = 0$$

2) Substitution method.

If the D.E. of the form $y' = (ax + by) \dots \dots \dots (1)$

Non-Separable D.E., then

$$\text{Suppose } ax + by = z \dots \dots \dots (2)$$

$$adx + bdy = dz$$

$$dy = \frac{dz - adx}{b} \dots \dots \dots (3)$$

$$\frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b} \dots \dots \dots (4)$$

By substituting equation (2) and (4) in D.E (1) we get.

$$\frac{\frac{dz}{dx} - a}{b} = z \Rightarrow \frac{dz}{dx} = bz + a$$

Is S.D.E can be solved by previous way.

Example: Solve:- $y' = (x + y)^2$ (is not SDE) $\dots \dots \dots (1)$

Solution: Suppose $x + y = z \dots \dots \dots (2)$

$$dx + dy = dz \Rightarrow 1 + y' = z' \Rightarrow y' = z' - 1 \dots \dots \dots (3)$$

By putting equ (3) and equ (2) in equ (1) we get $z' - 1 = (z)^2$

$$z' = (z^2 + 1) \text{ is SDE} \Rightarrow \left[\frac{dz}{dx} = (z^2 + 1) \right] \quad \frac{1}{1+z^2} \quad \Rightarrow \int \frac{dz}{1+z^2} = \int dx$$

$$\Rightarrow \tan^{-1}(z) = x + c, z = \tan(x + c), y = \tan(x + c) - x \text{ is G.S.}$$

Example: solve:- $(x + y)dx + dy = 0$

Solution:

$$(x + y)dx = -dy$$

$$(x + y) = -\frac{dy}{dx}$$

$$(x + y) = -y'$$

$$y' = -(x + y) \quad \dots\dots (1)$$

$$\text{Suppose } x + y = z \quad \dots\dots (2)$$

$$dx + dy = dz$$

$$1 + y' = z'$$

$$y' = z' - 1 \quad \dots\dots (3)$$

Put (2) & (3) in (1) we get

$$z' - 1 = -z$$

$$z' = -z + 1$$

$$\frac{dz}{dx} = -z + 1$$

$$\frac{dz}{-z+1} = dx$$

Take integration to both side we get

$$-\ln(-z + 1) = x + c$$

$$\ln(-z + 1) = -x - c \quad \dots\dots (4)$$

Take e for (4)

$$-z + 1 = e^{-x-c}$$

$$-z = e^{-x-c} - 1$$

$$z = -e^{-x-c} + 1$$

$$x + y = -e^{-x-c} + 1$$

$y = -e^{-x-c} + 1 - x$ is G.S

Example (H.W.):- Solve: $y' = x^2 - 8xy + 16y^2$

3) Homogenous D.E

Homogenous function: A function $F(x, y)$ is called Homogenous function of n-th degree if satisfy the relation.

$$\forall t \in R \Rightarrow F(tx, ty) = t^n f(x, y)$$

Example: 1) Test the function

$$f(x, y) = x^2 y, t \in R$$

$$\begin{aligned} F(tx, ty) &= (tx)^2(ty) \\ &= (t^2x^2)(ty) \\ &= t^3 f(x, y) \Rightarrow F \text{ is H.F. of 3-rd degree} \end{aligned}$$

Example: 2) Test the function

$$F(x, y) = \frac{x}{y} \sin\left(\frac{y}{x}\right) + \cosh\left(\frac{y}{x}\right), \quad t \in R$$

$$F(tx, ty) = \frac{tx}{ty} \sin\left(\frac{ty}{tx}\right) + \cosh\left(\frac{ty}{tx}\right),$$

$$F(tx, ty) = t^0 f(x, y) \quad F \text{ is H.F. of zero - degree.}$$

Example: 3) Test the function

$$F(x, y) = x$$

$$F(tx, ty) = (tx) = t(x) = t(F(x, y))$$

F is H.F. of first degree.

Homogeneous D.E : A D.E. $M(x, y) dx + N(x, y) dy = 0$ is called H.D.E if both functions M and N are H. functions of the same degree (i.e.) [the H. degree of M=the H. degree of N]

Example: 1) $(x^2 + y^2)dx + x.y dy = 0$.

$$M(x, y) = (x^2 + y^2)$$

$$M(tx, ty) = (tx^2 + ty^2) = t^2(x^2 + y^2) = t^2 F(x, y) \text{ is H.F. of 2-nd degree.}$$

$$N(x, y) = (xy)$$

$$\begin{aligned} N(tx, ty) &= (tx)(ty) \\ &= t^2(xy) = t^2 N(x, y) \quad N \text{ is H.F. of 2-nd degree} \end{aligned}$$

DE is H. of 2-nd degree .

Example:-Solve:-

$$(x^2 + yx)dx + y^3 dy = 0. \quad \text{H.W}$$

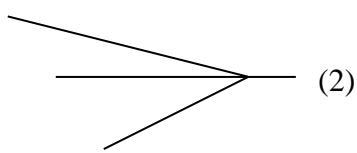
$$(x^2 + y)dx + y^2 dy = 0. \quad \text{H.W}$$

Note: Every homogeneous D. E of n-th degree can be reduced into separable D.E by using the relation

$$\left(V = \frac{y}{x} \right).$$

Example: solve: - $(x^2 + y^2)dx + xy dy = 0 \quad \text{---(1)}$ is HDE of 2-nd degree.

Solution: suppose $\frac{y}{x} = v$



(2)

$$dy = xdv + vdx$$

Sub. equ. (2) inequ. (1), we get the S.D.E.

$$(x^2 + x^2v^2)dx + x^2v(xdv + vdx) = 0$$

$$[x^2(1+v^2) + x^2v^2]dx + x^3v dv = 0$$

$$([x^2(1+v^2) + x^2v^2]dx + x^3v dv = 0) \quad) \quad \frac{1}{x^3(1+2v^2)}$$

$$\frac{dx}{x} + \frac{v}{(1+2v^2)}dv = 0$$

$$\ln x + \frac{1}{4}\ln(1+2v^2) = c$$

$$4\ln x + \ln(1+2v^2) = 4c$$

$$\ln x^4 + \ln(1+2v^2) = 4c$$

$$\ln[x^4(1+2v^2)] = 4c$$

$$x^4(1+2v^2) = c_1 \text{ where } e^{4c} = c_1$$

$$\text{Subst. } v = \frac{y}{x}$$

$$x^4(1+2\frac{y^2}{x^2}) = c_1$$

$$x^4 + 2y^2x^2 = c_1 \text{ is the G.solu.of DE (1).}$$

Examples:-Solve the following DEs. (H.W.)

$$1- 2e^x(1-\frac{y}{x})dx + (1+2e^x)\frac{y}{x}dy = 0$$

$$2- (2x+3y)dx + (y-x)dy = 0$$

$$3- (3x^2 - y^2)dx - xydy = 0$$

Example:Test the functions **in DE.(1)**

$$(x+y+1)dx + xdy = 0 --- (1)$$

$$M(x,y) = x + y + 1$$

$$M(tx,ty) = tx + ty + 1 = t(x, y + \frac{1}{t}) \neq t M(x,y) \text{ is non-homog.func.}$$

$$N(x,y) = x$$

$$N(tx,ty) = t x = t N(x,y) \text{ H.F.of first degree}$$

DE (1) is non-Homog. DE.

4) Non-Homogenous DE With Linear Coefficients:-

The general form of non-homogeneous DE with linear coefficients is
 $(ax + by + c)dx + (\alpha x + \beta y + \gamma)dy = 0 \dots\dots\dots(1)$

where a, b, c, α, β and γ are constants.

To change the Non-H.D.E. into H.D.E. or S.D.E., there exist two cases:-

Case1:- if $m_1 \neq m_2$ (two lines are intersected).

$$ax + by + c = 0 \dots\dots\dots(2) \Rightarrow m_1 = \frac{-a}{b}$$

$$\alpha x + \beta y + \gamma = 0 \dots\dots\dots(3) \Rightarrow m_2 = \frac{-\alpha}{\beta}$$

$$m_1 \neq m_2$$

(h, k) the intersection point

$$ah + bk + c = 0$$

$$\alpha h + \beta k + \gamma = 0 \quad \text{---} \quad (4)$$

Suppose $x = x_1 + h$ and $y = y_1 + k$

$$dx = dx_1 \text{ and } dy = dy_1 \quad \text{---} \quad (5)$$

Subst. equ (5) in D.E (1), we get the H.D.E

$$(a(x_1 + h) + b(y_1 + k) + c)dx_1 + (\alpha(x_1 + h) + \beta(y_1 + k) + \gamma)dy_1 = 0$$

$$[(ax_1 + by_1) + (bk + ah + c)]dx_1 + [(\beta y_1 + \alpha x_1) + (\beta k + \alpha h + \gamma)]dy_1 = 0$$

$$(ax_1 + by_1)dx_1 + (\beta y_1 + \alpha x_1)dy_1 = 0 \dots\dots\dots(6) \text{ is H.D.E.}$$

equ. (6) can be solved by homogenous method by supposes

$$v = \frac{y_1}{x_1}$$

$$\text{and } y_1 = x_1 v \quad \text{---} \quad (7)$$

$$dy_1 = x_1 dv + v dx_1$$

Subst equ. (7) in equ (6) we get the separable D.E., we can solve by integration immediately we get the G.solution .

In which contains two variable x_1, v subst. values of v ,

$$x_1 = x - h, \text{ and } y_1 = y - k$$

Example: solve $(2y - x - 5)dx + (3x + y + 1)dy = 0 \dots\dots\dots(1)$ (Non-H.D.E.)

Solution:

$$2y - x - 5 = 0$$

$$3x + y + 1 = 0$$

$$m_1 = \frac{-(-1)}{2} = \frac{1}{2}, m_2 = \frac{-(-3)}{-1} = -3$$

$$\Rightarrow \frac{1}{2} \neq -3 \Rightarrow m_1 \neq m_2 \exists (h, k) = (-1, 2)$$

$$2y - x - 5 = 0$$

$$3x + y + 1 = 0$$

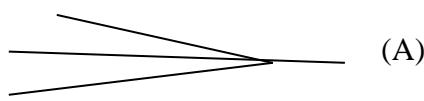
$$-7x - 7 = 0 \Rightarrow (x = -1),$$

$$(y = 2)$$

Suppose $x = x_1 + h$ and $y = y_1 + k$

$x = x_1 - 1$ and $y = y_1 + 2$

$dx = dx_1$ and $dy = dy_1$



Substequ. (A) inequ (1) we get the H. D. E.

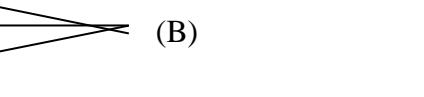
$$(2(y_1 + 2) - (x_1 - 1) - 5)dx_1 + (3(x_1 - 1) + (y_1 + 2) + 1)dy_1 = 0$$

$$(2y_1 + 4 - x_1 + 1 - 5)dx_1 + (3x_1 - 3 + y_1 + 2 + 1)dy_1 = 0$$

$$(2y_1 - x_1)dx_1 + (3x_1 + y_1)dy_1 = 0 \quad \text{---(2) is HDE}$$

Suppose $v = \frac{y_1}{x_1}$

and $y_1 = x_1v$



$$dy_1 = x_1dv + vdx_1$$

Substequ. (B) inequ (2) we get the separable DE

$$(2vx_1 - x_1)dx_1 + (3x_1 + vx_1)(x_1dv + vdx_1) = 0$$

$$[x_1(v^2 + 5v - 1)dx_1 + x_1^2(3 + v)dv = 0] \quad \frac{1}{(v^2 + 5v - 1)} \text{ Is SDE}$$

$$\int \frac{dx}{x} + \int \frac{3+v}{(v^2 + 5v - 1)} dv = 0$$

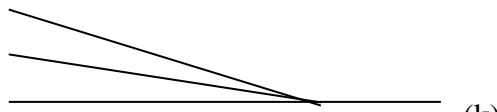
$$(\text{H.W}) \quad \text{solve } (x - 2y - 5)dx - (-3x + 6y + 1)dy = 0$$

Case 2: if $m_1 = m_2$ (the two lines are not intersected at the point, but are parallel).

Then suppose $ax + by = z$

$$adx + bdy = dz$$

$$dy = \frac{dz - adx}{b}$$



Subst. equ. (k) in D.E. (1) we get the S.D.E.

$$(z + c)dx + (mz + \gamma) \left(\frac{dz - adx}{b} \right) = 0$$

Can be solved, by integration immediately finally subst. the value of (z), we get G.solution.

m = multiple of (z)

Example: solve: $(3x - 6y + 5)dx + (12x - 24y - 2)dy = 0$ -----(1)

$$\text{Solution: } m_1 = \frac{-3}{-6} = \frac{1}{2}$$

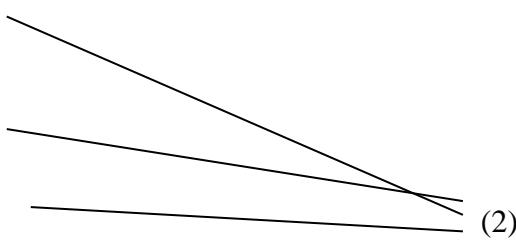
$$m_2 = \frac{-12}{-24} = \frac{1}{2} \Rightarrow m_1 = m_2$$

Then suppose

$$3x - 6y = z$$

$$3dx - 6dy = dz$$

$$dy = \frac{3dx - dz}{6}$$



Subst. equ. (2) in D.E. (1) we get the S.D.E.

$$(z + 5)dx + (4z - 2)\left(\frac{3dx - dz}{6}\right) = 0 \text{ is S.D.E.}$$

Example:- Solve the following D.Es.(H.W.)

$$1) (x + y + 2) dx + (-x - y + 2) dy = 0$$

$$2) (6x - 8y - 5) dy = (3x - 4y - 2) dx$$

Note / If the function of the form $f(x y)$ or the DE.of theform $y f(x y) dx + x g(x y) dy=0$

(1) Suppose $yx=v$

$$y = \frac{v}{x}$$

$$dy = \frac{x dv - v dx}{x^2}$$



Subst. equ.(2) in the DE.(1) ,we get $\frac{v}{x}f(v)dx + xg(v)\left(\frac{x dv - v dx}{x^2}\right)$ is SDE

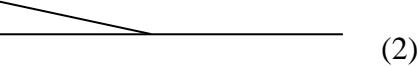
Example: solve:- $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$ ----- (1)

Solution :

Suppose $yx=v$

$$y = \frac{v}{x}$$

$$dy = \frac{x dv - v dx}{x^2}$$



Subst. equ.(2) in the D.E.(1)

$$\frac{v}{x}(v+1)dx + x(1+v+v^2)\left(\frac{x dv - v dx}{x^2}\right) = 0 \Rightarrow xv(v+1)dx + x(1+v+v^2)(xdv - vdx) = 0$$

$$(xv^2 + xv - xv - xv^2 - x v^3)dx + x^2(1+v+v^2)dv = 0$$

$$\int(-xv^3)dx + x^2(1+v+v^2)dv = \int 0 \quad S.D.E$$

$$(\int(-xv^3)dx + x^2(1+v+v^2)dv) = \int 0 \quad \frac{1}{-v^3 x^2} \Rightarrow \int \frac{dx}{x} - \int \frac{1+v+v^2}{v^3} dv$$

5) Exact D.E.

Exact DE: A D.E. $M(x, y) dx + N(x, y) dy = 0$ ----- (1) is called exact D.E. if $\exists df(x, y)$ such that

$$df(x, y) = M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy = 0 \text{ (by def. of exact function)}$$

$$\frac{\partial f}{\partial x} = M(x, y) \text{ and } \frac{\partial f}{\partial y} = N(x, y)$$

***** The necessary and sufficient conditions of D.E $M(x, y)dx + N(x, y)dy = 0$

is Exact D.E. if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ such that $M, N, M_x, M_y, N_x, N_y, \dots$ are continuous in R.

Example: solve: $(x+y) dx + (x-y) dy = 0$ ----- (1)

Solution: $M(x, y)dx + N(x, y)dy = 0$

$$\therefore M(x, y) = x + y \Rightarrow \frac{\partial M}{\partial y} = 1 \text{ and } N(x, y) = x - y \Rightarrow \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ than D.E. (1) is exact

$$df(x, y) = (x+y) dx + (x-y) dy = 0 \text{ s.t}$$

$$\frac{\partial f}{\partial x} = x + y \text{ -----(1)}$$

$$\frac{\partial f}{\partial y} = x - y \text{ -----(2)}$$

Integration both side of equ. (1) with respect to (x) and choosing the arbitrary function of (y) only

$$f(x, y) = \frac{x^2}{2} + yx + h(y) \text{ -----(4),}$$

Differentiation equ.(4) with respect to y

$$\frac{\partial f}{\partial y} = x + \frac{\partial}{\partial y} h(y) \text{ ----- (5)}$$

Put (1) in (5) we get

$$x-y = x + \frac{\partial}{\partial y} h(y)$$

$$-y = \frac{\partial}{\partial y} h(y) \text{ -----(6)}$$

Integrate both side of equ. (6) with respect to (y)

$$\int -y dy = \int \frac{\partial}{\partial y} h(y) dy$$

$$\frac{-y^2}{2} + c = h(y) \rightarrow h(y) = c - \frac{y^2}{2}, \text{ put } h(y) \text{ in (4) we get G.S as follow}$$

$$\therefore f(x, y) = \frac{x^2}{2} + yx - \frac{y^2}{2} + c$$

Example: solve: $2x(ye^{x^2} - 1)dx + e^{x^2} dy = 0$ ----- (1)

Solution: $\frac{\partial M}{\partial y} = 2xe^{x^2}$, $\frac{\partial N}{\partial x} = 2xe^{x^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ \therefore D.E. is exact, $\therefore df(x, y) = 2x(ye^{x^2} - 1)dx + e^{x^2}dy = 0$

$$\frac{\partial f}{\partial x} = 2x(ye^{x^2} - 1) \quad \text{--- (2)} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x^2} \quad \text{--- (3)}$$

Integration both side of equ. (3) with respect to (y) and choosing the arbitrary function of (x) only
 $f(x, y) = ye^{x^2} + Q(x)$ --- (*)

Differentiation equ.(*) with respect to x

$$\frac{\partial f}{\partial x} = y 2xe^{x^2} + \frac{\partial}{\partial x} Q(x) \quad \text{--- (4)} . \text{In both equation (2) and (4), we get}$$

$$2xye^{x^2} - 2x = y 2xe^{x^2} + \frac{\partial}{\partial x} Q(x) \Rightarrow Q'(x) = -2x \quad \text{--- (5)}$$

Integrate both side of equ. (5) with respect to (x)

$$\Rightarrow \int \frac{\partial}{\partial x} Q(x) dx = \int -2x dx$$

$$\Rightarrow Q(x) = -x^2 + c \Rightarrow f(x, y) = ye^{x^2} - x^2 + c \text{ be the G.solution.}$$

H.W.: Solve the following DES:-

$$(1) y^3 \sin 2x dx - \frac{3}{2} y^2 \cos 2x dy = 0 .$$

$$(2) (3x^2 + 3xy^2)dx + (3x^2y - 3y^2 + 2y)dy = 0 .$$

$$(3) (2ye^{2x} + 2x \cos y)dx + (e^{2x} - x^2 \sin y)dy = 0$$

Linear First Order Ordinary D.E

The general form of first order and first degree of ordinary DE is $a(x) \frac{dy}{dx} + b(x)y = c(x)$ --- (1); where

$a(x) \neq 0$ and a, b, c are functions of x only.

$$\frac{dy}{dx} + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}$$

$$\frac{dy}{dx} + p(x)y = Q(x) \text{ is the standard form of LFODE}$$

$$y' + p(x)y = Q(x) \quad \text{--- (2) (non-homog. & non-exact)}$$

$$y = \frac{\int Q(x)e^{\int p(x)dx}dx + c}{e^{\int p(x)dx}}, \text{ Or } y = e^{-\int p(x)dx} [\int Q(x)e^{\int p(x)dx}dx + c]$$

Is the general solution of non-homog. and non-exact D.E(liner first order)

Example: solve: $2xy' + xy = 3x^2$ ----- (1)

Solution: divide equ (1) by $2x$, we obtain,

$$y' + \frac{y}{2} = \frac{3}{2}x$$

By comparison with $y' + p(x)y = Q(x)$

$$p(x) = \frac{1}{2}, \quad Q(x) = \frac{3}{2}x$$

$$I.F = e^{\int p(x)dx} = e^{\int \frac{x}{2} dx} = e^{\frac{1}{2} \int x dx} = e^{\frac{x^2}{4}} \quad I.F = e^{\int p(x)dx} = e^{\int \frac{1}{2} dx}$$

$$y = e^{-\frac{x^2}{4}} \left[\int \frac{3}{2}x^2 e^{\frac{x^2}{4}} dx + c_1 \right]$$

Example: solve: $y' \sin x = y \cos x + \sin^2 x$ -----(1)

Solution: $y' \sin x - y \cos x = \sin^2 x$

$$y' - \frac{y \cos x}{\sin x} = \sin x \Rightarrow y' + p(x)y = Q(x); \quad p(x) = -\frac{\cos x}{\sin x} \quad \& \quad Q(x) = \sin x$$

$$y = (I.F)^{-1} \left[\int Q(x) e^{\int p(x)dx} dx + c \right]$$

$$I.F = e^{\int p(x)dx} = e^{\int \frac{\cos x dx}{\sin x}} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$y = \sin x \left[\int \sin x \frac{1}{\sin x} dx + c \right]$$

Bernoulli's Equation

Def: A DE $y' + p(x)y = Q(x)y^n$ -----(1) is said to be Bernoulli's Equation such that p, and Q are functions of (x) only and $n \neq 0, 1 (n \in R)$:

If $n=0 \Rightarrow y' + p(x)y = Q(x)$ is (F.O.L.D.E)

If $n=1 \Rightarrow y' + p(x)y = Q(x)y \Rightarrow y' + [p(x) - Q(x)]y = 0$ is (S.D.E)

$y' + p(x)y = Q(x)y^n$ ----- (1) (Bernolli's Equation)

To solve DE (1) we will divided it by (y^n) we obtain $y^{-n}y' + p(x)yy^{-n} = Q(x)$

$$y^{-n}y' + p(x)y^{1-n} = Q(x) \quad \text{---(2)}$$

Suppose $y^{1-n} = u \quad \text{--- (3)}$

$$(1-n)y^{1-n-1}y' = u'$$

$$y^n y' = \frac{u'}{(1-n)} \quad \text{--- (4)}$$

Subst both equ (3) & equ (4) in equ (2); we get an FOLDE

$$\frac{u'}{(1-n)} + p(x)u = Q(x)$$

$$\left[\frac{u'}{(1-n)} + p(x)u = Q(x) \right] \quad (1-n)$$

$$u' + (1-n)p(x)u = (1-n)Q(x) \quad \text{---(5) is FOLDE}$$

equ (5) can be solving by method of FOLDE. we get the value of u .Finally subst. the value of $y^{1-n} = u$, we get the general solution of Bernoulli's equ.

Example: solve: $(12e^{2x} y^2 - y)dx = dy$

Solution: $\frac{dy}{dx} = 12e^{2x}y^2 - y$

$$\frac{dy}{dx} + y = 12e^{2x}y^2 \quad \text{-----(1) Bernoulli's equ.}$$

Divide equ (1) by y^2 we obtain

$$y'y^{-2} + y^{-1} = 12e^{2x} \quad \text{-----(2)}$$

Let $y^{-1} = u \quad \text{-----(3)}$

Differentiation equ (3) with respect to x

$$-y^{-2}y' = u'$$

$$y^{-2}y' = -u' \quad \text{-----(4)}$$

Substequ (3) &equ (4) inequ.(2), we obtain a F.O.L.D.E.

$$-u' + u = 12e^{2x}$$

$u' - u = -12e^{2x}$ is a FOLDE $p(x) = -1$ & $Q(x) = -12e^{2x}$

$$u = e^{\int -1 dx} [\int -12e^{2x} e^{\int -1 dx} dx + c]$$

$$u = e^x [-12 \int e^x dx + c] \Rightarrow u = e^x [-12e^x + c] \Rightarrow$$

$$y^{-1} = -12e^{2x} + ce^x \Rightarrow \frac{1}{y} = -12e^{2x} + ce^x$$

$$y = \frac{1}{-12e^{2x} + ce^x} \text{ is the gernarl solution of Birnolli 's equ.}$$

Example: solve $\frac{dy}{dx} - \frac{4}{x}y = x\sqrt{y} \quad \text{-----(1)}$

Solution: $y' - \frac{4}{x}y = xy^{\frac{1}{2}} \Rightarrow y'y^{-\frac{1}{2}} - \frac{4}{x}y^{\frac{1}{2}} = x \quad \text{-----(2)}$

Suppose $-y^{\frac{1}{2}} = u \quad \text{--- (3)} \Rightarrow -\frac{1}{2}y^{-\frac{1}{2}}y' = u' \Rightarrow (y^{-\frac{1}{2}}y') = -2u' \quad \text{--- (4)} \quad , \quad \text{Subst (3) & (4) in equ(2)}$

we obtain FOLDE

$$[-2u' + \frac{4}{x}u = x] \quad (\frac{-1}{2})$$

$$u' - \frac{2}{x}u = -\frac{x}{2} \text{ is a FOLDE}$$

$$I.F = e^{\int p(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$u = x^2 \left[\int \frac{-x}{2} x^{-2} dx + c \right]$$

H.W: solve the following equations.

1) $2y' - xy = x$

$$2) \frac{dx}{dy} + \frac{3}{y}x = 2y$$

$$3) (x+1)y' - y = e^x(x+1)^2$$

Simultaneous Ordinary Differential Equation

Is a set of equations which contains only one independent variable and the number of equations are equal to the number of the dependent variables , as fallows:

$$\frac{dx}{dt} = f(x, t) \quad \dots \quad (1)$$

$$\frac{dy}{dt} = g(x, y, t) \quad \dots \quad (2)$$

We can solving system (*) by choosing the D.E in which contains only one dependent variable say equ.(1) & solving by integration immediately or by previous ways , we get the value of dependent variable and subst. in equ. (2) , and solving by previous way , we get the G. solution of system (*)

Example: solve: $\frac{dx}{x-t} = \frac{dy}{x+y} = \frac{dt}{2t}$

Solution :

$$\frac{dx}{x-t} = \frac{dt}{2t} \Rightarrow \frac{dx}{dt} = \frac{x-t}{2t} \quad \dots \quad (1)$$

$$\frac{dy}{x+y} = \frac{dt}{2t} \Rightarrow \frac{dy}{dt} = \frac{x+y}{2t} \quad \dots \quad (2)$$
(A)

To find the general Solution, we can choose the first equ.

$$\frac{dx}{dt} = \frac{x-t}{2t}$$

$$\frac{dx}{dt} - \frac{x}{2t} = \frac{-1}{2} \quad [\text{FOLDE}]$$

$$\frac{dx}{dt} + p(t)x = Q(t)$$

$$p(t) = \frac{-1}{2t}, \quad Q(t) = -\frac{1}{2}$$

$$x = \sqrt{t} \left[\int \left(\frac{-1}{2} \right) \frac{1}{\sqrt{t}} dt + c \right]$$

$$x = t^{\frac{1}{2}} \left[-\frac{1}{2} \frac{\sqrt{t}}{\frac{1}{2}} + c \right] \Rightarrow x = -t + ct^{\frac{1}{2}}$$

$$x = -t + ct^{\frac{1}{2}} \text{ subst. the value of (x) in equ. (2) in system A}$$

$$\frac{dy}{dt} = \frac{(ct^{\frac{1}{2}} - t) + y}{2t}$$

$$\frac{dy}{dt} = \frac{ct^{\frac{-1}{2}}}{2} - \frac{1}{2} + \frac{y}{2t}$$

$$\frac{dy}{dt} - \frac{1}{2t}y = \frac{ct^{\frac{-1}{2}}}{2} - \frac{1}{2} \text{ is F.O.L.D.E}$$

Chapter Three

Reduction of Higher Order Ordinary D.E.

Consider a D.E of higher order is $F(x, y, y', y'', \dots, y^n) = 0$ can be reduced into first order by using substitution. We can study this D.E of second order, then equ. (1) becomes $g(x, y, y', y'') = 0$ equ(2) can be solved by two cases:

Cases 1: If the dependent variable (y) does not (explicity) or (appears) in the D.E (2), then equ(2), becomes

$$g(x, y', y'') = 0 \quad \dots \quad (3)$$

Suppose $\frac{dy}{dx} = y' = p \dots \dots \dots (4)$

$$\frac{d^2y}{dx^2} = y'' = p' = \frac{dp}{dx} \quad \text{---(4)}$$

$$y'' = \frac{dp}{dx} \quad \text{---(5)}$$

Subst. both equ (4) and (5) in D.E (3) we get a relation between (x, p)

$g_2(x, p, \frac{dp}{dx}) = 0$ ----(6) can be solved by previous way and we get (p) subst. (p) by $(\frac{dy}{dx})$ and by integration immediately we get the g.solution

Cases 2: If the independent variable (x) does not (explicity) or (appears) in the D.E (2), then equ(2), becomes $g(y, y', y'') = 0 \dots (*)$

$$\frac{d^2y}{dx^2} = y'' = p' = \frac{dp}{dx} \Rightarrow y'' = \frac{dp}{dx} \cdot \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \cdot \frac{dp}{dy} \quad \text{--- (***) (by chain rule)}$$

Subst. both equ (***) and (****) in D.E (*) we get a relation between (y, p)

$g^{**}(x, p, p \frac{dp}{dy}) = 0$ -----(*****) can be solving by previous way and we get (p) subst. (p) by $(\frac{dy}{dx})$ and by integration immediately we get the g.solution

Examples: solve the following D.E.

$$1) xy'' + (y')^3 = 0 \dots\dots\dots(1)$$

$$3) x^2 y'' - (y')^2 - 2xy' = 0 \text{ H.W}$$

$$4) \quad \gamma''' - \gamma'' = 1$$

$$5) \quad 2y'' - (y')^2 = 0 \quad H.W.$$

Solution 1) since (y) not appears.

$$\begin{aligned} \frac{dy}{dx} &= y' = p \\ p' = y'' &= \frac{dp}{dx} \quad \xrightarrow{\text{(2)}} \text{Subst. equ (2) in equ (1) we get } xp' + p^3 = 0 \\ \{xp' + p^3 = 0 \quad S.D.E\} \quad \frac{dy}{xp^3} \end{aligned}$$

$$\int \frac{dp}{p^3} + \int \frac{dx}{x} = 0 \Rightarrow \left\{ \frac{1}{-2} p^{-2} + \ln x = c \right\}. -2$$

$$\frac{1}{p^2} - 2\ln x = -2c \quad \text{Suppose } c_1 = -2c$$

$$\frac{1}{p^2} + \ln x^{-2} = c_1 \Rightarrow \frac{1}{p^2} = c_1 - \ln x^{-2} \Rightarrow p = \frac{1}{\sqrt{c_1 - \ln x^{-2}}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{c_1 - \ln x^{-2}}}$$

$$y = \int \frac{1}{\sqrt{c_1 - \ln x^{-2}}} dx$$

Solution 2) since (x) not appears in equ.(*)

Then Suppose $\frac{dy}{dx} = y' = p$

$$\frac{d^2y}{dx^2} = y'' = p' = \frac{dp}{dx} \quad (**)$$

$$y'' = p \cdot \frac{dp}{dy} \quad --- (***)$$

Subst. both equ (***) and (****) in D.E (*) we get a relation between (y, p)

$$\left\{ \begin{array}{l} y p \frac{dp}{dy} + p^2 = 0 \\ \frac{dy}{p^2 y} \end{array} \right\} S.D.E$$

$$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$$

$$\ln p + \ln y = c$$

$$\ln(p \cdot y) = c$$

$$p \cdot y = e^c \text{ let } c_1 = e^c$$

$$p \cdot y = c_1$$

$$y dy = c_1 dx$$

Solution 3) since x and y don't appear in D.E (1), then solving by any cases. Suppose we can solving by first case.

$$y''' - y'' = 1 \quad --- (1)$$

$$\frac{dy}{dx} = y'' = p$$

$$p' = y''' = \frac{dp}{dx} \quad --- (2)$$

$$\frac{dp}{dx} - p = 1$$

$$\frac{dp}{dx} = 1 + p \quad S.D.E$$

$$\frac{dp}{1+p} = dx$$

$$\ln(1+p) = x + c$$

$$1 + p = k e^x \quad \text{let } e^c = k$$

$$p = k e^x - 1$$

$$y'' = k e^x - 1 \quad --- (3) \quad (y \text{ does n't appears})$$

$$\frac{dy}{dx} = y' = p$$

$$p' = y'' = \frac{dp}{dx}$$

(4)

Substequ (4) in D.E (3), we get

$$\frac{dp}{dx} = ke^x - 1$$

$$\int dp = \int (ke^x - 1) dx$$

$$p = ke^x - x + c$$

$$dy = (ke^x - x + c) dx$$

$$y = ke^x - \frac{x^2}{2} + xc + c_1$$

be the g.solution of D.E (1)

Example: solve H.W

$$1) xy'' + y' = 3x^2 - x$$

$$2) (1+x^2)y'' - 2xy' = 2x$$

$$3) y'' + 2y(1+y')^2 = 0$$

$$4) y'' + (y')^2 + y = 0$$

Higher Degree of ordinary D.E

O.D.E of the first order but of the higher degree. The general form of higher degree of O.D.E is $a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots + a_{n-1} p + a_n = 0$ where a_0, a_1, \dots, a_n are constant and $(a_0 \neq 0)$ ---(1). To solve equ.(1) there exist three cases:

Case1: equ. (1) Solvable for (p). if equ.(1) can be written of the form

$$(p - f_1(x, y))(p - f_2(x, y)) \dots (p - f_n(x, y)) = 0$$

$$p - f_1(x, y) = 0 \Rightarrow p = f_1(x, y) \Rightarrow \frac{dy}{dx} = f_1(x, y) \Rightarrow g_1(x, y) = c$$

$$p - f_2(x, y) = 0 \Rightarrow p = f_2(x, y) \Rightarrow \frac{dy}{dx} = f_2(x, y) \Rightarrow g_2(x, y) = c$$

Continue in this way, we get $g_n(x, y) = c$.

Then the general solution of this case is $[g_1(x, y) - c][g_2(x, y) - c] \dots [g_n(x, y) - c] = 0$; where c is constant.

Example: solve: $\left(\frac{dy}{dx}\right)^2 + x \frac{dy}{dx} + y \frac{dy}{dx} + xy = 0$ ---(1)

Solution: Suppose $\frac{dy}{dx} = p \Rightarrow p^2 + xp + yp + xy = 0 \Rightarrow p(p + x) + y(p + x) = 0$

$$(p + x)(p + y) = 0, \text{ or } p + x = 0 \Rightarrow dy + x dx = 0 \Rightarrow y + \frac{x^2}{2} = c$$

$$\text{and } p + y = 0 \Rightarrow dy / y + dx = 0 \Rightarrow \ln y + x = c$$

The general solution of D.E.O is $(y + \frac{x^2}{2} - c)(\ln y + x - c) = 0$ where c is a constant.

Solves the follows equation. (H.W)

$$(1) \quad yp^2 + (x - y)p - x = 0$$

$$(2) \quad x y p^2 + (x^2 + xy + y^2)p + x^2 + xy = 0$$

Case 2: equ.(1) solvable for y

Equ.(1): $a_0 p^n + a_1 p^{n-1} + \dots + a_{n-1} p + a_n = 0$ can be written of the form $F(x, y, p) = 0 \Rightarrow y = F(x, p)$ ---(2)

differentiation equ.(2) with respect to x and subst. $\frac{dy}{dx} = p$ we obtain a relation between x, p can be solved by

previous way and obtained the value of p.

Finally subst. the value of p in equ.(2), we get the g. solution, and the singular

Example: 1) solve: $3p^5 - py + 2 = 0$

$$\text{Solution: } 3p^5 - py + 2 = 0 \text{ ---(1)}$$

$$py = 3p^5 + 2 \Rightarrow y = 3p^4 + \frac{2}{p} \text{ ---(2)}$$

$$\frac{dy}{dx} = p = 12p^3 \frac{dp}{dx} + \frac{-2}{p^2} \Rightarrow p = 12p^3 \frac{dp}{dx} - \frac{2}{p^2} \frac{dp}{dx} \Rightarrow 12p \frac{dp}{dx} - \frac{2}{p^2} \frac{dp}{dx} - p = 0$$

$$2p \frac{dp}{dx} (6p^2 - \frac{1}{p^3}) - p = 0 \Rightarrow [2(6p^2 - \frac{1}{p^3}) \frac{dp}{dx} - 1] * \frac{1}{p} = 0 \Rightarrow 2(6p^2 - \frac{1}{p^3}) dp - dx = 0$$

2) Solve: $y = 2xp + \tan^{-1}(xp^2)$

Solution: $y = 2xp + \tan^{-1}(xp^2)$ ---(1)

$$\begin{aligned} \frac{dy}{dx} &= p = 2(x \frac{dp}{dx} + p) + \frac{1}{1+(xp^2)^2} \cdot (2px \frac{dp}{dx} + p^2) \\ p &= 2x \frac{dp}{dx} + 2p + \frac{2xp \frac{dp}{dx} + p^2}{1+x^2 p^4} \Rightarrow p + 2x \frac{dp}{dx} + \frac{2xp \frac{dp}{dx} + p^2}{1+x^2 p^4} = 0 \\ p(1+\frac{p}{(1+x^2 p^4)}) + 2x \frac{dp}{dx}(1+\frac{p}{1+x^2 p^4}) &= 0 \Rightarrow (1+\frac{p}{(1+x^2 p^4)})(p + 2x \frac{dp}{dx}) = 0 \\ p + 2x \frac{dp}{dx} = 0] \frac{dx}{xp} &\Rightarrow \int \frac{dx}{x} + 2 \int \frac{dp}{p} = 0 \Rightarrow \ln x + 2 \ln p = c_1 \Rightarrow \ln x + \ln p^2 = c_1 \\ \ln(xp^2) = c_1 &\Rightarrow xp^2 = e^{c_1} = c \Rightarrow p = \sqrt{\frac{c}{x}} \\ y = 2x \sqrt{\frac{c}{x}} + \tan^{-1}(c) &\text{ is the g. solution or } 1 + \frac{p}{1+x^2 p^4} = 0 \Rightarrow x^2 p^4 + p = 0 \end{aligned}$$

H.W: solve the following D.E

- (1) $xp^2 + x = 2yp$
- (2) $y + px = x^4 p^2$
- (3) $y = (2+p)x + p^2$

Case 3: equ.(1) solvable for (x) equ.(1) $a_0 p^n + a_1 p^{n-1} + \dots + a_{n-1} p + a_n = 0$ can be written of the form

$F(x, y, p) = 0 \Rightarrow x = F(y, p) \dots$ (2) differentiation equ.(2) with respect to y and subst. $\frac{dx}{dy} = \frac{1}{p}$ we obtain

the a relation between y, p can be solved by previous way and obtain the value of p. finally subst. the value of p in equ.(2) we get the g. solution .

Example: solve: $y = 3px + 6y^2 p^2 \dots$ (1)

Solution: since equ.(1) solvable for x, then

$$3px = y - 6y^2 p^2$$

$$3x = \frac{y}{p} - 6y^2 p$$

$$3 \frac{dx}{dy} = 3 \cdot \frac{1}{p} = \frac{p - y \frac{dp}{dy}}{p^2} - 6y^2 \frac{dp}{dy} - 12py \Rightarrow \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2} \frac{dp}{dy} - 6y^2 \frac{dp}{dy} - 12yp$$

$$[\frac{2}{p} + \frac{y}{p^2} \frac{dp}{dy} - 6y^2 \frac{dp}{dy} - 12yp = 0] * p^2 \Rightarrow (2p + y \frac{dp}{dy}) + (6y^2 p^2 \frac{dp}{dy} + 12yp^3) = 0$$

$$(2p + y \frac{dp}{dy}) + 6yp^2(y \frac{dp}{dy} + 2p) = 0 \Rightarrow (2p + y \frac{dp}{dy})(1 + 6yp^2) = 0$$

$$2p + y \frac{dp}{dy} = 0] * \frac{dy}{py} \Rightarrow 2 \frac{dy}{y} + \frac{dp}{p} = 0 \Rightarrow \ln y^2 + \ln p = c \Rightarrow \ln(y^2 p) = c$$

and $y^2 p = c_1 \Rightarrow p = c_1 / y^2 \Rightarrow y = 3 \frac{c_1}{y^2} x + 6y^2 \frac{c_1^2}{y^4}$ be the general solution.

$$\text{or } 1 + 6yp^2 = 0 \Rightarrow 6yp^2 = -1 \Rightarrow p^2 = \frac{-1}{6y} \Rightarrow p = \pm \sqrt{\frac{-1}{6y}} \notin \mathbb{C}$$

H.W: solve (1) $p^3 - p(y - 3) + x = 0$

$$(2) 3py + 6x^2 p^2 = x$$

$$(3) 3px + 6x^2 p = xy$$

Clairaut's Equation: an equation of the form $y = px + f(p)$ --- (1) where f is a function of (p) only is called Clairaut's equation and we can obtain the general solution by subst. p by c as follows: $y = cx + f(c)$ is the g. solution of equ.(1) where c is a constant differentiation equ.(1) with respect to x and subst. $\frac{dy}{dx} = p$, we obtain

$$\frac{dy}{dx} = p = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}$$

$$\frac{dp}{dx}(x + f'(p)) = 0$$

$$dp/dx = 0 \Rightarrow dp = 0 \Rightarrow p = c, \therefore y = cx + f(c)$$

$$\text{or } x + f'(p) = 0 \Rightarrow f(p) = x^2/2 + c.$$

Example: solve: $y = px + \sqrt{4 + p^2}$ --- (1)

Solution: since equ.(1) is Clairaut's equation. Then $p = c$

$$y = cx + \sqrt{4 + c^2}$$

Example: solve: $p = \sin(y - xp)$

Solution: $p = \sin(y - xp)$ --- (1)

$$\sin^{-1} p = \sin^{-1} \sin(y - xp) \Rightarrow \sin^{-1} p = (y - xp) \Rightarrow y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p \quad \text{--- (2) is Clairaut's equation.}$$

$$\frac{dy}{dx} = p = p + x \frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}} \cdot \frac{dp}{dx} \Rightarrow \frac{dp}{dx} \left(x + \frac{1}{\sqrt{1-p^2}} \right) = 0 \Rightarrow \frac{dp}{dx} = 0 \Rightarrow p = c$$

$$\therefore y = cx + \sin^{-1} c \text{ is the general solution}$$

$$\text{or } x + \frac{1}{\sqrt{1-p^2}} = 0 \Rightarrow \frac{1}{\sqrt{1-p^2}} = -x \Rightarrow \sqrt{1-p^2} = \frac{-1}{x} \Rightarrow 1-p^2 = \frac{1}{x^2} \Rightarrow 1-\frac{1}{x^2} = p^2$$

$$\Rightarrow p = \pm \sqrt{1-\frac{1}{x^2}} \text{ be a singular solution.}$$

H.W: solve the following D.E.

$$(1) (y - px)^2 = 1 + p^2$$

$$(2) p^3 - y + xp = 0$$

$$(3) y - p - p^2 = xp$$

Higher order ordinary D.E with constant coefficients:

The general form of higher order O.D.E is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_3 \frac{d^3 y}{dx^3} + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{d^1 y}{dx^1} + a_0 y = Q(x) \quad \text{--- (1) where } a_1, a_2, \dots, a_n \text{ are constants.}$$

$$\text{An } a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = Q(x) \quad \text{--- (1)}$$

If $Q(x) \neq 0$, then equ.(1) is called non-Homogeneous D.E with constant coefficient if $Q(x) = 0$, then equ.(1) becomes

$$\text{an } \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{--- (2) is called a H.D.E with c.c.}$$

If at least one of the coefficients a_0, a_1, \dots, a_n is a function of x only then equ.(1) becomes a_n

$$(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = Q(x) \quad \text{--- (3) equ.(3) is called NON.H.D.E}$$

with variable coefficients or if $Q(x) = 0$ in equ.(3), then equ.(3) becomes

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (4)$$

Equ.(4) is called a H.D.E with V.C.

Operators: (D)

Is a differentiation of any dependent variable with respect to independent variable.

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, D^3 = \frac{d^3}{dx^3}, \dots, D^n = \frac{d^n}{dx^n}$$

$$\text{Or } Dy = \frac{dy}{dx}, D^2 y = \frac{d^2 y}{dx^2}, D^3 y = \frac{d^3 y}{dx^3}, \dots, D^n y = \frac{d^n y}{dx^n}$$

Properties:

$$(1) D^n D^m (f(x)) = D^n D^m (f(x)) = D^{n+m} f(x)$$

$$(2) (D^n + D^m) f(x) = (D^n + D^m) f(x) = D^m f(x) + D^n f(x)$$

$$(3) D(f + g)(x) = Df(x) + Dg(x)$$

$$(4) \text{if } C \text{ is constant then } D(Cf(x)) = C Df(x)$$

$$(5) (D-a)(D-b)y = (D-b)(D-a)y \text{ where } a, b \text{ are constants.}$$

$$(6) (D-a(x))(D-b)y \neq (D-b)(D-a(x))y \text{ where at least one of them (a) or (b) is a function of (x) only.}$$

$$\text{Ex: } (D-2)(D-3)e^x = (D-3)(D-2)e^x.$$

$$\text{Ex: } (D-2x)(D-3)e^x \neq (D-3)(D-2x)e^x$$

$$\text{Proof (5)} : (D-a)(D-b)y = D^2 y - Dby - aDy + aby$$

$$= D^2 y - bDy - aDy + aby \quad (*)$$

$$(D-b)(D-a)y = D^2 y - Day - bDy + aby$$

$$= D^2 y - aDy - bDy + aby \quad (**)$$

$$\therefore (D-a)(D-b)y = (D-b)(D-a)y \text{ Where } a, b \text{ are constants.}$$

$$\text{Proof (6)} : (D-a(x))(D-b)y = D^2 y - Dby - a(x)Dy + a(x)by$$

$$(D-a(x))(D-b)y = D^2 y - bDy - a(x)Dy + a(x)by \quad (1)$$

$$(D-b)(D-a(x))y = D^2 y - D(a(x)y) - bDy + ba(x)y$$

$$(D-b)(D-a(x))y = D^2 y - a(x)Dy - yDa(x) - bDy + ba(x)y \quad (2)$$

Since equ(1) \neq equ(2) then

$$(D-a(x))(D-b)y \neq (D-b)(D-a(x))y$$

Note: we can write higher order O.D.E with constant coefficient or with the variable coefficient (Homog.or.non-Homog) by using the operator

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = Q(x) \quad (1)$$

$$a_n D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y + a_0 y = Q(x)$$

$$(a^n D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y) y = Q(x) \quad (*)$$

$$f(D)y = Q(x) \text{ Where}$$

$$f(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0$$

$$f(D)y = 0 \quad \text{--- (***) Homog.L.D.E}$$

With C.C. ($y \neq 0$) $\Rightarrow f(D) = 0$ the $f(D)$ is called the characteristic equation.

$[a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_2(x)D^2 + a_1(x)D + a_0(x)]y = Q(x) \quad \text{--- (** non-Homg. with V.C)}$

Or $[a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_2(x)D^2 + a_1(x)D + a_0(x)]y = 0 \quad \text{--- (***) Homog. With V.C.}$

Reduction of higher order O.D.E in to first order D.E with c.c.

$$a_n D^n + a_{n-1} D^{n-1} + \dots + a_3 D^3 + a_2 (D^2 + a_1 D + a_0) y = Q(x) \quad \text{--- (1)}$$

Where a_0, a_1, \dots, a_n are constants and Q is a function of (x) only we study the second order and non-Homog.D.E with C.C as follows:

$$(D^2 + aD + b)y = Q(x) \quad \text{--- (1)}$$

$$f(D)y = 0 \Rightarrow y \neq 0 \Rightarrow f(D) = 0$$

$$(D - m_1)(D - m_2)y = Q(x) \quad \text{--- (2)}$$

Where $b = m_1 m_2$ and $a = -(m_1 + m_2)$

$$\begin{array}{r} -m_1 D \\ -m_2 D \\ \hline -(m_1 + m_2)D \end{array}$$

$$\text{Suppose } (D - m_2)y = u(x) \quad \text{--- (3)}$$

Subst. eqn.(3) in eqn.(2), we get a.F..L.D.E

$$(D - m_1)u = Q(x)$$

$$Du - m_1 u = Q(x)$$

$$\frac{du}{dx} - m_1 u = Q(x) \Rightarrow p(x) = -m_1 \Rightarrow Q(x) = Q(x) \Rightarrow y = u$$

$$u' + p(x)u = Q(x)$$

$$I_f = e^{-m_1 \int dx} = e^{-m_1 x}$$

$$u e^{-m_1 x} = \int Q(x) e^{-m_1 x} dx + c_1$$

$$u = e^{m_1 x} [\int Q(x) e^{-m_1 x} dx + c_1] \quad \text{--- (4)}$$

Subst. equ (3) in equ(4), we get L.F.O.D.E.

$$\frac{dy}{dx} - m_2 y = e^{m_1 x} [\int Q(x) e^{-m_1 x} dx + c_1]$$

$$I_f = e^{-m_2 x}$$

$$y e^{-m_2 x} = \{ \int e^{m_1 x} [\int Q(x) e^{-m_1 x} dx + c_1] e^{-m_2 x} dx + c_2 \}$$

$$y = e^{m_2 x} \{ \int e^{m_1 x} [\int Q(x) e^{-m_1 x} dx + c_1] e^{-m_2 x} dx + c_2 \}$$

be the g. solution where c_1, c_2 are constants.

$$\text{Ex.: solve: } y'' + y' - 2y = e^x \quad \text{--- (1)}$$

$$\text{Solution: } \frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = e^x \quad \text{--- (1)}$$

$$D^2y + Dy - 2y = e^x \quad \dots (1)$$

$$(D^2 + D - 2)y = e^x \quad \dots (1)$$

$$(D + 2)(D - 1)y = e^x \quad \dots (2)$$

$$\text{Suppose } (D - 1)y = u(x) \quad \dots (3)$$

Subst. equ.(3) in equ.(2), we get $(D + 2)u = e^x$ is (F.O.L.D.E)

$$Du + 2u = e^x$$

$$\frac{du}{dx} + 2u = e^x \quad (p(x) = 2, Q(x) = e^x)$$

$$I_f = e^{\int p(x)dx} = e^{2\int dx} = e^{2x}$$

$$\boxed{I_f = e^{2x}}$$

$$u e^{2x} = \int e^x e^{2x} dx + c_1$$

$$u = e^{-2x} \left[\int e^{3x} \frac{3}{2} dx + c_1 \right]$$

$$u = e^{-2x} \left[\frac{1}{3} e^{3x} + c_1 \right] \text{ where } c \text{ is constant}$$

$$u = \frac{1}{3} e^x + c_1 e^{-2x} \quad \dots (4)$$

Subst. equ(4) in equ.(3), we get

$$(D - 1)y = \frac{1}{3} e^x + c_1 e^{-2x}$$

$$\frac{dy}{dx} - y = \frac{1}{3} e^x + c_1 e^{-2x} \quad (\text{L.F.O.D.E})$$

$$(p(x) = -1, Q(x) = \frac{1}{3} e^x + c_1 e^{-2x})$$

$$I_f = e^{-x}$$

$$y e^{-x} = \int \left(\frac{1}{3} e^x + c_1 e^{-2x} \right) \cdot e^{-x} dx + c_2$$

$$y = e^x \cdot \left[\int \left(\frac{1}{3} e^x + c_1 e^{-2x} \right) \cdot e^{-x} dx + c_2 \right]$$

$$y = e^x \cdot \left(\frac{1}{3} x - \frac{c_1}{3} e^{-3x} + c_2 \right)$$

$$y = e^x \left(\frac{1}{3} x - \frac{1}{3} c_1 e^{-3x} + c_2 \right)$$

$$y = \frac{1}{3} x e^x - \frac{1}{3} c_1 e^{-2x} + c_2 e^x \text{ is the g. solution where } c_2 \text{ is constant.}$$

Example: $(D^3 - D)y = 2\cos^2 x$

Solution: $(D^3 - D)y = 2\cos^2 x \quad \dots (1)$

$$D(D^2 - 1)y = 2\cos^2 x$$

$$D(D-1)(D+1)y = 2\cos^2 x \quad \dots \quad (2)$$

$$\text{Suppose } (D-1)(D+1)y = u \quad \dots \quad (3)$$

Substequ.(3) in equ.(2), we get .

$$Du = 2\cos^2 x$$

$$\frac{du}{dx} = 2\cos^2 x \text{ is S.D.E}$$

$$\int du = 2 \int \cos^2 x \, dx$$

$$u = 2 \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx$$

$$u = x + \frac{1}{2} \sin 2x + c_1 \quad \dots \quad (4)$$

Substequ.(4) in equ.(3), we get

$$(D-1)(D+1)y = x + \frac{1}{2} \sin 2x + c_1 \quad \dots \quad (5)$$

$$\text{Suppose } (D+1)y = g \quad \dots \quad (6)$$

Substequ.(6) in equ.(5), we get

$$(D-1)g = x + \frac{1}{2} \sin 2x + c_1 \text{ L.F.O.D.E}$$

$$\frac{d}{dx} g - g = x + \frac{1}{2} \sin 2x + c_1$$

$$ge^{-x} = \int (x + \frac{1}{2} \sin 2x + c_1)e^{-x} \, dx + c_2$$

$$ge^{-x} = \int x e^{-x} \, dx + \frac{1}{2} \int \sin(2x)e^{-x} \, dx + c_1 \int e^{-x} \, dx + c_2$$

$$ge^{-x} = [\int x e^{-x} \, dx + \frac{1}{2} \int e^{-x} \sin 2x \, dx + c_1 \int e^{-x} \, dx + c_2] \quad \dots \quad (7)$$

Substequ.(6) in equ.(7), we get

$$(D-1)y = e^x [\int x e^{-x} \, dx + \frac{1}{2} \int e^{-x} \sin 2x \, dx + c_1 \int e^{-x} \, dx + c_2]$$

L.F.O.D.E

$$\text{Solve: H.W: } (D^2 + 4D + 4)y = e^{-2x} \sec^2 x$$

Homogenous linear D.E. with C.C.

The general from of higher-order O.D.E with C.C. is

$$(a_n D^n + a_{n-1} D^{n-1} + \dots + a_3 D^3 + a_2 D^2 + a_1 D + a_0) y = 0$$

We can study the L.H.D.E of the second order with C.C.

$$y'' + ay' + by = 0 \quad \dots \quad (1)$$

$$(D^2 + aD + b)y = 0 \quad \dots \quad (1)$$

Note: if $(D^2 + aD + b)y = Q(x) \quad \dots \quad (*)$

Is non-Homog. L.D.E. with C.C. we can find the general solution of equ.(*) such that contains the complementary solution of $(D^2 + aD + b)y = 0 \dots (**)$ H.D.E with C.C. in which denoted by y_c , and the particular solution of equ.(*) (non-Homog),denoted by y_p ,then $y_6 = y_c + y_p$.

We study the H.D.E. with C.C.

$$(D^2 + aD + b)y = 0 \dots (1)$$

$f(D)y = 0 \Rightarrow f(D) = 0$ is a characteristic equ.

$$f(D) = D^2 + aD + b = 0$$

$$f(m) = m^2 + am + b = 0$$

$$m_{1,2} = \frac{-a \mp \sqrt{a^2 - 4b}}{2} \dots (2)$$

Or $f(D)y = 0$

$$(D^2 + aD + b)y = 0$$

$$(D - m_1)(D - m_2)y = 0 \dots (3) \text{ Such that}$$

m_1, m_2 are roots

$$m_1 m_2 = b$$

$$-(m_1 + m_2) = a$$

$$\text{Suppose } (D - m_2)y = u \dots (4)$$

Substequ.(4) in equ.(3) , we get S.D.E.

$$(D - m_1)u = 0$$

$$Du - m_1 u = 0$$

$$\frac{du}{dx} - m_1 u = 0$$

$$\frac{du}{u} - m_1 dx = 0$$

$\ln u - m_1 x = C$,where c is constant

$$\ln u = C + m_1 x$$

$$u = e^c \cdot e^{m_1 x} \quad e^c = c^* \rightarrow \text{where } c^* \text{ is constant} \quad u = c^* e^{m_1 x} \dots (5)$$

Substequ.(4) in equ.(5) , we get

$$(D - m_2)y = c^* e^{m_1 x}$$

$$\frac{dy}{dx} - m_2 y = c^* e^{m_1 x} \dots (6) \quad \text{L.F.O.D.E}$$

$$p(x) = -m_2$$

$$Q(x) = c^* e^{m_1 x}$$

$$y \cdot e^{-m_2 x} = \int c^* e^{m_1 x} e^{-m_2 x} dx + c_1$$

$$y = e^{+m_2 x} [c^* \int e^{(m_1 - m_2)x} dx + c] \dots (7)$$

We find the relation between two equ.(2)&(3). There exist three cases:

Case1: if $(a^2 - 4b > 0)$ then there exists two distinct (different) real roots $(m_1 \neq m_2 \in IR)$ in equ (7)

$$y = e^{m_2 x} [c * \int e^{(m_1 - m_2)x} dx + c_1]$$

$$y = e^{m_2 x} \left[\frac{c *}{m_1 - m_2} e^{(m_1 - m_2)x} + c_1 \right]$$

$$y = \frac{c *}{m_1 - m_2} e^{m_1 x} + c_1 e^{m_2 x}$$

$$y = c_2 e^{m_1 x} + c_1 e^{m_2 x}, \text{ where } c_2 = \frac{c *}{m_1 - m_2}$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Be a complementary solution of H.L.D.E with C.C in which contains two real distinct roots.

Case 2:

If $(a^2 - 4b = 0)$ then there exist two (repeated) real roots

$(m_1 = m_2 = m \in IR)$. In equ.(7)

$$y = e^{mx} [c * \int e^{(m-m)x} dx + c_1]$$

$$y = e^{mx} [c * \int e^{(m-m)x} dx + c_1]$$

$$y = e^{mx} (c * x + c_1)$$

$$yc = e^{mx} (c_1 + c_2 x) \quad \text{where } c * = c_2$$

Or

$$yc = c_1 e^{mx} + c_2 x e^{mx}$$

Or

$$yc = (c_1 x + c_2) e^{mx}$$

Be a complementary solution of H.L.D.E with C.C in which contains two equal roots (repeated real roots).

Case 3: if $(a^2 - 4b < 0)$ then there exist two complex roots

$$m_{1,2} = a \pm ib \begin{cases} m_1 = a + ib \\ m_2 = a - ib \end{cases}$$

$$b \neq 0, m_{1,2} \in C$$

In first case

$$yc = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$yc = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$

$$yc = c_1 e^{ax} e^{ibx} + c_2 e^{ax} \cdot e^{-ibx}$$

By Euler's formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$yc = e^{ax} [c_1 (\cos bx + i \sin bx) + c_2 (\cos bx - i \sin bx)]$$

$$yc = e^{ax} [(c_1 + c_2) \cos bx + (ic_1 + ic_2) \sin bx]$$

$$yc = e^{\alpha x} [K_1 \cos bx + K_2 \sin bx]$$

Where $c_1 + c_2 = K$ and $ic_1 + ic_2 = K_2$

be a complementary solution of H.L.D.E with C.C in which contains two complex roots .

Or

$$yc = K_1 e^{\alpha x} \cos bx + K_2 e^{\alpha x} \sin bx$$

Examples:

Solve the following H.D.E.(find the g. solution or the complementary solution)

$$(1) y'' + 5y' + 6y = 0$$

$$(2) (D - 3)^2 y = 0$$

$$(3) (D^2 + D + 1)y = 0$$

$$(4) (D - 2)(D - 3)(D - 1)^4 y = 0$$

$$(5) (D^3 + 4D^2 + D - 6)y = 0$$

$$(6) (D^2 + 5D + 6)(D^2 + D + 1)y = 0$$

$$(7) (y''' - 4y'' + 4)y = 0$$

$$(8) (D^3 - 1)^2 y = 0$$

$$\text{Solution (1): } (D^2 + 5D + 6)y = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m + 3)(m + 2) = 0$$

$$m = -3, m = -2 \in R$$

$$m_1 \neq m_2 \in R$$

$$-2 \neq -3 \in R$$

$$yc = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\boxed{yc = c_1 e^{-3x} + c_2 e^{-2x}}$$

$$\text{Solution (2): } (D - 3)^2 y = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, 3$$

$$yc = e^{mx} (c_1 x + c_2)$$

$$yc = e^{3x} (c_1 x + c_2)$$

Or

$$yc = e^{3x} (c_1 + c_2 x)$$

$$\text{Solution (*): } (D - 3)^3 y = 0$$

$$(m - 3)^3 = 0$$

$$yc = e^{3x} (c_1 + c_2 x + c_3 x^2)$$

$$\text{Solution (3): } m^2 + m + 1 = 0$$

$$A = 1, B = 1, C = 1$$

$$m_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\therefore m_{1,2} = a \pm ib \therefore a = \frac{-1}{2}$$

$$b = \frac{\sqrt{3}}{2}$$

$$yc = e^{ax} (K_1 \cos bx + K_2 \sin bx)$$

$$yc = e^{\frac{-1}{2}x} [K_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + K_2 \sin\left(\frac{\sqrt{3}}{2}x\right)]$$

Solution (4): $(D - 2)(D - 3)(D - 1)^4 y = 0$

$$(m - 2)(m - 3)(m - 1)^4 = 0$$

$$m_1 = 2 \in R$$

$$m_2 = 3 \in R$$

$$m_3 = -1, -1, -1, -1 \in R$$

Or

$$m_3 = m_4 = m_5 = m_6 = -1$$

$$yc = c_1 e^{2x} + c_2 e^{3x} + e^{-x} (c_3 + c_4 x + c_5 x^2 + c_6 x^3)$$

Solution (5): $(D^3 + 4D^2 + D - 6)y = 0$

$$(D + 3)(D^2 + D - 2)y = 0$$

$$(D + 3)(D + 2)(D - 1)y = 0$$

$$(m + 3)(m + 2)(m - 1) = 0$$

$$\Rightarrow m + 3 = 0 \Rightarrow m_1 = -3$$

$$\Rightarrow m + 2 = 0 \Rightarrow m_2 = -2$$

$$\Rightarrow m - 1 = 0 \Rightarrow m_3 = 1$$

$$\therefore yc = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^x$$

Solution (6): $(D^2 + 5D + 6)(D^2 + D + 1)y = 0$

$$(D + 3)(D + 2)(D^2 + D + 1)y = 0$$

$$(m + 3)(m + 2)(m^2 + m + 1) = 0$$

$$\Rightarrow m + 3 = 0 \Rightarrow m_1 = -3$$

$$\Rightarrow m + 2 = 0 \Rightarrow m_2 = -2$$

$$\Rightarrow m^2 + m + 1 = 0 \Rightarrow m_{3,4} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2}$$

$$\Rightarrow m_{3,4} = \frac{-1}{2} \mp i \frac{\sqrt{3}}{2}$$

$$\therefore yc = c_1 e^{-3x} + c_2 e^{-2x} + e^{\frac{-1}{2}x} [K_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + K_2 \sin\left(\frac{\sqrt{3}}{2}x\right)]$$

Solution (7): $(y''' - 4y'' + 4y) = 0$

$$D^4 y - 4D^2 y + 4Y = 0$$

$$(D^2 - 2)^2 y = 0$$

$$(m^2 - 2)^2 = 0$$

$$(m^2 - 2)(m^2 - 2) = 0$$

$$m^2 - 2 = 0 \Rightarrow m^2 = 2 \Rightarrow m_{1,2} = \mp\sqrt{2}$$

$$m^2 - 2 = 0 \Rightarrow m^2 = 2 \Rightarrow m_{3,4} = \mp\sqrt{2}$$

$$yc = e^{\sqrt{2}x} (c_1 + c_2 x) + e^{-\sqrt{2}x} (c_3 x + c_4)$$

Or

$$yc = (c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}) + x (c_3 e^{\sqrt{2}x} + c_4 e^{-\sqrt{2}x})$$

Solution (8): $(D^3 - 1)y = 0$

$$(D - 1)(D^2 + D + 1)y = 0$$

$$(m - 1)(m^2 + m + 1) = 0$$

$$m - 1 = 0 \Rightarrow m_1 = 1 \in IR$$

$$m^2 + m + 1 = 0 \Rightarrow m_{2,3} = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2} \in C$$

$$yc = c_1 e^x + e^{\frac{-1}{2}x} [c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)]$$

Solution (9): $(D^3 - 1)^2 y = 0$

$$(m - 1)^2 (m^2 + m + 1)^2 = 0$$

$$m - 1 = 0 \Rightarrow m_{1,2} = -1, 1 \in R$$

$$\boxed{yc_1 = e^x (c_1 x + c_2)}$$

$$m_{3,4} = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2} \in C$$

$$yc_2 = e^{\frac{-1}{2}x} \left[k_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + k_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right] + x e^{\frac{-1}{2}x} \left[k_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + k_4 \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$$

$$yc = e^x (c_1 x + c_2) + e^{\frac{-1}{2}x} \left[k_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + k_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right] + x e^{\frac{-1}{2}x} \left[k_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + k_4 \sin\left(\frac{\sqrt{3}}{2}x\right) \right] H.W:$$

solve

$$(1) y''' - y'' - 8y' + 12y = 0$$

$$(2) (D^3 + 3D^2 + 3D + 1)y = 0$$

$$(3) (D^3 + 2D^2 - 5D - 6)y = 0$$

$$(4) (y''' - 16y)^2 = 0$$

$$(5) (D+1)^3 (D^2 + 2D + 1) (D^2 - D + 2)^3 (D+7)(D-5)y = 0$$

How to find a particular solution of L. Non-Homog D.E. with C.C.:

Consider $(a_n D^n + a_{n-1} D^{n-1} + \dots + a_3 D^3 + a_2 D^2 + a_1 D + a_0) y = Q(x)$

BE A L.NON-H. D.E With C.C, where a_0, a_1, \dots, a_n Constants and Q is a function of (x) only.

There exist three methods:

- (1) the variation of parameters method
- (2) the operators method
- (3) Un determinant coefficients method

(1) The variation of parameters method:

We study non-H.L.D.E of second order with C.C.:

$$y'' + ay' + by = Q(x) \quad (1)$$

$$D^2 + aD + b = Q(x) \quad (1)$$

To find the general solution of D.E (1), we obtain (yc) and (yp)

$$y_G = y_c + y_p \text{ to find } (yc) \text{ (be a complementary solution) of equ.(1) suppose } (D^2 + aD + b)y = 0 \quad (2)$$

$$m_2 + am + b = 0$$

$y_c = c_1 y_1 + c_2 y_2 \quad (3)$ (where y_1, y_2 are two linearly independent. Solutions) since y_1, y_2 are two linearly independent solution. Solution then

$$\begin{cases} y_1'' + ay_1' + by_1 = 0 \\ y_2'' + ay_2' + by_2 = 0 \end{cases} \quad (*)$$

We can find a particular solution of D.E(1) by changing two arbitrary constants c_1, c_2 in to $v_1(x), v_2(x)$

$$y_p = v_1 y_1 + v_2 y_2 \quad (4) \text{ be a particular solution of D.E(1)}$$

$$y_p = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$$

$$\text{Suppose } [y_1 v_1' + y_2 v_2' = 0] \quad (A)$$

$$y_p' = v_1 y_1' + v_2 y_2' \quad (5)$$

$$y_p'' = v_1 y_1'' + y_1' v_1' + v_2 y_2'' + y_2' v_2' \quad (6)$$

Subst. equ (4, 5 and 6) in D.E (1), we get

$$v_1 y_1'' + y_1' v_1' + v_2 y_2'' + y_2' v_2' + a(v_1 y_1' + v_2 y_2') + b(v_1 y_1 + v_2 y_2) = Q(x)$$

$$v_1(y_1'' + ay_1' + by_1) + v_2(y_2'' + ay_2' + by_2) + v_1' y_1' + v_2' y_2' = Q(x)$$

$$v_1(0) + v_2(0) + v_1' y_1' + v_2' y_2' = Q(x)$$

$$v_1' y_1' + v_2' y_2' = Q(x) \quad (B)$$

We can solve both equ. A and B by grammer's method.

$$v_1' y_1' + v_2' y_2' = Q(x) \quad (B)$$

$$v_1' y_1' + v_2' y_2' = 0 \quad (A)$$

$$v'_1 = \frac{\begin{vmatrix} 0 & y_2 \\ Q(x) & y'_2 \\ \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \\ \end{vmatrix}} \Rightarrow v'_1 = \frac{-y_2 Q(x)}{w(y_1, y_2)}$$

$$v_1 = \int \frac{-y_2 Q(x)}{w(y_1, y_2)} dx$$

And

$$v'_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & Q(x) \\ \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \\ \end{vmatrix}} \Rightarrow v'_2 = \frac{y_1 Q(x)}{w(y_1, y_2)}$$

$$v_2 = \int \frac{y_1 Q(x)}{w(y_1, y_2)} dx$$

$$y_p = (-\int \frac{y_2 Q(x)}{w(y_1, y_2)} dx) y_1 + (\int \frac{y_1 Q(x)}{w(y_1, y_2)} dx) y_2$$

$$y_G = y_c + y_p$$

Example: - solve $y'' + y' = e^x$

Solution:- $(D^2 + D) y = e^x$

$$(D^2 + D) y = 0$$

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m_1 = 0, m_2 = -1$$

$$y_c = c_1 e^{0x} + c_2 e^{-x}$$

$$y_c = c_1 1 + c_2 e^{-x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_1 = 1$$

$$y_2 = e^{-x}$$

$$y_p = v_1 1 + v_2 e^{-x}$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \\ \end{vmatrix} = y_1 y'_2 - y_2 y'_1$$

$$w(1, e^{-x}) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \\ \end{vmatrix} = -e^{-x}$$

$$v_1 = \int \frac{y_1 Q(x)}{w(y_1, y_2)} dx \Rightarrow v_1 = \int \frac{-e^{-x} e^x}{-e^{-x}} dx \Rightarrow v_1 = e^x$$

$$v_2 = \int \frac{y_1 Q(x)}{w(y_1, y_2)} dx \Rightarrow v_2 = \int \frac{e^x}{-e^{-x}} dx \Rightarrow v_2 = -\int e^{2x} dx \quad v_2 = -\frac{1}{2} e^{2x}$$

$$y_p = e^x - \frac{1}{2} e^x \Rightarrow y_p = \frac{1}{2} e^x$$

$$y_G = y_C + y_p \Rightarrow y_G = c_1 1 + c_2 e^{-x} + \frac{1}{2} e^x$$

H.W.

Solve the following D.Es.

- 1) $y'' + 4y = \sec 2x$
- 2) $y'' + y = \tan x$
- 3) $(D^2 + 4)y = \sec^2 2x$
- 4) $y'' + y' + y = x^2 e^{-x}$

Operator's method

To study n-th order O.D.E with C.C by using operators method (D) dependent on the type of Q(x) and by using some theorems.

Theorem: - 1) $f(D)e^{bx} = f(b)e^{bx}$

2) $f(D^2)\cos bx = f(-b^2)\cos bx$ where b is constant.

3) $f(D^2)\sin bx = f(-b^2)\sin bx$

4) $f(D)\{e^{bx} y\} = e^{bx} \{f(D+b)y\}$

Proof (1):- $f(D)e^{bx} = (P_n D^n + P_{n-1} D^{n-1} + \dots + P_3 D^3 + P_2 D^2 + P_1 D^1 + P_0) e^{bx}$ where p_0, p_1, \dots, p_n are constant

$$f(D)e^{bx} = (P_n (D^n e^{bx}) + P_{n-1} (D^{n-1} e^{bx}) + \dots + P_3 (D^3 e^{bx}) + P_2 (D^2 e^{bx}) + P_1 (D^1 e^{bx}) + P_0 e^{bx})$$

$$(D^1 e^{bx}) = b e^{bx}$$

$$\begin{aligned} (D^2 e^{bx}) &= D(D e^{bx}) \\ &= D(b e^{bx}) \\ &= b(D e^{bx}) \\ &= b^2 e^{bx} \end{aligned}$$

$$\begin{aligned} (D^3 e^{bx}) &= D(D^2 e^{bx}) \\ &= D(b^2 e^{bx}) \\ &= b^2(D e^{bx}) \\ &= b^3 e^{bx} \end{aligned}$$

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$$(D^{n-1} e^{bx}) = b^{n-1} e^{bx}$$

$$(D^n e^{bx}) = b^n e^{bx}$$

$$f(D)e^{bx} = (P_n(b^n e^{bx}) + P_{n-1}(b^{n-1} e^{bx}) + \dots + P_3(b^3 e^{bx}) + P_2(b^2 e^{bx}) + P_1(b e^{bx}) + P_0 e^{bx})$$

$$f(D)e^{bx} = (P_n b^n + P_{n-1} b^{n-1} + \dots + P_3 b^3 + P_2 b^2 + P_1 b + P_0) e^{bx}$$

$$f(D)e^{bx} = f(b)e^{bx}$$

Type of Q(x)

1) e^{bx}
2) $\cos bx$ or $\sin bx$
3) x^m
4) $e^{bx} \cos bx$ or $e^{bx} \sin bx$ or $e^{bx} x^m$
5) $x^m \cos bx$ or $x^m \sin bx$
6) $e^{bx} x^m \cos bx$ or $e^{bx} x^m \sin bx$

There exist six cases.

Case1:- If $Q(x) = e^{bx}$ then the particular solution is [$f(D)y = Q(x) \Rightarrow f(D)y = e^{bx}$]

$$y_p = \frac{1}{f(D)} Q(x) \Rightarrow y_p = \frac{1}{f(D)} e^{bx}$$

$$\text{i) If } f(b) \neq 0 \text{ then } y_p = \frac{1}{f(D)} e^{bx} \text{ (By using theorem } f(D)e^{bx} = f(b)e^{bx} \text{)}$$

Example 1: - solve: $y'' + 2y' + y = e^x$

Solution: $y'' + 2y' + y = e^x$

$$(D^2 + 2D + 1)y = e^x$$

$$(D+1)^2 y = e^x$$

$$\text{Suppose } (D+1)^2 y = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

$$y_c = e^{-x} (c_1 + c_2 x)$$

$$y_p = \frac{1}{f(D)} e^x$$

$$y_p = \frac{1}{(D^2 + 2D + 1)} e^x$$

$$y_p = \frac{1}{(1^2 + 2*1 + 1)} e^x$$

$$y_p = \frac{1}{4} e^{bx}$$

$$y_G = e^{-x} (c_1 + c_2 x) + \frac{1}{4} e^x$$

Example 2):- solve: $y'' + y' + 4y = e^{-x}$

$$(D^2 + D + 4)y = e^{-x}$$

$$m^2 + m + 4 = 0$$

$$m_{1,2} = \frac{-1 \pm \sqrt{1^2 - 16}}{2}$$

$$m_{1,2} = \frac{-1 \pm i\sqrt{15}}{2}$$

$$y_c = e^{-\frac{1}{2}x} [k_1 \cos(\frac{\sqrt{5}}{2}x) + k_2 \sin(\frac{\sqrt{5}}{2}x)]$$

$$y_p = \frac{1}{f(D)} e^{bx}$$

$$y_p = \frac{1}{(D^2 + D + 4)} e^{-x}$$

$$y_p = \frac{1}{((-1)^2 + (-1) + 4)} e^{-x}$$

$$y_p = \frac{1}{4} e^{-x}$$

$$y_G = e^{-\frac{1}{2}x} [k_1 \cos(\frac{\sqrt{5}}{2}x) + k_2 \sin(\frac{\sqrt{5}}{2}x)] + \frac{1}{4} e^{-x}$$

H.W: solves the following.

$$1) y'' + y' - 2y = e^{5x}$$

$$2) y''' - y'' = e^{3x}$$

ii) If $f(b) = 0$ then $y_p = \frac{1}{f(b)=0} e^{bx}$ undefined, and $y_p = \frac{1}{(D-b)^r g(D)} e^{bx}$

Where b is a root in the complementary solution and (r) is the number of repeated root g(D) the remainder terms in f(D).

By using theorem $f(D)\{e^{bx} y\} = e^{bx} \{f(D+b)y\}$

$$y_p = e^{bx} \frac{1}{(D-b+b)^r g(D+b)} \{1\}$$

$$y_p = e^{bx} \frac{1}{(D)^r g(D+b)}$$

$$y_p = \frac{e^{bx}}{g(b)} (D)^{-r} \quad , \text{ where } g(D+b) = g(b)$$

$$y_p = \frac{e^{bx}}{g(b)} \frac{x^r}{r!} \quad , \text{ where } (D)^{-r} = \frac{x^r}{r!}$$

It is particular solution.

Examples:- $y''' - y'' = e^x$

Solution:- $(D^3 - D^2)y = e^x$

$$(D^3 - D^2)y = 0$$

$$D^2(D-1)y = 0$$

$$m^2(m-1) = 0 \Rightarrow m_{1,2} = 0, m_3 = 1$$

$$y_c = (c_1 + c_2x) + c_3e^x$$

$$y_p = \frac{1}{f(D)}Q(x)$$

$$y_p = \frac{1}{D^3 - D^2}e^x$$

$$y_p = \frac{1}{(D-b)^r g(D)}e^{bx}$$

$$y_p = e^x \frac{1}{(D-1+1)^r g(D+1)}$$

$$y_p = \frac{e^x}{D(D+1)^2}$$

$$y_p = \frac{e^x}{D*1} \Rightarrow y_p = e^x D^{-1}$$

$$y_p = e^x x$$

$$y_G = (c_1 + c_2x) + c_3e^x + e^x x$$

Example:- $(D + 2)^3(D - 1)y = e^{-2x}$

Solution:- $(m + 2)^3(m - 1) = 0$

$$m_{1,2,3} = -2, m_4 = 1$$

$$y_c = e^{-2x}(c_1 + c_2x + c_3x^2) + c_4e^x$$

$$y_p = \frac{1}{(D+2)^3(D-1)}e^{-2x}$$

$$y_p = e^{-2x} \frac{1}{(D)^3(D-3)}$$

$$y_p = \frac{e^{-2x}}{-3} D^{-3}$$

$$y_p = \frac{e^{-2x}}{-3} \frac{x^3}{3!}$$

$$y_G = e^{-2x}(c_1 + c_2x + c_3x^2) + c_4e^x + \frac{e^{-2x}}{-3} \frac{x^3}{3!}$$

H.W: Solve the following solution.

$$1) (D-1)(D+2)(D-3)y = e^{3x}$$

$$2) (D-1)(D-2)^2 y = (e^{2x} + 2e^x + 3e^{-x})$$

Case 2:- If $Q(x) = \cos ax$ or $\sin ax$ then $f(D^2)y = \cos ax$ or $\sin ax$

$$\Rightarrow y_p = \frac{1}{f(D^2)} \cos ax \text{ or } \sin ax$$

By using theorem $f(D^2) \cos ax = f(-a^2) \cos ax$ or $f(D^2) \sin ax = f(-a^2) \sin ax$

There exist two branches

i) If $f(-a^2) \neq 0$ then $y_p = \frac{1}{f(-a^2)} \cos ax \text{ or } \sin ax$

Example:- $y'' - 4y = \cos 2x$

Solution:- $(D^2 - 4)y = 0$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{D^2 - 4} \cos 2x$$

$$y_p = \frac{1}{-2^2 - 4} \cos 2x$$

$$y_p = \frac{-1}{8} \cos 2x$$

$$y_G = c_1 e^{2x} + c_2 e^{-2x} + \frac{-1}{8} \cos 2x$$

Example:- $(D^2 + 3D - 4)y = \sin 2x$

Solution:- $(D^2 + 3D - 4)y = 0$

$$m^2 + 3m - 4 = 0$$

$$(m+4)(m-1) = 0$$

$$m_1 = -4, m_2 = 1$$

$$y_c = c_1 e^{-4x} + c_2 e^x$$

$$y_p = \frac{1}{D^2 + 3D - 4} \sin 2x$$

$$y_p = \frac{1}{-2^2 + 3D - 4} \sin 2x$$

$$y_p = \frac{1}{3D - 8} \sin 2x$$

$$y_p = \frac{3D + 8}{9D^2 - 64} \sin 2x$$

$$y_p = \frac{3D + 8}{9(-2^2) - 64} \sin 2x$$

$$y_p = \frac{3D + 8}{-100} \sin 2x$$

$$y_p = \frac{-1}{100} (3D \sin 2x + 8 \sin 2x)$$

$$y_p = \left(\frac{-6}{100} \cos 2x + \frac{-8}{100} \sin 2x \right)$$

$$y_G = c_1 e^{-4x} + c_2 e^x + \left(\frac{-6}{100} \cos 2x + \frac{-8}{100} \sin 2x \right)$$

H.W: Solve the following solution.

$$y'' - 9y = 5\cos 3x$$

ii) If $f(-a^2) = 0$ then $y_p = \frac{1}{f(-a^2)} \cos ax$ or $\sin ax$

can be changing $\cos ax$ or $\sin ax$ into Euler's formula

$$e^{ix} = \cos x + i \sin x \text{ or } (e^{i\alpha x} = \cos \alpha x + i \sin \alpha x)$$

$$y_p = \frac{1}{f(D^2)} e^{ax} \text{ of the 1-st cases}$$

Can be solved by first cases. Finally e^{ax} changed into $(\cos ax + i \sin ax)$ then choose the real part if the problem contains $(\cos ax)$ or choose the imaginary part if the problem contains $(\sin ax)$

Example:- $(D^2 + 9)y = \sin 3x$

Solution:- $(D^2 + 9)y = 0$

$$m^2 + 9 = 0$$

$$m^2 = -9 \Rightarrow m = \pm 3i$$

$$y_c = [k_1 \cos 3x + k_2 \sin 3x]$$

$$y_p = \frac{1}{(D^2 + 9)} \sin 3x$$

$$y_p = \frac{1}{(-3^2 + 9)} \sin 3x$$

$$y_p = \frac{1}{0} \sin 3x \quad \text{undefined}$$

$$e^{i3x} = \cos 3x + i \sin 3x$$

$$y_p = \frac{1}{D^2 + 9} e^{3ix}$$

$$y_p = \frac{1}{(D + 3i)(D - 3i)} e^{3ix}$$

$$y_p = \frac{e^{3ix}}{6i} \frac{x}{1!}$$

$$y_p = \frac{1}{6i} \frac{x}{1!} (\cos 3x + i \sin 3x)$$

$$y_p = \frac{x}{6i} \cos 3x - \frac{x}{6} \sin 3x$$

$$y_p = \frac{x}{6i} \cos 3x$$

$$y_G = k_1 \cos 3x + k_2 \sin 3x + \left(\frac{x}{6i} \cos 3x \right)$$

Case 3:- If $Q(x) = \text{polynomials function i.e. } Q(x) = x^m \Rightarrow y_p = \frac{1}{f(D)} x^m$ can be solved by using the series

$$\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$\text{Or } \frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

Example:- solve : $y'' + 2y' + 3y = x^3$

Solution:- $(D^2 + 2D + 3)y = 0$

$$m^2 + 2m + 3 = 0$$

$$m_{1,2} = \frac{-2 \pm \sqrt{-8}}{2}$$

$$m_{1,2} = \frac{-2}{2} \pm \frac{2i\sqrt{2}}{2}$$

$$m_{1,2} = -1 \pm i\sqrt{2}$$

$$y_c = e^{-x} (k_1 \cos(\sqrt{2}x) + k_2 \sin(\sqrt{2}x))$$

$$y_p = \frac{1}{D^2 + 2D + 3} x^3$$

$$y_p = \frac{1}{3[1 + (\frac{D^2}{3} + 2\frac{D}{3})]} x^3$$

$$y_p = \frac{1}{3}[1 + (\frac{D^2}{3} + 2\frac{D}{3})]^{-1} x^3$$

$$y_p = \frac{1}{3}[1 - (\frac{D^2}{3} + 2\frac{D}{3}) + (\frac{D^2}{3} + 2\frac{D}{3})^2 + (\frac{D^2}{3} + 2\frac{D}{3})^3] x^3$$

$$y_p = \frac{x^3}{3} - \frac{1}{9} D^2 x^3 - \frac{2}{9} D x^3 + \frac{1}{3} (\frac{D^4}{9} + \frac{4}{9} D^3 + \frac{4}{9} D^3 + \frac{2}{9} D^2) x^3$$

$$y_p = \frac{x^3}{3} - \frac{6}{9} x - \frac{2}{3} x^2 + \frac{24}{27} x - \frac{48}{81} \text{ is particular solution .}$$

$$y_G = e^{-x} (k_1 \cos(\sqrt{2}x) + k_2 \sin(\sqrt{2}x)) + \frac{x^3}{3} - \frac{6}{9} x - \frac{2}{3} x^2 + \frac{24}{27} x - \frac{48}{81}$$

Case 4:- if $Q(x) = e^{ax} \cos bx$ or $e^{ax} \sin bx$ or $Q(x) = e^{ax} x^m$ then

$y_p = \frac{1}{f(D)} e^{ax} \cos bx$ or $(\sin bx)$ or $y_p = \frac{1}{f(D)} e^{ax} x^m$ can be solved by using the theorem

$$f(D)\{e^{bx} y\} = e^{bx} f(D+y)$$

$y_p = e^{ax} \frac{1}{f(D+a)} \cos bx$ or $(\sin bx)$ can be solving by second cases or $y_p = e^{ax} \frac{1}{f(D+a)} x^m$ can be solved by third cases

Example: - solve: $(D^2 - 4)y = e^{3x} \sin 2x$

Solution:- $(D^2 - 4)y = 0$

$$m^2 - 4 = 0 \Rightarrow m_{1,2} = \pm 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{D^2 - 4} e^{3x} \sin 2x$$

$$y_p = e^{3x} \frac{1}{(D+3)^2 - 4} \sin 2x$$

$$y_p = e^{3x} \frac{1}{D^2 + 6D + 9 - 4} \sin 2x$$

$$y_p = e^{3x} \frac{1}{D^2 + 6D + 5} \sin 2x$$

$$y_p = e^{3x} \frac{1}{-2^2 + 6D + 5} \sin 2x$$

$$y_p = e^{3x} \frac{1}{1+6D} \sin 2x$$

$$y_p = e^{3x} \frac{1-6D}{1-36D^2} \sin 2x$$

$$y_p = e^{3x} \frac{1-6D}{1-36(-(2)^2)} \sin 2x$$

$$y_p = \frac{e^{3x}}{145} \sin 2x - \frac{12e^{3x}}{145} \cos 2x$$

$$y_G = \frac{e^{3x}}{145} \sin 2x - \frac{12e^{3x}}{145} \cos 2x + c_1 e^{2x} + c_2 e^{-2x}$$

Case 5:- If $Q(x) = x^m \cos bx$ or $(x^m \sin bx)$ then $y_p = \frac{1}{f(D)} x^m \cos bx$ or $(x^m \sin bx)$

We can solved by changing $\sin bx$ or $\cos bx$ into Euler's formula ($e^{ib} = \cos bx + i \sin bx$)

$$y_p = \frac{1}{f(D)} x^m e^{ib} \quad (*)$$

$y_p = e^{ibx} \frac{1}{f(D+ib)} x^m$ can be solving by third cases. Finally we can changing e^{ibx} into $\cos bx + i \sin bx$ and chooses the real part if equ (*) contain $(\cos bx)$, but if equ(*) contain $(\sin bx)$, then choose the imaginary part.

Example:- $(D^2 + 3D + 2)y = x \sin 2x$

Solution:- $y_c = H.W$

$$y_p = \frac{1}{(D^2 + 3D + 2)} x \sin 2x$$

$$y_p = \frac{1}{(D^2 + 3D + 2)} x e^{i2x}$$

$$y_p = e^{i2x} \frac{1}{(D + 2i)^2 + 3(D + 2i) + 2} x$$

$$y_p = e^{i2x} \frac{1}{D^2 + 4Di - 4 + 3D + 6i + 2} x$$

$$y_p = e^{i2x} \frac{1}{D^2 + 4Di - 4 + 3D + 6i + 2} x$$

$$y_p = e^{i2x} \frac{1}{D^2 + 4Di + 3D + 6i - 2} x$$

$$y_p = e^{i2x} \frac{1}{(6i - 2)[1 + (\frac{D^2}{6i - 2} + \frac{4Di}{6i - 2} + \frac{3D}{6i - 2})]} x$$

$$y_p = \frac{e^{i2x}}{(6i - 2)} [1 - (\frac{D^2}{6i - 2} + \frac{4Di}{6i - 2} + \frac{3D}{6i - 2})] x$$

$$y_p = \frac{e^{i2x}}{(6i - 2)} \left[x - \frac{4i}{(6i - 2)} - \frac{3}{(6i - 2)} \right]$$

$$y_p = \frac{(\cos 2x + i \sin 2x)}{(6i - 2)} \left[x - \frac{4i}{(6i - 2)} - \frac{3}{(6i - 2)} \right]$$