Introduction:

The mathematical idea of a vector plays an important role in many areas of physics.

* Thinking about a particle traveling through space, we imagine that its speed and direction of travel can be represented by a vector v in 3-dimensional space.

* A static structure such as a bridge has loads which must be calculated at various points. These are also vectors, giving the direction and magnitude of the force at those isolated points.

* In the theory of electromagnetism, Maxwell's equations deal with vector fields in 3-dimensional space which can change with time. Thus at each point of space and time, two vectors are specified, giving the electrical and the magnetic fields at that point.

* In quantum mechanics, a given experiment is characterized by an abstract space of complex functions. Each function is thought of as being itself a kind of vector.

* Fluid flow in two and three dimensions is compactly represented using concepts from vector analysis.

Looking at above examples, we see that linear algebra comes up in physics involving "classical physics". In any case, it is clear that the theory of linear algebra is very basic to any study of physics.

Chapter one

Matrices

Definition: A matrix is a rectangle array of numbers or variables denoted by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The *rows* of such a matrix A are the m horizontal lists of scalars:

$$(a_{11}, a_{12}, \ldots, a_{1n}), \quad (a_{21}, a_{22}, \ldots, a_{2n}), \quad \ldots, \quad (a_{m1}, a_{m2}, \ldots, a_{mn})$$

and the *columns* of A are the n vertical lists of scalars:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{bmatrix}$$

Note that:

1. the element a_{ij}, called the ij-entry or ij-element, appears in row i and column j. We frequently denote such a matrix by simply writing A=[a_{ij}].

2. a matrix with m rows and n columns is called an m by n matrix, written m×n. The pair of numbers m and n is called the size or the degree of the matrix and denoted by m×n.

3. two matrices A and B are equal, written A=B, if they have the same size and if corresponding elements are equal.

4. a matrix consists of vectors that arranged as columns

Example:

$$\mathbf{A} = \begin{bmatrix} 4 & -7 & 5 & 0 \\ -2 & 0 & 11 & 8 \\ 19 & 1 & -3 & 12 \end{bmatrix}$$ is a matrix with 3 rows and 4 columns.

The 12 entries of the matrix are referenced by the row and column.

The (2,3) entry of A is 11. We may also write $a_{23} = 11$.

The matrix A has degree 3 × 4 (three by four).

Example: A force acts on a particle toward north and west of acceleration in magnitude 5m\sec^2 and 10m\sec^2 respectively. Write a matrix representing the acceleration components corresponding to their force components.

Matrix addition and scalar multiplication:

The sum of two matrices A=[a_{ij}] and B=[b_{ij}] of the same size m×n is the matrix obtained by adding corresponding elements from A and B. That is,

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{1n}+b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

The *product* of the matrix A by a scalar k, written $k \cdot A$ or simply kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Example:

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

Observe that $A + B$ and kA are also $m \times n$ matrices. We also define

$$-A = (-1)A \qquad \text{and} \qquad A - B = A + (-B)$$

The matrix $-A$ is called the *negative* of the matrix A, and the matrix $A - B$ is called the *difference* of A and B. The sum of matrices with different sizes is not defined.

Example:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 & 8 \\ 1 & -3 & -7 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 1+4 & -2+6 & 3+8 \\ 0+1 & 4+(-3) & 5+(-7) \end{bmatrix} = \begin{bmatrix} 5 & 4 & 11 \\ 1 & 1 & -2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3(1) & 3(-2) & 3(3) \\ 3(0) & 3(4) & 3(5) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 9 \\ 0 & 12 & 15 \end{bmatrix}$$

$$2A - 3B = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 8 & 10 \end{bmatrix} + \begin{bmatrix} -12 & -18 & -24 \\ -3 & 9 & 21 \end{bmatrix} = \begin{bmatrix} -10 & -22 & -18 \\ -3 & 17 & 31 \end{bmatrix}$$

Basic properties of matrices under the operations of matrix addition and scalar multiplication follow.

THEOREM 2.1: Consider any matrices A, B, C (with the same size) and any scalars k and k'. Then

(i) $(A + B) + C = A + (B + C)$, (v) $k(A + B) = kA + kB$,

(ii) $A + 0 = 0 + A = A$, (vi) $(k + k')A = kA + k'A$,

(iii) $A + (-A) = (-A) + A = 0$, (vii) $(kk')A = k(k'A)$,

(iv) $A + B = B + A$, (viii) $1 \cdot A = A$.

Two matrices A and B are equal if all their corresponding entries are equal.

Example: Given that the following matrices are equal, find the values of *x* and *y*.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} x & 2 \\ 3 & y \end{bmatrix}$$

Example: Given that the following matrices are equal, find the values of *x*, *y*, and *z*.

$$A = \begin{bmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} x & 0 \\ 6 & y+4 \\ \frac{z}{3} & 1 \end{bmatrix}$$

Definition: The m × n matrix whose entries are all 0 is denoted $O_{m\times n}$. Its called the zero matrix.

Matrix multiplication:

Suppose $A = [a_{ik}]$ and $B = [b_{kj}]$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix whose ij-entry is obtained by multiplying the ith row of A by the jth column of B. That is,

$$\begin{bmatrix} a_{11} & \dots & a_{1p} \\ . & \dots & . \\ a_{i1} & \dots & a_{ip} \\ . & \dots & . \\ a_{m1} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ . & \dots & . \\ . & c_{ij} & . \\ . & \dots & . \\ c_{m1} & \dots & c_{mn} \end{bmatrix}$$

where $$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot -1+2\cdot 4+3\cdot 1 & 1\cdot 0+2\cdot 2+3\cdot 3 \\ -1\cdot -1+0\cdot 4+4\cdot 1 & -1\cdot 0+0\cdot 2+4\cdot 3 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 5 & 12 \end{pmatrix}$$

Remark: Matrix multiplication is not commutative. That is, $AB \neq BA$, in general.

Example: Let $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$. Show that AB≠BA.

THEOREM 2.2: Let A, B, C be matrices. Then, whenever the products and sums are defined,

(i) $(AB)C = A(BC)$ (associative law),

(ii) $A(B+C) = AB + AC$ (left distributive law),

(iii) $(B+C)A = BA + CA$ (right distributive law),

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

We note that $0A = 0$ and $B0 = 0$, where 0 is the zero matrix.

Definition: A matrix with non-zero entries only on the diagonal is called diagonal matrix.

Example: The following matrix is a diagonal matrix.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Definition: A matrix A is called square if it has the same number of rows and columns.

Definition: An Upper Triangular Matrix is a matrix whose the elements below the diagonal are zeros.

$$\begin{pmatrix} 1 & 7 & -2 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

Definition: A Lower Triangular Matrix is a matrix whose the elements above the diagonal are zeros.

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

Definition: A scalar matrix is a diagonal matrix in which the diagonal elements are equal.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Definition: Identity matrix is a square matrix, which has ones on the main diagonal and zeros elsewhere and denoted by I_n.

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition: The transpose of the matrix A, denoted A^t, is obtained from A by making the first row of A into the first column of A^t, the second row of A into the second column of A^t, and so on.

Example:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{pmatrix} \qquad A^t = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

THEOREM 2.3: Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined,

(i) $(A+B)^T = A^T + B^T$, (iii) $(kA)^T = kA^T$,

(ii) $(A^T)^T = A$, (iv) $(AB)^T = B^T A^T$.

Definition: A symmetric matrix is a square matrix that satisfies $A = A^t$.

Exercises:

1. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find AA^T and A^TA. What can you say about matrices MM^T and M^TM in general? Explain.

2. Let A=$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$. Find each of AA^t and A^tA.

3. Find the transpose of the half of $\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$

Powers of matrices:

Let A be an n-square matrix over a field K. *Powers* of A are defined as follows:

$$A^2 = AA, \qquad A^3 = A^2A, \qquad \ldots, \qquad A^{n+1} = A^nA, \qquad \ldots, \qquad \text{and} \qquad A^0 = I$$

Example:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} \quad \text{and} \quad A^3 = A^2A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$. Then

$$f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercises:

1- Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{pmatrix}$, $C = \begin{pmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{pmatrix}$, $a = 4,\ b = -7$

Show that

(a) $A + (B + C) = (A + B) + C$ $\qquad$ $(c)\,(AB)C = A(BC)$

(b) $(a + b)C = aC + bC$ $\qquad$ $(d)\ a(B - C) = aB - aC$

2- Let $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$. Compute A^3 *and* $A^2 - 2A + I_2$.

Determinants:

Definition: Let A be a square matrix. The determinant function on the set of square matrices, denoted by det(A) or $|A|$, is defined as the sum of all signed elementary products from A.

∗ If A=[a], then det(A)=a.

∗ Consider the 2×2 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

∗ Give the 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\det(A)=(a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23} a_{31} + a_{13}\, a_{21}\, a_{32})-(a_{13}\, a_{22}\, a_{31} + a_{11}\, a_{23}\, a_{32} + a_{12}\, a_{21}\, a_{33})$

Similarly, we can find the determinant of square matrices of every degree.

Properties of determinants:

Let A be a square matrix.

1- If A has a zero row or column, then det(A) = 0.

2- det(A) = det(A^t)

3- If A is an $n \times n$ upper triangular, lower triangular or diagonal matrix, then det(A) is the product of the entries on the main diagonal of the matrix; that is det(A) = $a_{11}\, a_{22} \dots a_{nn}$.

4- If B is the matrix that results when a row or a column of A is multiplied by a scalar k , then det(B) = k det(A).

5- If B is the matrix that results when two rows or two columns of A is interchanging , then det(B) = - det(A).

6- If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column of A is added to another column , then det(B) = det(A).

7- If two rows or two columns in A are proportional, then det(A)=0.

8- det (k A) = k^n det(A).

Example: Evaluate the determinant of each of the following matrices:

1- $\begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$

2- $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{pmatrix}$

3- $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{pmatrix}$

4- $\begin{pmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

Evaluating determinant by using cofactors expansion method:

The determinant of an nxn matrix A can be completed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products, that is for each $1 \leq i, j \leq n$,

$$\det(A) = \underbrace{a_{1j}C_{1j} + a_{2j}C_{2j} + \ \dots + a_{nj}C_{nj}}_{cofactors\ expansion\ along\ the\ jth\ column}$$

and

$$\det(A) = \underbrace{a_{i1}C_{i1} + a_{i2}C_{i2} + \ \dots + a_{in}C_{in}}_{cofactors\ expansion\ along\ the\ ith\ row}$$

, where C_{ij} is the determinant of A after removing the ith row and jth column.

Example: Let A= $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$. Evaluate det(A) by cofactors expansion along the first row of A.

Definition: If A is any n×n matrix and C_{ij} is the factor of a_{ij}, then the transpose of the following matrix is called adjoint of A and is denoted by adj(A).

$$\begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

Example: Find adjA for $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$.

Solution:

The cofactors of A are

$C_{11} = 12$ $C_{12} = 6$ $C_{13} = -16$

$C2_{1} = 4$ $C_{22} = 2$ $C_{23} = 16$

$C_{31} = 12$ $C_{32} = -10$ $C_{33} = 16$

Thus the adjoint of A is $adj(A) = \begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}^t = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$

Exercises: Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{pmatrix}$. Find the following:

1. The matrix of cofactors.
2. $adj(A)$

Inverse of matrices

Definition: A square matrix A is invertible if there exists a matrix B of the same size such that $AB = BA = I$.

The matrix B is called the inverse of A and denoted by $B = A^{-1}$.

Example: Suppose tat $A = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$.

Is A invertible? If ″ yes, find it

Example: Does the following matrix have an inverse?

$$A = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Example: Let $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$.

1. Compute the inverse of the each of A and B.

3- Use the matrix A and B in exercise 2 to verify

(a) $(A + B)^t = A^t + B^t$ (b) $(B^t)^{-1} = (B^{-1})^t$

4- Use the matrix A and B in exercise 2 to compute A^3 and $A^2 - 2A + I_2$.

5- Find $(5A^t)^{-1}$.

Remark: For a square matrix A, if det(A)≠0, then A is invertible. Otherwise, A is not invertible.

Properties of the Inverse

Let A and B be square matrices.

1- $(A^{-1})^{-1} = A$.

2- $(AB)^{-1} = B^{-1}A^{-1}$.

3- $(A^{-1})^t = (A^t)^{-1}$.

Methods for finding matrices inverse:

There are three methods for finding the invers of an invertible matrix A:

1. Using the definition of matrix inverse:

In this way, we give an unknown matrix A^{-1} of the same size of A. and then we use the equation AA^{-1}=I to find A^{-1} which is an inverse of A.

2. $A^{-1} = \frac{1}{det(A)}\, adj(A)$.

3. By using row operation by producing a matrix of the form [A| I] and find [I | A^{-1}].

Example: Find the inverse of matrix $\begin{pmatrix} 4 & -3 \\ 5 & 7 \end{pmatrix}$ by using two ways.

Example: Use row operations to find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$.

Let $[A|\ I] = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$.

we added -2 times the first row to the second and -1 times the first row to the third, we get:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

We added 3times the second row to the third to get

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

$We\ multiplied\ the\ third\ row\ by - 1$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

We added 3 times the third row to the second and -3 times the third row to the first:

$$\left(\begin{array}{ccc|ccc}1 & 2 & 0 & -14 & 6 & 3\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

We added -2 times the second row to the first:

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -40 & 16 & 9\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{pmatrix}$.

Exercises: Find the inverse of the following given matrices

(1) $A = \begin{pmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{pmatrix}$ (2) $B = \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 3 & 0 & 0\\ 1 & 3 & 5 & 0\\ 1 & 3 & 5 & 7\end{pmatrix}$ (3) $C = \begin{pmatrix}a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{pmatrix}$

2-Evalute the determinant of the following matrices

(a) $A = \begin{pmatrix}2 & -1 & 3\\ 1 & 2 & 4\\ 5 & -3 & 6\end{pmatrix}$, (b) $B = \begin{pmatrix}2 & 1 & 3 & 1\\ 1 & 0 & 1 & 1\\ 0 & 2 & 1 & 0\\ 0 & 1 & 2 & 4\end{pmatrix}$

3-Given that $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = -6$, find $\begin{vmatrix}3a & 3b & 3c\\ -d & -e & -f\\ 4g & 4h & 4i\end{vmatrix}$

4. Solve the equation $\begin{vmatrix}x & 5 & 7\\ 0 & x+1 & 6\\ 0 & 0 & 2x-1\end{vmatrix} = 0.$

System of linear equations

Definition: The equation of the straight line in the xy-plane can be represented algebraically by an equation of the form $a_1x + a_2y = b$.

Where a_1, a_2 *and* b are real constants and $a_1,$ *and* a_2 are not both zero. An equation of this form is called a linerar equation in the variables x and y.

Definition: A linear equation of n variables $x_1, x_2, \ldots, x_n$ is the expression in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$, where $a_1, a_2, \ldots, a_n$ *and* b are real constants.

Example: the equations

$x + 3y = 7$, $y = \frac{1}{2}x + 3z + 1$ and $x_1 - 2x_2 - 3x_3 + x_4 = 7$ are linear.

But the equations

$x + 3\sqrt{y} = 5$, $3x + 2y - z - xz = 4$ and $y = \sin x$ are not linear.

Definition: A system of linear equations consists of a finite number of linear equations of variables $x_1, x_2, \ldots, x_n$. An arbitrary system of m linear equations in n unknowns can be written as:

$$\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\
\vdots & & \vdots & & \vdots & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m
\end{array}$$

where a_{ij} and b_i are known constants for every i=1, 2, …, m and j=1, 2, …, n.

Remark: Every system of linear equations has either no solutions, exactly one solution or infinitely many solutions.

Operations for used to solve systems of linear equations.

1- Multiplying an equation by a nonzero constant.
2- Interchanging two equations.
3- Adding a multiple of one equation to another.

These operations correspond to the following operations on the rows of the matrix.

1- Multiplying a row by a nonzero constant.
2- Interchanging two rows
3- Adding a multiple of one row to another row.

Example: The following is a 3×3 system of linear equations:

$$-2y + z = -1$$

$$x + 2z = 3$$

$$x + 2y + z = 5$$

Matrix form for a system of linear equation:

The general form of an m×n system of linear equations is:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m \end{array} \qquad (1)$$

The above system can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (2)$$

If we set A=$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$, X=$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and B=$\begin{bmatrix} b \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, then system(2) becomes:

$$AX=B \qquad (3)$$

The formulas (1), (2) and (3) are equivalent.

Note: If A is a square matrix and

1. detA=0, then either the system (1) has no solution or it has an infinite number of solutions.
2. detA≠0, then the system (1) has exactly one solution.

Methods for solving the $n \times n$ systems of linear equations:

1. Cramer's rule

If $AX = b$ is a system of n linear equations in n unknowns such that det(A) ≠0, then the system has a unique solution. This solution is $x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ \dots, x_n = \frac{\det(A_n)}{\det(A)}$.

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix b.

Example: Using Cramer's rule to solve the linear system:

$$x + 2z = 6$$
$$-3x + 4y + 6z = 30$$
$$-x - 2y + 3z = 8$$

Solution: The matrices A, A_1, A_2 and A_3 are

$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix}$ and $A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}$.

The determinant of each of A, A_1, A_2 and A_3 are:

det (A)=

=44

, $\det(A_1) =$

=-40

, $\det(A_2) =$

=72

and det (A_3)=

=152

Thus the solution of above system is:

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11} \text{ and } z = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}.$$

2. Matrix inversion method:

In this way, we multiply both sides of the system AX=B by A^{-1} we get the solution $X=A^{-1}B$ of the system.

Example: Using matrix invertible to solve the system of linear equations

$$\mathrm{x} + 2\mathrm{y} + 3\mathrm{z} = 5$$

$$2\mathrm{x} + 5\mathrm{y} + 3\mathrm{z} = 3$$

$$\mathrm{x} + 8\mathrm{z} = 17$$

Solution. In matrix form this system can be written as $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Now $X = A^{-1}B = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

The solution of the above system is $x_1 = 1,\ x_2 = -1,\ x_3 = 2.$

3. Gaussian elimination method:

In this way, we consider the matrix $[A \quad B]$ and we make A un upper triangular matrix to get the solution of the given system directly.

Example: Solve the system:

$$x + y + 2z = 9$$

$$2x + 4y - 3z = 1$$

$$3x + 6y - 5z = 0$$

Solution: Now $[A \quad B]=\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$

Add -2 times the first row to the second row to obtain

$$[A \quad B]=\begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 3 & 6 & -5 & 0\end{bmatrix}$$

Add -3 times the first row to the third row to obtain

$$[A \quad B]=\begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 3 & -11 & -27\end{bmatrix}$$

Add $\frac{-3}{2}$ times the second row to the third row to obtain

$$[A \quad B]=\begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2}\end{bmatrix}$$

Multiply the third row by -2 to get:

$$[A \quad B]=\begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 0 & 1 & 3\end{bmatrix}$$

Thus the solution of the above system is $z = 3\,, y \;=\; 2\; and\; x = 1$.

Exercises: Solve the system of equations:

1- $4x + 5y \quad = 2$

$11x + y + 2z = 3$

$x + 5y + 2z = 1$

2- $-x - 4y + 2z + w \;= -32$

$2x - y + 7z + 9w = 14$

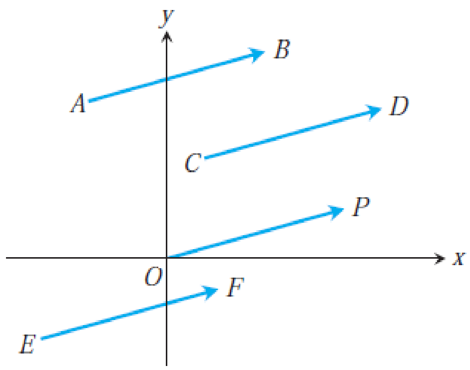
$-x + y + 3z + w = 11$

$x - 2y + z - 4w = -4$

Chapter two: Vectors

DEFINITIONS Vector, Initial and Terminal Point, Length

A **vector** in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B; its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.



DEFINITION Component Form

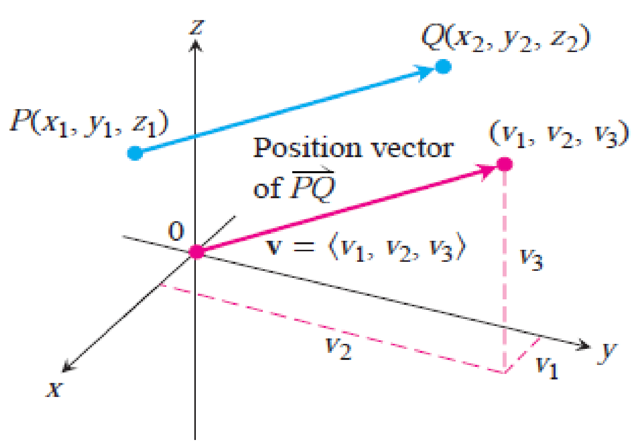
If **v** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle .$$

If **v** is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle .$$

Some times the vector v in three dimensional space is written as $v=v_1i+v_2j+v_3k$ or $v=v_xi+v_yj+v_zk$.



The only vector with length 0 is the **zero vector** $\mathbf{0} = \langle 0, 0 \rangle$ or $\mathbf{0} = \langle 0, 0, 0 \rangle$. This vector is also the only vector with no specific direction.

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example:

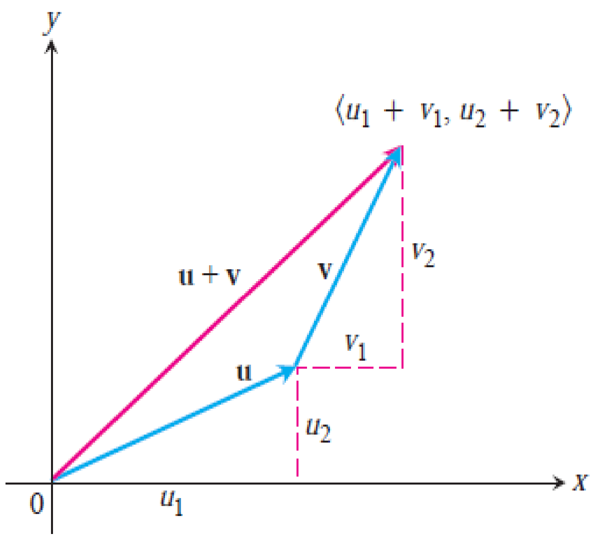
Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

DEFINITIONS Vector Addition and Multiplication of a Vector by a Scalar

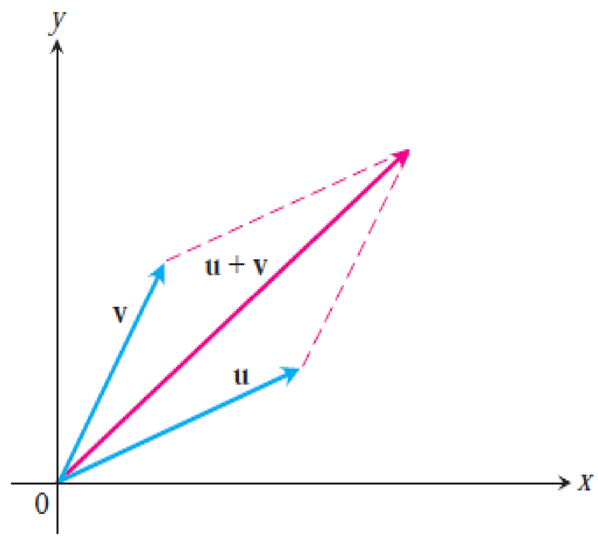
Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$



(a)



(b)

Example:

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

Basic properties of vectors under the operations of vector addition and scalar multiplication:

For any vectors u, v and w in R^3 and any scalars k and k′ in R,

(i) $(u+v)+w=u+(v+w)$,

(ii) $u+0=u$,

(iii) $u+(-u)=0$,

(iv) $u+v=v+u$,

(v) $k(u+v)=ku+kv$,

(vi) $(k+k')u=ku+k'u$,

(vii) $(kk')u=k(k'u)$,

(viii) $1u=u$.

Vectors between two points:

A vector from a point $P_1(a_1, b_1, c_1)$ to an other point $P_2(a_2, b_2, c_2)$ is defined by:

$\overrightarrow{P_1P_2}=(a_1-a_2, b_1-b_2, c_1-c_2)$

Zero vector:

The zero vector is denoted by $\vec{0}=(0, 0, 0)$.

Unit vector:

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.

Example:

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

DEFINITION Dot Product

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("**u** dot **v**") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

THEOREM 1 Angle Between Two Vectors

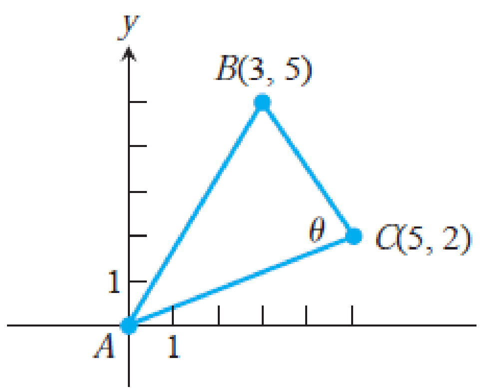
The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Example:

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Example: Find the angle θ in the following triangle



DEFINITION Orthogonal Vectors

Vectors **u** and **v** are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example:

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector **u**

Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$
3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0}\cdot\mathbf{u} = 0.$

Cosine Law:

Let u and v be two vectors. Then $\mathbf{u\cdot v=|u||v|\cos\theta}$, where θ is the angle between u and v.

Vector projection of **u** onto **v**:

The **vector projection** of u onto a nonzero vector v is the vector determined by $\text{proj}_{\,\mathrm{v}}\mathrm{u} = \frac{\mathrm{u}\cdot\mathrm{v}}{|\mathrm{v}|^2}\mathrm{v}.$

Example: Find the vector projection of u=(

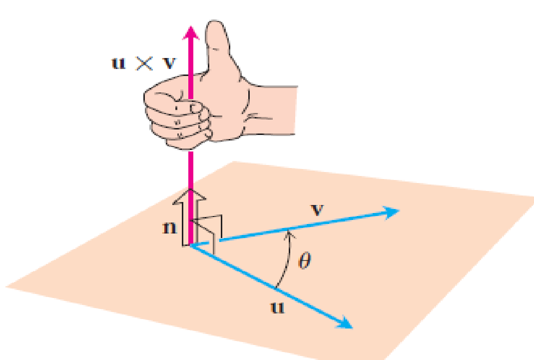
Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of **u** in the direction of **v**.

The Cross Product

We start with two nonzero vectors **u** and **v** in space. If **u** and **v** are not parallel, they determine a plane. We select a unit vector **n** perpendicular to the plane by the **right-hand rule**.

DEFINITION Cross Product

$$\mathbf{u}\times\mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin\theta)\,\mathbf{n}$$



Calculating Cross Products Using Determinants

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

Exercise: Let u and v are not parallel vectors. Show that u×v is perpendicular to both u and v.

Sine Law: For any two vectors u and v, $|u \times v| = |u||v|sin\theta$, where θ is the angle between them.

Calculating Cross Products Using Determinants

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example:

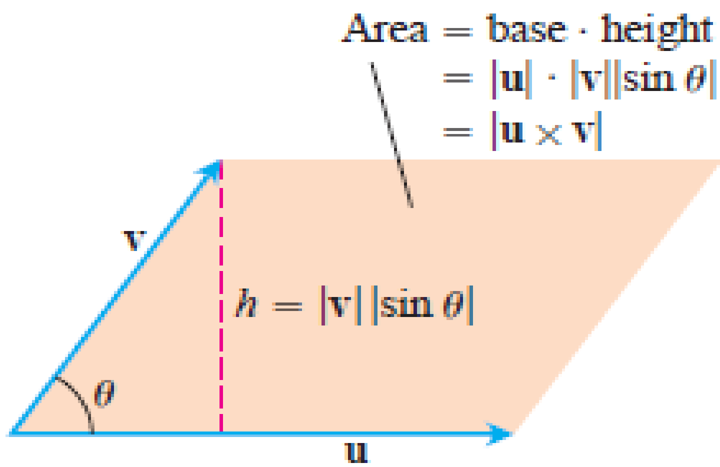
Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Example:

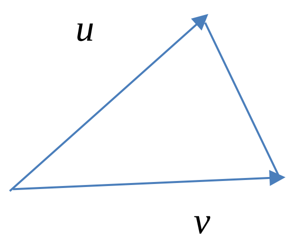
Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

Area and volume using the cross product:

1. Area of parallelograms:

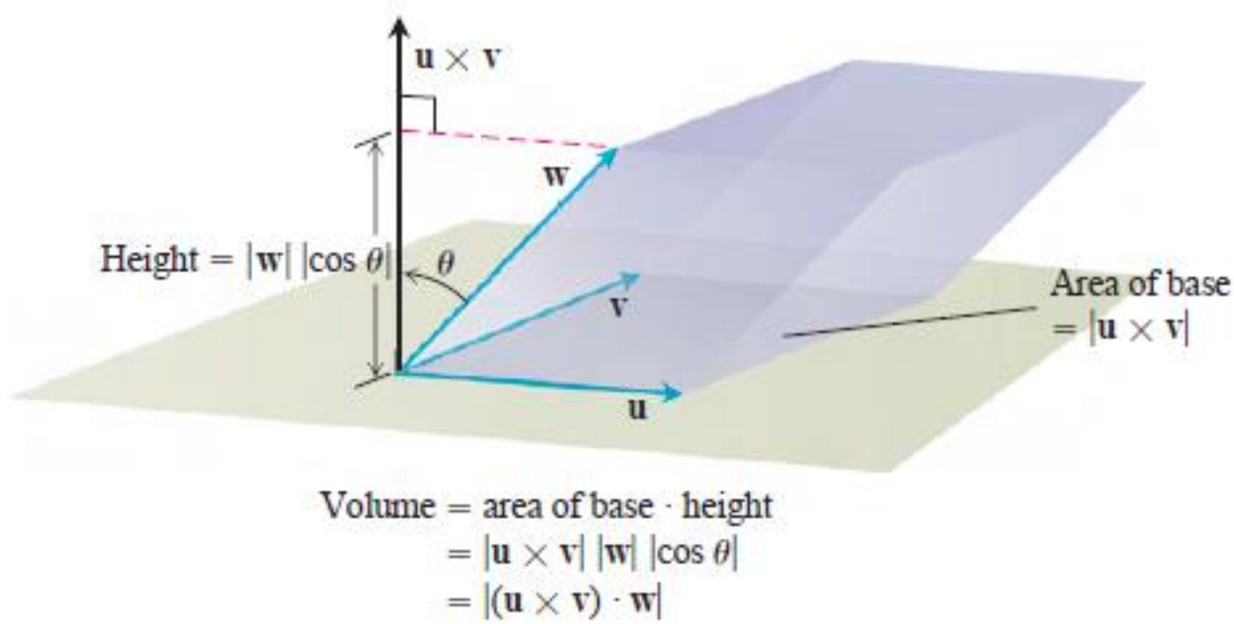


2. Area of triangles:

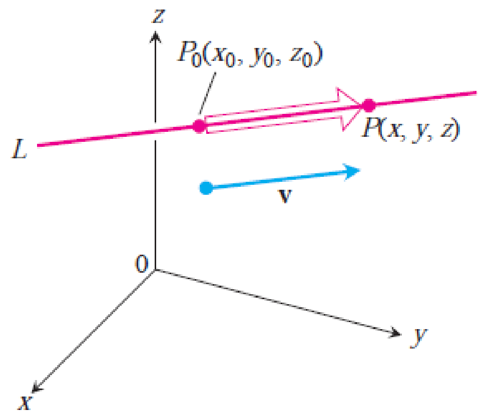


From the first part , the area of above triangle can be obtained as: Area=$\frac{1}{2}|u \times v|$

3. Volume of Parallelograms:



Lines and Line Segments in Space



In the plane, a line is determined by a poin
space a line is determined by a point and a ı

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$ (Figure 12.35). Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter t. The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \qquad (1)$$

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \qquad (3)$$

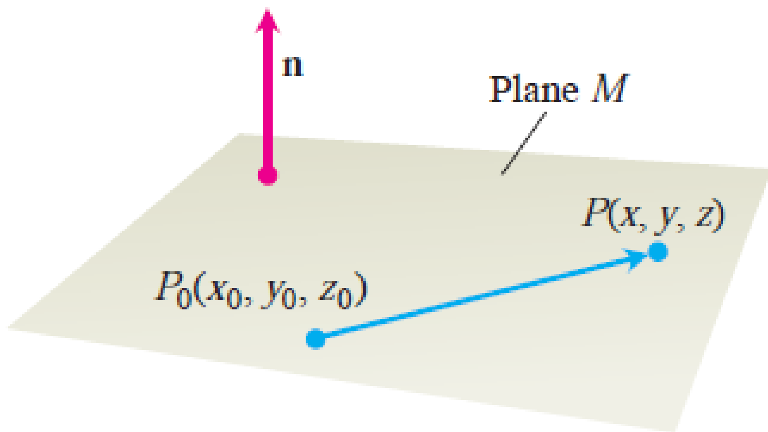
Example:

Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Example:

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

An Equation for a Plane in Space



From the figure $\overrightarrow{P_0P}$ and n are parallel. Thus $\overrightarrow{P_0P}$ ·n=0

The plane through P_0 normal to the vector n=(a, b, c) is:

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

Example:

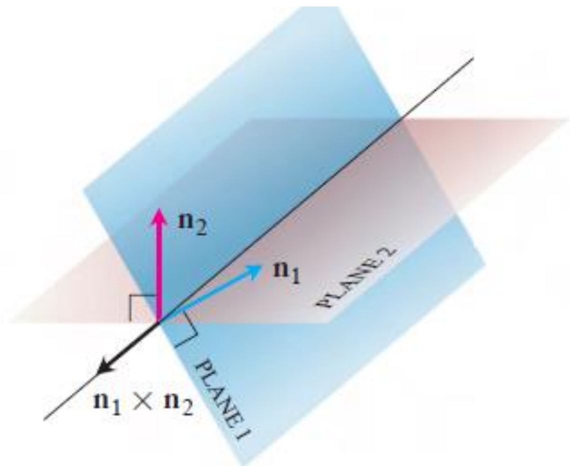
Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Example:

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k. Two planes that are not parallel intersect in a line.



Example:

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Example:

Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Intersection of a Line and a Plane

Example:

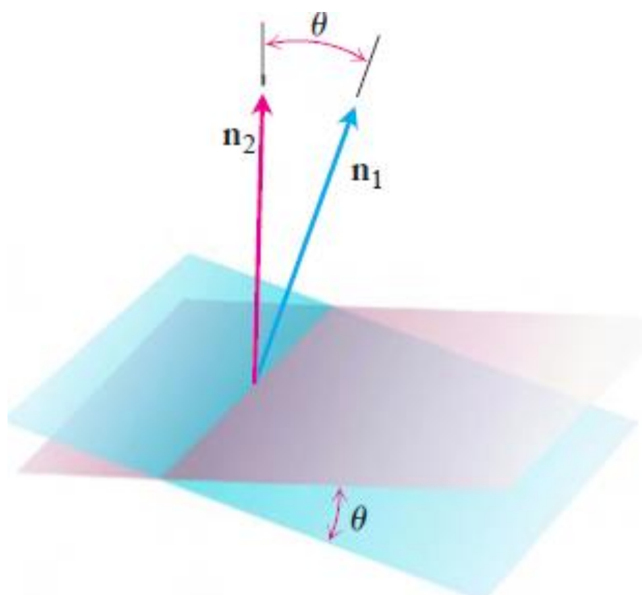
Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Angles Between Planes

The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors (Figure 12.42).



Example:

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.