Question bank:

Q1\ Verify that $\sqrt{2}\,|z| \geq |\text{Re } z| + |\text{Im } z|$.

Q2\ In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + i| = 1$; *(b)* $|z + i| \leq 3$; *(c)* $|z - 4i| \geq 4$.

Q3\ Sketch the set of points determined by the condition

(a) $\text{Re}(z - i) = 2$; *(b)* $|2z + i| = 4$.

Q4\ If $z1z2 = 0$, then at least one of the numbers $z1$ and $z2$ must be zero.

Q5\ Show that:

(a) z is real if and only if $\bar{z} = z$;

(b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

Q6\ Show that the hyperbola $x^2 - y^2 = 1$ can be written as $\bar{z}^2 + z^2 = 2$.

Q7\ Find the principal argument Arg z when

(a) $z = -2 - 2i$ *(b)* $z = (\sqrt{3} - i)$.

Q8\ Establish the equation

$$1 + z + z^2 + \cdot\cdot\cdot + z^n = \frac{1-z^{n+1}}{1-z},\ z \neq 1.$$

Q9\ Sketch the following sets and determine which are regions:

(a) $|z - 2 + i| \leq 1$; *(b)* $|2z + 3| > 4$;

(c) $\text{Im}z > 1$; *(d)* $\text{Im } z = 1$;

(e) $0 \leq \arg z \leq \pi/4$

(f) $|z - 4| \geq |z|$.

Q10\ For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \frac{1}{1-z^2}$; *(b)* $f(z) = \text{Arg}(z-i)$

Q11\ Prove that

(a) $\lim_{z\to z0} \text{Re } z = \text{Re } z0$

(b) $\lim_{z\to z0} \bar{z} = \overline{z0}$;

Q12\ Show that a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdot\cdot\cdot + a_n z^n$ of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

Q13\ Let f denote the function whose values are

$$f(z) = \begin{cases} \frac{\bar{z}}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that f is not differentiable every where.

Q14\ *Let f: C→C is defined by* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Is f an analytic function on C?

Q15\ Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ When:

(a) $u(x, y) = 2x(1 - y)$; (b) $u(x, y) = 2x - x3 + 3xy2$;

(c) $u(x, y) = \sinh x \sin y$; (d) $u(x, y) = y/(x2 + y2)$.

Q16\ Show that if v and V are harmonic conjugates of u(x, y) in a domain D, then v(x, y) and V (x, y) can differ at most by an additive constant.

Q17\ For the functions f and paths C, evaluate the integral of f (z) = (z + 2)/z and C is:

(a) the semicircle z = 2 eiθ (0 ≤ θ ≤ π);

(b) the semicircle z = 2 eiθ (π ≤ θ ≤ 2π);

(c) the circle z = 2 eiθ (0 ≤ θ ≤ 2π).

Ans. (a) −4 + 2πi; (b) 4 + 2πi; (c) 4πi.

Q18\ f (z) = z − 1 and C is the arc from z = 0 to z = 2 consisting of

(a) the semicircle z = 1 + eiθ (π ≤ θ ≤ 2π);

(b) the segment z = x (0 ≤ x ≤ 2) of the real axis.

Ans. (a) 0 ; (b) 0.

Q19\ Give an example to apply Cauchy Goursat Theorem.

Q20\ Is the addition of two convergent series give an other convergent series?

EXERCISES

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

   (a) $f(z) = 3x + y + i(3y - x)$; (b) $f(z) = \sin x \cosh y + i \cos x \sinh y$;

   (c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; (d) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

   (a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i(x^2 - y^2)$; (c) $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

   (a) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$; (b) $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$; (c) $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

   Ans. (a) $z = 0, \pm i$; (b) $z = 1, 2$; (c) $z = -2, -1 \pm i$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughtout D.

EXERCISES

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

   (a) $u(x, y) = 2x(1 - y)$; (b) $u(x, y) = 2x - x^3 + 3xy^2$;
   (c) $u(x, y) = \sinh x \sin y$; (d) $u(x, y) = y/(x^2 + y^2)$.

   Ans. (a) $v(x, y) = x^2 - y^2 + 2y$; (b) $v(x, y) = 2y - 3x^2y + y^3$;
   (c) $v(x, y) = -\cosh x \cos y$; (d) $v(x, y) = x/(x^2 + y^2)$.

2. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

4. Use Theorem 2 in Sec. 26 to show that v is a harmonic conjugate of u in a domain D if and only if $-u$ is a harmonic conjugate of v in D. (Compare with the result obtained in Exercise 3.)

   Suggestion: Observe that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D if and only if $-if(z)$ is analytic there.

EXERCISES

1. Show that

   (a) $\exp(2 \pm 3\pi i) = -e^2$; (b) $\exp\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$;

   (c) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \overline{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

   Ans. $f'(z) = 2z \exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

   (a) $e^z = -2$; (b) $e^z = 1 + \sqrt{3}i$; (c) $\exp(2z - 1) = 1$.

   Ans. (a) $z = \ln 2 + (2n+1)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;

   (b) $z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;

   (c) $z = \frac{1}{2} + n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

EXERCISES

1. Show that

   (a) $\operatorname{Log}(-ei) = 1 - \frac{\pi}{2}i$; (b) $\operatorname{Log}(1 - i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

2. Show that

   (a) $\log e = 1 + 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;

   (b) $\log i = \left(2n + \frac{1}{2}\right)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;

   (c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that

   (a) $\operatorname{Log}(1 + i)^2 = 2\operatorname{Log}(1 + i)$; (b) $\operatorname{Log}(-1 + i)^2 \ne 2\operatorname{Log}(-1 + i)$.

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

8. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (4), Sec. 30.]

9. Show that

(*a*) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(*b*) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2 + i}$$

EXERCISES

1. Show that

(*a*) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $(-1)^{1/\pi} = e^{(2n+1)i} \quad (n = 0, \pm 1, \pm 2, \ldots)$.

2. Find the principal value of

(*a*) i^i; (*b*) $\left[\frac{e}{2}(-1 - \sqrt{3}i)\right]^{3\pi i}$; (*c*) $(1-i)^{4i}$.

Ans. (*a*) $\exp(-\pi/2)$; (*b*) $-\exp(2\pi^2)$; (*c*) $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; *(b)* $\tan^{-1}(1+i)$; *(c)* $\cosh^{-1}(-1)$; *(d)* $\tanh^{-1} 0$.

Ans. *(a)* $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3 \; (n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i \; (n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots)$.

EXERCISES

1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

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Ans. *(a)* $\exp(-\pi/2)$; *(b)* $-\exp(2\pi^2)$; *(c)* $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.