# Numerical Analysis Questions

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1. Suppose that 1.414 is used as an approximation to $\sqrt{2}$. Find the absolute error and relative errors.

2. If 0.333 is approximate value of $\frac{1}{3}$, find the absolute, relative and percentage errors.

3. Use the formula
$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$
to approximate the derivative of $f(x) = 3x^2$ at $x = 1$ using $h = 0.1$. Compute both the absolute true error $|E_t|$, and absolute relative true error $|\in_a|$.

4. Provide a general formula for determining both the absolute true error and absolute relative true error when approximating the derivative of $f(x) = x^2$ at $x = a$ using a value $h$ in the expression
$$f'(x) \approx \frac{(a+h)^2-a^2}{h}.$$

5. Using $|\in_a|$ from the previous exercise, what is the greatest value of $h$ that can be used to approximate the derivative of $f(x) = x^2$ at $x = 4$ with an error of no more than 1%?

1. Starting with the intervals $[0.5, 0.6]$, use 3 steps of the Bisection method to find a smaller interval containing a root of $f(x) = e^x - 2cos(x)$. What will be the length of the final interval after 10 steps of the method?

2. Rearranging the equation $x^3 = 0.5$, gives the iterative formula $x_{n+1} = g(x_n)$, where $g(x) = (2x^2)^{-1}$.
(a) Starting with $x_0 = 1$, compute the $x_n$ up to $n = 6$, and describe what is happening.
(b) Explain the behavior of (a) by evaluating $g'(x)$ at $x = 2^{-\frac{1}{3}}$.

   (c) Repeat (a) and (b) for the two iterative schemes $x_{n+1} = \frac{x_n}{2} + \frac{1}{4x_n^2}$ and $x_{n+1} = \frac{2x_n}{3} + \frac{1}{6x_n^2}$. In particular: which of them gives faster convergence, and why? [Work to four decimal places.]

3. Apply the Regula-Falsi method for 3 steps starting with $x_0 = 0$, $x_1 = 1$ to find approximations to the root of $f(x) = \frac{1}{x+1} - \frac{3}{4}$.

4. If we are asked to find the root of $f(x) = x^2 - 1$ in $[0, 2]$, give a function $g(x)$ for which the root is a fixed point.

5. State the criteria for the function $g(x)$ to have a fixed point in the interval $[a, b]$. Does the function $g(x) = \sqrt{x}$ have a fixed point in the interval $[0.5, 2]$?

6. State the criteria for a function $g(x)$ used in a fixed point iteration method to converge on an interval $[a, b]$. Does the function $g(x) = \frac{1}{2}x + \frac{1}{x}$ meet these criteria on the interval $[1, 2]$?

7. Solve $4\cos(x) = 2^x$ with an accuracy of $10^{-1}$, by using Newton-Raphson method.

8. Find the positive root of $\cos(x) - x\, e^x = 0$ for two steps, use false position method.

9. Use Secant method to write the iterative formula for finding $\sqrt{8}$.

10. For what values of $x_0$ and $x_1$ can the secant method be used to solve the following equation?

$$f(x) = \frac{4x - 7}{x - 2} = 0.$$

11. Can Newton-Raphson method be used to solve $f(x) = 0$ if $f(x) = \sqrt{x - 3}$ and the starting value is $x_0 = 4$? Explain your answer.

12. Find the order of convergence for $x_{n+1} = \frac{1}{2}\, x_n(1 + \frac{a}{x_n^2})$ at $\sqrt{a}$.

13. Which of the following iteration forms
    1. $x_{n+1} = x_n(3 - 3cx_n + c^2x_n^2)$
    2. $x_{n+1} = x_n(2 - cx_n)$

   should be used for finding an approximate value of – ? Explain your answer.

14. Find $a$, $b$ and $c$ so that the order of the iteration form $x_{n+1} = ax_n^{-2} + cd^2x_n^{-5}$ for $\sqrt[3]{d}$ becomes as high as possible.

15. Suppose a zero of $g(x) = x$ is approximated by bisection. How many iterations of the bisection method on $[-1, 4]$ are sufficient to compute a solution to $g(x) = x$ to error less than $10^{-6}$ ?

16. Find the conditions on $\alpha$ to ensure that the iteration $x_{n+1} = x_n - \alpha f(x_n)$, $n = 0, 1, \cdots$ will converge linearly to the zero of $f$ if started near the zero.

17. Starting with the interval $[1.1, 1.2]$, use 3 steps of the bisection method to find a smaller interval containing a root of $f(x) = e^x - 2 - x$. Give the interval after the 3 steps, the estimate for the root, and the maximum error.

18. Newton-Raphson method converge to $r$ of order two if $f(r) = 0$ and $f'(r) \neq 0$. What special properties must a function $f$ have if Newton-Raphson method applied to $f$ converges cubically to a zero of $f$?

19. Show that the sequence $p_n = \frac{1}{n}$ converges linearly to $p = 0$, and determine the number of the terms required to have $|p_n - p| < 5 \times 10^{-2}$.

20. The cubic equation $x^3 - 2x - 5 = 0$ has one real root that is near to $x = 2$. The equation can be rewritten in the following manner:
$(i)\ x = \frac{1}{2}(x^3 - 5)$ $\quad (ii)\ x = \frac{5}{x^2-2}$ $\quad (iii)\ x = \sqrt[3]{x^2 + 5}$.
Choose the form which satisfies the condition $|g'(x)| < 1$ and find the root correct to 3 dp.

21. The cubic equation $x^3 - 3x - 20 = 0$, has one real root that is near to $x_0 = 0.3$. The equation can be rewritten in the following manner:
$(i)\ x\frac{1}{3}(x^3 - 20)$ $\quad (ii)\ x = \frac{20}{x^2-3}$ $\quad (iii)\ x = \sqrt{3 + \frac{20}{x}}$ $\quad (iv)\ x = \sqrt[3]{3x + 20}$.
Choose the form which satisfies the condition $|g'(x_0)| < 1$ and find the root correct to 4 dp. Which of them gives rise to very rapid convergence?

22. Given the following variations of the equation, $x^4 + x^2 - 80 = 0$,
$(i)\ x = \sqrt[4]{80 - x^2}$ $\quad (ii)\ x = \sqrt{80 - x^4}$ $\quad (iii)\ x = \sqrt{\frac{80}{1+x^2}}$.
Which of them gives rise to a convergent sequence? Find the real root of the equation correct to 4 dp. Take $x_0 = 3$.

23. To locate the root of $e^{-x} - cos(x) = 0$ that is near to 1.29, using iteration, we could rewrite the equation as,
$(i)\ x = cos^{-1}(e^{-x})$ $(ii)\ x = -log(cos(x)) = log(sec(x))$
$(iii)\ x = x - 0.01(e^{-x} - cos(x))$.
Which of these three forms (if any) would yield a convergence iteration scheme? Which would converge the fastest?

24. Starting with $x_0 = 6$, perform ten iterations using each of the recurrence relations:
$(i)\ x_{n+1} = \sqrt{5x_n - 4}$ $(ii)\ x_{n+1} = \frac{x_n-4}{2x_n-5}$
which of $(i)$ and $(ii)$ has the higher rate of convergence?

25. Determine which of the following iterative functions, $g(x)$, can be used to locate the zeros of the equation $x^3 + 2x - 1 = 0$ on the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$:
$(i)\ x = \frac{1}{2}(1 - x^3)$ $(ii)\ x = \frac{1-2x}{(}x^2)$ $(iiii)\ x = \frac{x^3}{1-x}$
$(iv)\ x = \sqrt[3]{1 - 2x}$ $(v)\ x = x - 0.2(x^3 + 2x - 1)$ $(vi)\ x = \frac{x^3+2x-1}{3x^3+2}$.

26. Consider the polynomial, $p(x) = x^5 - 6x^4 + 8x^3 + 8x^2 + 4x - 40$. Starting with initial approximation $x_0 = x3$, evaluate p(3) and p′(3). Using Homer's method with Newton-Raphson method, compute the next two approximations correct to 2 dp.

27. Starting with $x_0 = 8$, perform 5 iterations using the following iterative formula: $x_{n+1} = 0 - \frac{20}{x_n}$. Use Aitken's iterative method with $x_3$, $x_4$ and $x_5$, to improve the rate of convergence.

28. Use Aitken's iterative method to find the root of $e^x = 5x$ near to $x_0 = 0.3$. Compare the result method with iteration $x_{n+1} = \frac{1}{5}e^x$ starting with $x_0 = 0.3$.

29. Given $f(x) = x^3 - c$, use Newton's method to establish the recurrence relation,

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{c}{x_n^2}\right]$$

for the cubic root of a number $a$. With $c = 9$ and $x_0 = 3$, perform the iterations 6 times to find the cubic root of 9.