Example 4 Evaluating $\iint_R 3x\,dA$ over a Polar Rectangular Region

$$R = \{(r,\theta): 1 \le r \le 2 \text{ and } 0 \le \theta \le \pi\}$$

Example 5 Evaluating $\iint_R (x+y)\,dA$ over a Polar Rectangular Region

$$R = \{(x,y): 1 \le x^2 + y^2 \le 4 \text{ and } x \le 0\}$$

Example 6: Use polar coordinates to evaluate

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{\frac{3}{2}}\,dy\,dx$$

Example 7: Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$

Example 8: Find the volume of the solid bounded above by the surface $z = 3 + r$ and below by the region enclosed by the cardioid $r = 1 + \sin\theta$

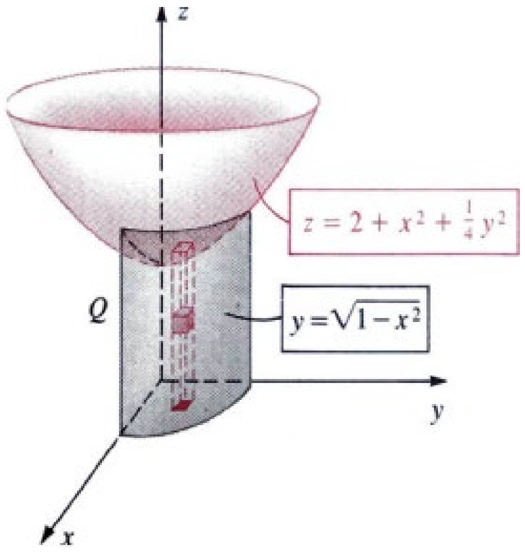
Example 6: Find symmetric equations of the line in the xy-plane that passes through the points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$.

In Problems 1–22, find a vector equation, parametric equations, and symmetric equations of the indicated line.

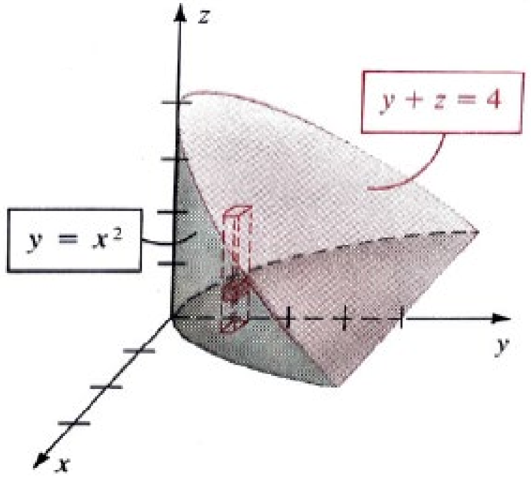
1. Containing (2, 1, 3) and (1, 2, −1).
2. Containing (1, −1, 1) and (−1, 1, −1).
3. Containing (1, 3, 2) and (2, 4, −2).
4. Containing (−2, 4, 5) and (3, 7, 2).
5. Containing (−4, 1, 3) and (−4, 0, 1).
6. Containing (2, 3, −4) and (2, 0, −4).
7. Containing (1, 2, 3) and (3, 2, 1).
8. Containing (7, 1, 3) and (−1, −2, 3).
9. Containing (1, 2, 4) and (1, 2, 7).
10. Containing (−3, −1, −6) and (−3, 1, 6).
11. Containing (2, 2, 1) and parallel to 2**i** − **j** − **k**.
12. Containing (−1, −6, 2) and parallel to 4**i** + **j** − 3**k**.
13. Containing (1, 0, 3) and parallel to **i** − **j**.
14. Containing (2, 1, −4) and parallel to **i** + 4**k**.
15. Containing (−1, −2, 5) and parallel to −3**j** + 7**k**.
16. Containing (−2, 3, −2) and parallel to 4**k**.
17. Containing (−1, −3, 1) and parallel to −7**j**.

Example 4 Express $\int\int_Q\int f(x,y,z)dV$ as an iterated integral if Q is the region in the first octant bounded by the coordinate planes and the graphs of

$$z-2=x^2+\frac{1}{4}y^2, \text{and } x^2+y^2=1.$$



Example 5 Find the volume of the solid that is bounded by the cylinder $y=x^2$ and by the planes $y+z=4$ and $z=0$.



H.W 1: Evaluate $\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\,dzdydx$.

H.W 2: Evaluate $\int_0^4\int_0^{\frac{4-x}{2}}\int_0^{\frac{12-3x-6y}{4}} 1\,dzdydx$, using the order of integration $dV=dydxdz$.

Let v be any nonzero vector. Then $u = v/|v|$ is the unit vector having the same direction as v.

Example 3: Find the unit vector having the same direction as $v = 2i - 3j$.

Example 4: Find the vector v whose direction is $\frac{5\pi}{4}$ and whose magnitude is 7.

H.W.

In Problems 7–18, find the magnitude and direction of the given vector.

7. $\mathbf{v} = (4, 4)$
8. $\mathbf{v} = (-4, 4)$
9. $\mathbf{v} = (4, -4)$
10. $\mathbf{v} = (-4, -4)$
11. $\mathbf{v} = (\sqrt{3}, 1)$
12. $\mathbf{v} = (1, \sqrt{3})$
13. $\mathbf{v} = (-1, \sqrt{3})$
14. $\mathbf{v} = (1, -\sqrt{3})$
15. $\mathbf{v} = (-1, -\sqrt{3})$
16. $\mathbf{v} = (1, 2)$
17. $\mathbf{v} = (-5, 8)$
18. $\mathbf{v} = (11, -14)$

In Problems 19–26, write in the form $a\mathbf{i} + b\mathbf{j}$ the vector $\mathbf{v}$ that is represented by $\overrightarrow{PQ}$. Sketch $\overrightarrow{PQ}$ and $\mathbf{v}$.

19. $P = (1, 2)$; $Q = (1, 3)$
20. $P = (2, 4)$; $Q = (-7, 4)$
21. $P = (5, 2)$; $Q = (-1, 3)$
22. $P = (8, -2)$; $Q = (-3, -3)$
23. $P = (7, -1)$; $Q = (-2, 4)$
24. $P = (3, -6)$; $Q = (8, 0)$
25. $P = (-3, -8)$; $Q = (-8, -3)$
26. $P = (2, 4)$; $Q = (-4, -2)$

27. Let $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (-5, 4)$. Find the following:
    (a) $3\mathbf{u}$ (b) $\mathbf{u} + \mathbf{v}$ (c) $\mathbf{v} - \mathbf{u}$ (d) $2\mathbf{u} - 7\mathbf{v}$
    Sketch these vectors.
28. Let $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = -4\mathbf{i} + 6\mathbf{j}$. Find the following:
    (a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $3\mathbf{u}$
    (d) $-7\mathbf{v}$ (e) $8\mathbf{u} - 3\mathbf{v}$ (f) $4\mathbf{v} - 6\mathbf{u}$
    Sketch these vectors.
29. Show that the vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors.
30. Show that the vector $(1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$ is a unit vector.
31. Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} \neq \mathbf{0}$, then $\mathbf{u} = (a/\sqrt{a^2 + b^2})\mathbf{i} + (b/\sqrt{a^2 + b^2})\mathbf{j}$ is a unit vector having the same direction as $\mathbf{v}$.

THE DOT PRODUCT

Definition 2: **Dot Product** Let $u = (a_1, b_1) = a_1 i + b_1 j$ and $v = (a_2, b_2) = a_2 i + b_2 j$. Then the dot product of u and v denoted $\boldsymbol{u} \cdot \boldsymbol{v}$, is defined by

$$\mathbf{u} \cdot \boldsymbol{v} = a_1 a_2 + b_1 b_2$$

REMARK. The dot product of two vectors is a scalar. For this reason the dot product is often called the **scalar product**. It is also called the **inner product**.

Example 5: If $u = (1,3)$ and $v = (4, -7)$, find $\mathbf{u} \cdot \boldsymbol{v}$.

Theorem 1. For any vectors **u**, **v**, **w**, and scalar α,

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(iii) $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(v) $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Definition 3 **ANGLE BETWEEN TWO VECTORS** Let u and v be two nonzero vectors. Then the angle φ between u and v is defined to be the smallest angle between the representations of u and v that have the origin as their initial points.

Theorem 2. Let u and v be two nonzero vectors. Then if φ is the angle between them,

$$\cos\varphi = \frac{\mathbf{u}\cdot\boldsymbol{v}}{|\mathbf{u}||\boldsymbol{v}|}.$$

we could define the dot product $\mathbf{u}\cdot\boldsymbol{v}$ by

$$\mathbf{u}\cdot\boldsymbol{v} = |\mathbf{u}||\boldsymbol{v}|\cos\varphi$$

Example 6 Find the cosine of the angle between the vectors u = 2i + 3j and v = - 7i + j.

Definition 4 **PARALLEL VECTORS** Two nonzero vectors u and v are parallel if the angle between them is 0 or π .

Example 7 Show that the vectors u = (2, -3) and v = (-4, 6) are parallel.

Definition 5 **ORTHOGONAL VECTORS** The nonzero vectors u and v are called orthogonal (or perpendicular) if the angle between them is $\frac{\pi}{2}$.

Example 8 Show that the vectors u = 3i - 4j and v = 4i + 3j are orthogonal.

Theorem 3 The nonzero vectors $\mathbf{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\mathbf{u} \cdot \boldsymbol{v} = \mathbf{0}$.

Theorem 4 Let v be a nonzero vector. Then for any other vector u, the vector

$$w = u - \left[\frac{u \cdot v}{|v|^2}\right] v \text{ is orthogonal to } v.$$

Proof: H.W.

H.W.

In Problems 1–10, calculate both the dot product of the two vectors and the cosine of the angle between them.

1. $\mathbf{u} = \mathbf{i} + \mathbf{j}$; $\mathbf{v} = \mathbf{i} - \mathbf{j}$
2. $\mathbf{u} = 3\mathbf{i}$; $\mathbf{v} = -7\mathbf{j}$
3. $\mathbf{u} = -5\mathbf{i}$; $\mathbf{v} = 18\mathbf{j}$
4. $\mathbf{u} = \alpha\mathbf{i}$; $\mathbf{v} = \beta\mathbf{j}$; α, β real and nonzero
5. $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$; $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$
6. $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$; $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$
7. $\mathbf{u} = -3\mathbf{i} + 4\mathbf{j}$; $\mathbf{v} = -2\mathbf{i} - 7\mathbf{j}$
8. $\mathbf{u} = 4\mathbf{i} + 5\mathbf{j}$; $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$
9. $\mathbf{u} = 11\mathbf{i} - 8\mathbf{j}$; $\mathbf{v} = 4\mathbf{i} - 7\mathbf{j}$
10. $\mathbf{u} = -13\mathbf{i} + 8\mathbf{j}$; $\mathbf{v} = 2\mathbf{i} + 11\mathbf{j}$

In Problems 13–20, determine whether the given vectors are orthogonal, parallel, or neither. Then sketch each pair.

13. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j}$; $\mathbf{v} = -6\mathbf{i} - 10\mathbf{j}$

14. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$; $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$

15. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$; $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j}$

16. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$; $\mathbf{v} = -6\mathbf{i} + 4\mathbf{j}$

17. $\mathbf{u} = 7\mathbf{i}$; $\mathbf{v} = -23\mathbf{j}$

18. $\mathbf{u} = 2\mathbf{i} - 6\mathbf{j}$; $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$

19. $\mathbf{u} = \mathbf{i} + \mathbf{j}$; $\mathbf{v} = \alpha\mathbf{i} + \alpha\mathbf{j}$; α real

20. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$; $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

Example 3: Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ dose not exist.

Example 4: Find $\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$.

Example 5: Using the epsilon and delta to prove that $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$.

H.W: Prove that $\lim_{(x,y)\to(1,2)} x^2 + 2y = 5$.

H.W: Show that the function $\frac{3x^3y}{x^4+y^4}$, has no limit as (x, y) approaches $(0,0)$.

Example 4: Calculate the volume of the region beneath the surface $z = xy^2 + y^3$ and over the rectangle $R = \{(x, y): 0 \le x \le 2 \; and \; 1 \le y \le 3\}$.

H.W: Evaluate the following integrals

1. $\int_1^3 \int_2^4 (40 - 2xy)dydx$
2. $\int_1^2 \int_0^3 (1 + 8xy)dxdy$.
3. Let R denote the rectangle $\{(x, y): -1 \le x \le 0 \; and \; -2 \le y \le 3\}$.

   Calculate the double integrals

a) $\iint_R (x + y)dA$

b) $\iint_R (y^2 - x^2)dA$

c) $\iint_R (3x^2 - 5y^2)dA$