Numerical Methods for Differential Equations

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CHAPTER 1

Numerical Methods for Ordinary Differential Equations

1.1 Introduction

Numerical methods are becoming more and more important in engineering applications, simply not only because of the difficulties encountered in finding exact analytical solutions but also, because of the ease with which numerical techniques can be used in conjunction with modern high-speed digital computers. Several numerical procedures for solving initial value problems involving first-order ordinary differential equations are discussed in this chapter.

1.2 Euler's Method for Ordinary Differential Equations

Euler's method is a numerical technique used to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0. \tag{1.1}$$

We derive Euler method as follows:

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now, since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition$y(x_0) = y_0$.

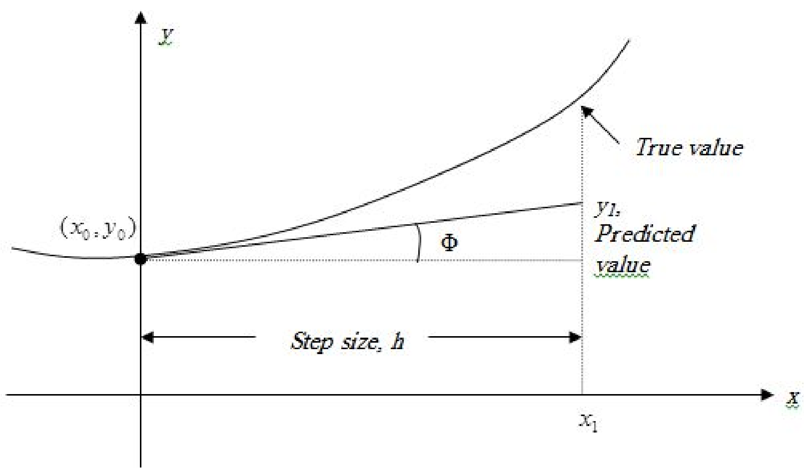

Figure 1.1: *Graphical interpretation of the first step of Euler's method.*

So the slope at $x = x_0$ as shown in Figure 1.1 is:

$$Slope = \frac{Rise}{Run} = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0).$$

From here $y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$.
Calling $x_1 - x_0$ the step size h, we get:

$$y_1 = y_0 + hf(x_0, y_0). \tag{1.2}$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by:

$$y_2 = y_1 + f(x_1, y_1)h, x_2 = x_1 + h.$$

Based on the above equations, if we now know the value of $y(x_n) = y_n$ at x_n, then

$$y_{n+1} = y_n + hf(x_n, y_n). \tag{1.3}$$

This formula is known as **Euler's method** and is illustrated graphically in Figure 1.2. In some books, it is also called the *Euler-Cauchy method*.

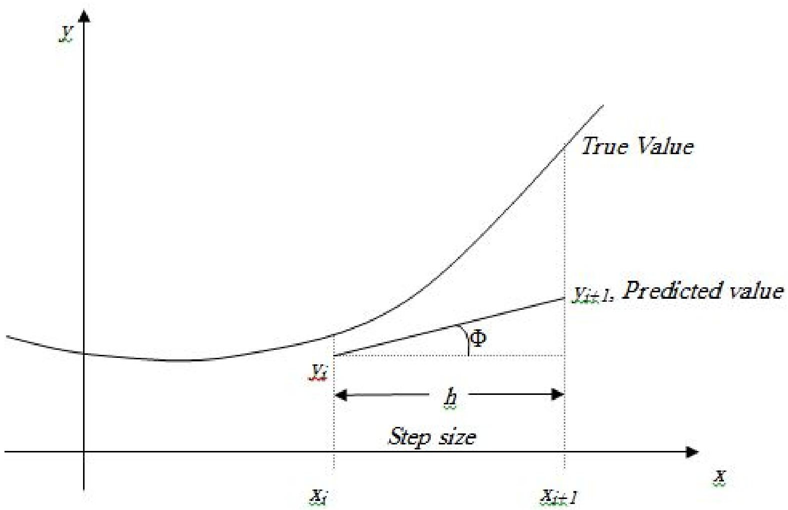

Figure 1.2: *General graphical interpretation of Euler's method.*

Example 1.1. Solve the initial value problem $y' = x + 2y$, with $y(0) = 0$ numerically, finding a value for the solution at $x = 1$, and using steps of size $h = 0.25$.

Solution: Clearly, the description of the problem implies that the interval we'll be finding a solution on is [0,1]. The differential equation given tells us the formula for $f(x, y)$ required by the Euler Method, namely:

$$f(x, y) = x + 2y$$

and the initial condition tells us the values of the coordinates of our starting point: $x_0 = 0$ and $y_0 = 0$.

We now use the Euler method formulas (1.3) to generate values for x_1 and y_1.
The x-iteration formula, with $n = 0$ gives us: $x_1 = x_0 + h = 0 + 0.25 = 0.25$.
And the y-iteration formula, with $n = 0$ gives us:

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= y_0 + h(x_0 + 2y_0) = 0 + 0.25(0 + 2 * 0) = 0. \end{aligned}$$

Summarizing, the second point in our numerical solution is: $x_1 = 0.25$ and $y_1 = 0$.
We now move on to get the next point in the solution, (x_2, y_2).
The x-iteration formula, with $n = 1$ gives us: $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$.
And the y-iteration formula, with n = 1 gives us:

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= y_1 + h(x_1 + 2y_1) = 0 + 0.25(0.25 + 2 * 0) = 0.0625. \end{aligned}$$

Summarizing, the third point in our numerical solution is: $x_2 = 0.5$ and $y_2 = 0.0625$.
We now move on to get the fourth point in the solution, (x_3, y_3).
The $x-$iteration formula, with $n = 2$ gives us: $x_3 = x_2 + h = 0.5 + 0.25 = 0.75$.
And the y-iteration formula, with $n = 2$ gives us:

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= y_2 + h(x_2 + 2y_2) = 0.0625 + 0.25(0.5 + 2 * 0.0625) = 0.21875. \end{aligned}$$

Summarizing, the fourth point in our numerical solution is: $x_3 = 0.75$ and $y_3 = 0.21875$.
We now move on to get the fifth point in the solution, (x_4, y_4).
The x-iteration formula, with $n = 3$ gives us: $x_4 = x_3 + h = 0.75 + 0.25 = 1$.
And the y-iteration formula, with $n = 3$ gives us:

$$\begin{aligned} y_4 &= y_3 + hf(x_3, y_3) \\ &= y_3 + h(x_3 + 2y_3) = 0.21875 + 0.25(0.75 + 2 * 0.21875) = 0.515625. \end{aligned}$$

Summarizing, the fourth point in our numerical solution is: $x_4 = 1$ and $y_4 = y(x_4) = y(1) = 0.515625$.

Example 1.2. solve $\frac{dy}{dx} = 1 - y, \;\; y(0) = 0$, using Euler's method. Find y at $x = 0.1$ and $x = 0.2$ Compare the results with the exact solution.

Solution: In the given differential equation $f(x, y) = 1 - y$.
Also we have, $x_0 = 0, \;\; y_0 = 0, \;\; h = 0.1$. Put $n = 0$ in (1.3), we have

$$\begin{aligned} y_1 = y(0.1) &= y_0 + hf(x_0, y_0) \\ &= 0 + (0.1)(1) = 0.1. \end{aligned}$$

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$.
Put $n = 1$ in (1.3), we have

$$\begin{aligned} y_2 = y(0.2) &= y_1 + hf(x_1, y_1) \\ &= (0.1) + (0.1)(1 - 0.1) = 0.19. \end{aligned}$$

Hence, $y(0.1) = 0.1$ and $y(0.2) = 0.19$.
The exact solution of the given equation:

$$\frac{dy}{dx} = 1 - y \text{ is given by}$$

$$\frac{dy}{1-y} = dx \Longrightarrow ln(1-y) = x + c.$$

Put $x = 0$ and $y = 0$, we get $c = 0$.
$\therefore \quad 1 - y = e^x \Longrightarrow y = 1 - e^x$.
$\therefore \quad y(0.1) = 1 - e^{0.1} = 0.1052 \text{ and } y(0.2) = 1 - e^{0.2} = 0.2214$.

Example 1.3. Solve by Euler's method, the differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ with the initial condition $y(0) = 1$ and find y when $x = 0.1$.

Solution: Homework.

1.3 Modified Euler Method

Let $y_n^{'} = f(x_n, y_n)$. Then, an approximation for y at the end of the increment is $\tilde{y}_{n+1} = y_n + y_n^{'}h$ and an estimate for the slope at the end of the increment is $\tilde{y}_{n+1}^{'} = f(x_{n+1}, \tilde{y}_{n+1})$.
We can now set:

$$y_{n+1} = y_n + \frac{1}{2}\left(y_n^{'} + \tilde{y}_{n+1}^{'}\right)h.$$

The error can be found from:

$$y_{n+1} = y_n + y_n^{'}h + \frac{1}{2}y_n^{''}h^2 + O\left(h^3\right),$$

and since

$$y_{n+1} = y_n + y_n^{'}h + \frac{1}{2}\left[\frac{y_{n+1}^{'} - y_n^{'}}{h} + O\left(h\right)\right]h^2 + O\left(h^3\right),$$

or

$$y_{n+1} = y_n + \left(\frac{y_{n+1}^{'} + y_n^{'}}{2}\right)h + O\left(h^3\right).$$

Hence, the local error is $O\left(h^3\right)$ and the global error is $O\left(h^2\right)$. Another way to write our results is:

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + h, y_n + k_1), \\ y_{n+1} &= y_n + \frac{1}{2}\left(k_1 + k_2\right). \end{aligned} \tag{1.4}$$

Example 1.4. Using modified Euler's method, solve $\frac{dy}{dx} = 1 + xy$, with $y(0) = 2$. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$.

Solution: In the given differential equation $f(x, y) = 1 + xy$.
We have, $x_0 = 0$, $y_0 = 2$, and $h = 0.1$.
From (1.4), put $n = 0$, we have

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = (0.1)f(0, 2) = 0.1[1 + (0)(2)] = 0.1 \\
k_2 &= hf(x_0 + h, y_0 + k_1) = (0.1)f(0.1, 2.1) \\
&= (0.1)[1 + (0.1)(2.1)] = 0.121 \\
y_1 &= y(0.1) = y_0 + \frac{1}{2}(k_1 + k_2) \\
&= 2 + \frac{1}{2}(0.1 + 0.121) = 2.1105.
\end{aligned}$$

From (1.4), put $n = 1$, we have

$$\begin{aligned}
k_1 &= hf(x_1, y_1) = (0.1)f(0.1, 2.1105) = 0.1[1 + (0.1)(2.1105)] = 0.1211 \\
k_2 &= hf(x_1 + h, y_1 + k_1) = (0.1)f(0.2, 2.2316) \\
&= (0.1)[1 + (0.2)(2.2316)] = 0.1446 \\
y_2 &= y(0.2) = y_1 + \frac{1}{2} + (k_1 + k_2) \\
&= 2.1105 + \frac{1}{2}(0.1211 + 0.1446) = 2.2434.
\end{aligned}$$

From (1.4), put $n = 2$, we have

$$\begin{aligned}
k_1 &= hf(x_2, y_2) = (0.1)f(0.2, 2.2434) = 0.1[1 + (0.2)(2.2434)] = 0.1449 \\
k_2 &= hf(x_2 + h, y_2 + k_1) = (0.1)f(0.3, 2.3883) \\
&= (0.1)[1 + (0.3)(2.3883)] = 0.1716 \\
y_3 &= y(0.3) = y_2 + \frac{1}{2} + (k_1 + k_2) \\
&= 2.2434 + \frac{1}{2}(0.1449 + 0.1716) = 2.4017.
\end{aligned}$$

Example 1.5. Consider the initial value problem $y' = t + y$, $y(0) = 1$. Find the approximate value of y(0.3) using the modified Euler Method with a stepsize of $h = 0.1$.

Solution: For this problem $f(t, y) = t + y$ and the initial conditions are $t_0 = 0$ and $y_0 = 1$.

In the first step, we have

$$\begin{aligned}
k_1 &= hf(t_0, y_0) = (0.1)f(0,1) = 0.1,\\
k_2 &= hf(t_0 + h, y_0 + k_1) = (0.1)f(0.1, 1 + 0.1) = 0.12,\\
\text{and } \ t_1 &= 0.1,\\
y(0.1) \approx y_1 &= y_0 + \frac{1}{2}(k_1 + k_2)\\
&= 1 + (0.5)(0.1 + 0.12) = 1.11.
\end{aligned}$$

In the second step, we have

$$\begin{aligned}
k_1 &= hf(t_1, y_1) = (0.1)f(0.1, 1.11) = 0.121,\\
k_2 &= hf(t_1 + h, y_1 + k_1) = (0.1)f(0.2, 1.231) = 0.1431,\\
\text{and } \ t_2 &= 0.2,\\
y(0.2) \approx y_2 &= y_1 + \frac{1}{2}(k_1 + k_2)\\
&= 1.11 + (0.5)(0.121 + 0.1431) = 1.2421.
\end{aligned}$$

In the third step, we have

$$\begin{aligned}
k_1 &= hf(t_2, y_2) = (0.1)f(0.2, 1.2421) = 0.1442,\\
k_2 &= hf(t_2 + h, y_2 + k_1) = (0.1)f(0.3, 1.2421 + 0.1442) = 0.1686,\\
\text{and } \ t_3 &= 0.3,\\
y(0.3) \approx y_3 &= y_2 + \frac{1}{2}(k_1 + k_2)\\
&= 1.2421 + (0.5)(0.1442 + 0.1686) = 1.3985.
\end{aligned}$$

Example 1.6. Solve the first order initial value proble $y' = ty + 1$, with $y_0 = y(0) = 1, \quad 0 \leq t \leq 1, \quad h = 0.25$, by using modified Euler's method.

Solution: Homework

1.4 Taylor's Series Method

Consider the 1^{st} order ordinary differential equation (ODE) (1.1). Then, we could use a Taylor series about $x = x_0$ and obtain the complete solution, or,

$$\begin{aligned}
y(x) &= y(x_0) + y'(x_0)(x - x_0) + \frac{1}{2!}y''(x_0)(x - x_0)^2 + \frac{1}{3!}y'''(x_0)(x - x_0)^3\\
&+ \ldots + \frac{1}{N!}y^{(N)}(x_0)(x - x_0)^N + \ldots. \qquad (1.5)
\end{aligned}$$

Since $y(x_0) = y_0$ and $y'(x_0) = f(x_0, y_0) = y'_0$, then we can find the first two terms. For the second derivative:

$$\begin{aligned}
y''(x) = f'(x, y) = \frac{d^2y}{dx^2} = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}\right] &= f_x(x, y) + f_y(x, y)y'\\
&= f_x(x, y) + f_y(x, y)f(x, y).
\end{aligned}$$

Hence

$$\begin{aligned} y''(x_0) &= f_x(x_0, y_0) + f_y(x_0, y_0)y'_0 \\ &= f_x(x_0, y_0) + f_y(x_0, y_0)f(x_0, y_0). \end{aligned}$$

Similarly, the third derivative is:

$$\begin{aligned} y'''(x) &= f''(x, y) = \frac{d^3y}{dx^3} \\ &= f_{xx}(x,y) + f_{xy}(x,y)y' + f_{yx}(x,y)y' + f_{yy}(x,y)(y')^2 + f_x(x,y)f_y(x,y) + f_{yy}^2(x,y)y' \\ &= f_{xx}(x,y) + f_{xy}(x,y)f(x,y) + f_{yx}(x,y)f(x,y) + f_{yy}(x,y)f^2(x,y) + f_x(x,y)f_y(x,y) \\ &+ f_{yy}^2(x,y)f(x,y) \\ &= f_{xx}(x,y) + 2f_{xy}(x,y)f(x,y) + f_{yy}(x,y)f^2(x,y) + f_x(x,y)f_y(x,y) + f_{yy}^2(x,y)f(x,y). \end{aligned}$$

Hence

$$\begin{aligned} y'''(x_0) = {} & f_{xx}(x_0,y_0) + 2f_{xy}(x_0,y_0)f(x_0,y_0) + f_{yy}(x_0,y_0)f^2(x_0,y_0) + f_x(x_0,y_0)f_y(x_0,y_0) \\ & + f_{yy}^2(x_0,y_0)f(x_0,y_0). \end{aligned}$$

If we truncate at the third derivative:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \ldots + \text{Error},$$

and

$$\text{Error} = \frac{1}{4!}y^{(4)}(\zeta), \text{ where } x_n \le \xi \le x_{n+1}.$$

Example 1.7. Use Taylor's method to solve the equation $\frac{dy}{dx} = 3x + y^2$ to approximate y when $x = 0.1$, given that $y = 1$ when $x = 0$.

Solution: Here, $(x_0, y_0) = (0, 1)$ and $y' = \frac{dy}{dx} = 3x + y^2$.
Form equation (1.5), we have:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(4)} + \ldots,$$

where

$$\begin{aligned} y' &= \frac{dy}{dx} = 3x + y^2 \Rightarrow y'(0) = 1, \\ y'' &= 3 + 2y\,y' \Rightarrow y''(0) = 5, \\ y''' &= 2(y')^2 + 2y\,y'' \Rightarrow y'''(0) = 12, \end{aligned}$$

and

$$y^{(4)} = 6y'\,y'' + 2yy''' \Rightarrow y^{(4)}(0) = 54.$$

Hence, the required Taylor series in (1.5) becomes:

$$\begin{aligned} y(x) &= 1 + x + \frac{5}{2!}x^2 + \frac{12}{3!}x^3 + \frac{54}{4!}x^4 + \dots \\ &= 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4 + \dots . \end{aligned}$$

When $x = 0.1$, we have:

$$(y0.1) = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4 + ... \simeq 1.12722.$$

Example 1.8. Use the fifth order Taylor series method with a single integration step to determine $y(0.2)$. Given that

$$y' = \frac{dy}{dx} = x^2 - 4y\,, \quad y(0) = 1.$$

The analytical solution of the differential equation is:

$$y = \frac{31}{32}e^{-4x} + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}.$$

Compute also the estimated error and compare it with the actual error.

Solution: The Taylor series solution up to and including the term with h^4 is given by:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(4)}. \tag{1.6}$$

The given differential equation is:

$$y' = x^2 - 4y\,, \quad y(0) = 1.$$

Hence,

$$\begin{aligned} y' &= x^2 - 4y \Rightarrow y'(0) = -4, \\ y'' &= 2x - 4y' \Rightarrow y''(0) = 16, \\ y''' &= 16\,y' - 8x + 2 \Rightarrow y'''(0) = -62, \\ &\text{and} \\ y^{(4)} &= -64\,y' + 32x - 8 \Rightarrow y^{(4)}(0) = 348. \end{aligned}$$

For $h = 0.2$, from equation (1.6), we get:

$$y(0.2) = 1 + 0.2(-4) + \frac{1}{2!}(16)(0.2)^2 + \frac{1}{3!}(-62)(0.2)^3 + \frac{1}{4!}(248)(0.2)^4 = 0.4539.$$

Example 1.9. Use the fifth order Taylor's series, find the solution of the differential equation $xy' = x - y$, $y(2) = 2$ at $x = 2.1$.

1.5 Runge-Kutta Method

Runge-Kutta methods are a family of single-step, explicit, numerical techniques used for solving a first-order ordinary differential equation. Various types of Runge-Kutta methods are classified according to their order. For instance, second-order Runge-Kutta methods use the slope at two points; third-order methods use three-points, and so on.

1.5.1 Runge-Kutta Method Of Order Two

In the Runge-Kutta method of order two, we consider up to the second derivative term in the Taylor series expansion and then substitute the derivative terms with the appropriate function values in the interval. Consider the Euler's method is given by:

$$y_{i+1} = y_i + f\left(x_i, y_i\right) h, \tag{1.7}$$

where

$$x_0 = 0, \;\; y_0 = y(x_0) \;\text{ and }\; h = x_{i+1} - x_i.$$

To understand the Runge-Kutta 2^{nd} order method, we need to derive Euler's method from the Taylor series.

$$\begin{aligned} y_{i+1} &= y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left.\frac{d^2y}{dx^2}\right|_{x_i,y_i} (x_{i+1} - x_i)^2 \\ &+ \frac{1}{3!} \left.\frac{d^3y}{dx^3}\right|_{x_i,y_i} (x_{i+1} - x_i)^3 + \ldots \\ &= y_i + f(x_i, y_i)\,(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)\,(x_{i+1} - x_i)^2 \\ &+ \frac{1}{3!} f''(x_i, y_i)\,(x_{i+1} - x_i)^3 + \ldots. \end{aligned}$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f\left(x_i, y_i\right) h,$$

are Euler's method and hence can be considered to be the Runge-Kutta 1^{st} order method.
The true error in the approximation is given by:

$$E_t = \frac{f'\left(x_i, y_i\right)}{2!} h^2 + \frac{f''\left(x_i, y_i\right)}{3!} h^3 + \ldots. \tag{1.8}$$

So what would a 2^{nd} order method formula look like. It would include one more term of the Taylor series as follows:

$$y_{i+1} = y_i + f\left(x_i, y_i\right) h + \frac{1}{2!} f'\left(x_i, y_i\right) h^2. \tag{1.9}$$

Let us take a generic example of a first order ordinary differential equation

$$\frac{dy}{dx} = e^{-2x} - 3y, \;\; y(0) = 5.$$

Since $f(x,y) = e^{-2x} - 3y$ and y is a function of x,

$$\begin{aligned} f'(x,y) &= \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial}{\partial x}\left(e^{-2x} - 3y\right) + \frac{\partial}{\partial y}\left[\left(e^{-2x} - 3y\right)\right]\left(e^{-2x} - 3y\right) \\ &= -2e^{-2x} + (-3)\left(e^{-2x} - 3y\right) = -5e^{-2x} + 9y. \end{aligned}$$

The 2^{nd} order formula for the above example would be:

$$\begin{aligned} y_{i+1} &= y_i + f(x_i, y_i)\,h + \frac{1}{2!}f'(x_i, y_i)\,h^2 \\ &= y_i + \left(e^{-2x_i} - 3y_i\right)h + \frac{1}{2!}\left(-5e^{-2x_i} + 9y_i\right)h^2. \end{aligned}$$

However, we already see the difficulty of having to find $f'(x,y)$ in the above method. What Runge and Kutta did was write the 2^{nd} order method as

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)\,h, \tag{1.10}$$

where

$$k_1 = f(x_i, y_i) \;\text{ and }\; k_2 = f(x_i + p_1h, y_i + q_{11}k_1h).$$

This form allows one to take advantage of the 2^{nd} order method without having to calculate $f'(x,y)$.

So how do we find the unknowns a_1, a_2, p_1 and q_{11}. Equating Equation (1.9) and (1.10), gives three equations:

$$a_1 + a_2 = 1, \quad a_2p_1 = \frac{1}{2} \;\text{ and }\; a_2q_{11} = \frac{1}{2}.$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally, the value of a_2 is chosen to evaluate the other three constants. The three values generally used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, are known as Heun's Method, the midpoint method and Ralston's method, respectively.

Heun's Method

Here, $a_2 = \frac{1}{2}$ is chosen, giving:

$$a_1 = \frac{1}{2}, \;\; p_1 = 1, \;\; q_{11} = 1$$

resulting in:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2),$$

where

$$k_1 = hf\left(x_n, y_n\right) \quad \text{and} \quad k_2 = hf\left(x_n + h, y_n + k_1\right).$$

This method is graphically explained in Figure 1.3.

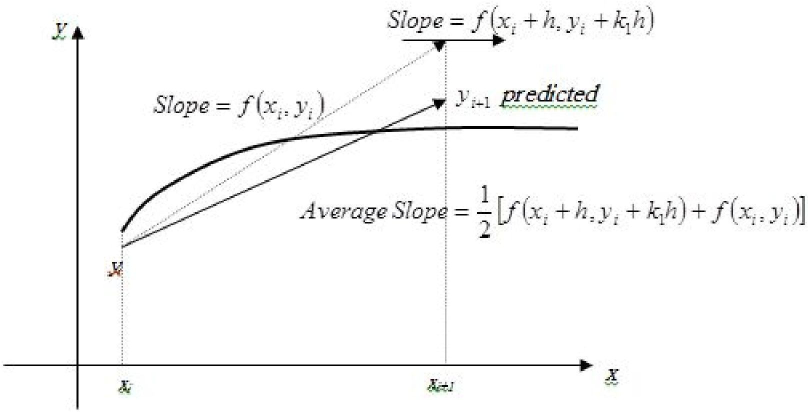

Figure 1.3: *Runge-Kutta 2^{nd} order method (Heun's method).*

Midpoint Method

Here, $a_2 = 1$ is chosen, giving:

$$a_1 = 0, \;\; p_1 = \frac{1}{2}, \quad q_{11} = \frac{1}{2},$$

resulting in:

$$y_{i+1} = y_i + k_2,$$

where,

$$k_1 = hf\left(x_i, y_i\right) \quad \text{and} \quad k_2 = hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right).$$

Ralston's Method

Here, $a_2 = \frac{2}{3}$ is chosen, giving:

$$a_1 = \frac{1}{3}, \quad p_1 = \frac{3}{4}, \;\; q_{11} = \frac{3}{4},$$

resulting in:

$$y_{n+1} = y_n + \frac{1}{3}\left(k_1 + 2k_2\right),$$

where,

$$k_1 = hf\left(x_n, y_n\right) \quad \text{and} \quad k_2 = hf\left(x_n + \frac{3}{4}h, y_n + \frac{3}{4}k_1\right).$$

Example 1.10. Use the second-order Runge-Kutta method with $h = 0.1$, to find y_1 and y_2 for $\frac{dy}{dx} = -x\,y^2$, $y(2) = 1$.

Solution: from $f(x, y) = -x\,y^2$, the second-order Range-Kutta method is

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$.

For n=0:

$x_0 = 2$ and $y_0 = 1$, hence,

$k_1 = hf(x_0, y_0) = 0.1f(2, 1) = 0.1(-2)(1)^2 = -0.2$

and

$k_2 = f(x_0 + h, y_0 + k_1) = 0.1f(2.1, 0.8) = 0.1(-2.1)(0.8)^2 = -0.1344.$

Hence,

$$\begin{aligned} y_1 = y(2.1) &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 1 + \frac{1}{2}(-0.2 - .1344) = 1 - 0.0328 = 0.8328. \end{aligned}$$

For n=1:

$x_1 = 2.1$ and $y_1 = 0.8328$, hence

$k_1 = hf(x_1, y_1) = 0.1f(2.1, 0.8328) = 0.1(-2.1)(0.8328)^2 = -0.1456,$

and $\quad k_2 = hf(x_1 + h, y_1 + k_1) = 0.1f(2.2, 0.6872) = 0.1(-2.2)(0.6872)^2 = -0.1039.$

Hence,

$$y_2 = y(2.2) = y_1 + \frac{1}{2}(k_1 + k_2) = 0.8328 + \frac{1}{2}(-0.1456 - 0.1039) = 0.70805.$$

Example 1.11. Use Runge-Kutta method of order two to evaluate $y(0,1)$ and $y(0,2)$ given that $y' = x + y$, where $y = 1$.

Solution: Given $f(x, y) = x + y$.

To find $y(0.1)$**:**

Here $x_0 = 0$, $y_0 = 1$ and let $h = 0.1$.

Now

$$\begin{aligned} k_1 &= hf(x_0, y_0) = (0.1)f(0, 1) = (0.1)[0 + 1] = 0.1, \\ k_2 &= hf\,(x_0 + h, y_0 + k_1) = (0.1)f(0.1, 1.1) \\ &= (0.1)[0.1 + 1.1] = 0.12. \end{aligned}$$

$$\begin{aligned} \therefore \qquad y_1 = y(0.1) &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 1 + \frac{1}{2}[0.1 + 0.12] = 1.11. \end{aligned}$$

To find $y(0.4)$**:**

Now $h = 0.1$, $x_1 = x_0 + h = 0 + 0.1 = 0.1$, and $y_1 = 1.11$.

$$\begin{aligned} k_1 &= hf(x_1, y_1) = (0.1)f(0.1, 1.11) = (0.1)[0.1 + 1.11] = 0.121, \\ k_2 &= hf\,(x_1 + h, y_1 + k_1) = (0.1)f(0.2, 1.231) \\ &= (0.1)[0.2 + 1.231] = 0.1431. \end{aligned}$$

$$\therefore \qquad y_2 = y(0.2) = y_1 + \frac{1}{2}(k_1 + k_2)$$
$$= 1.11 + \frac{1}{2}[0.121 + 0.1431] = 1.24205.$$

Example 1.12. Evaluate $y(1.1)$ and $y(1.2)$ using midpoint method for the initial value problem $y' = \frac{dy}{dx} = x^2 + y^2, \quad y(1) = 0.$

Solution: Given $f(x, y) = x^2 + y^2$.
To find $y(1.1)$:
Here $x_0 = 1$, $y_0 = 0$ and let $h = 0.1$.
Now

$$k_1 = hf(x_0, y_0) = (0.1)f(1, 0) = (0.1)[1^2 + 0^2] = 0.1,$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(1.05, 0.05)$$
$$= (0.1)[(1.05)^2 + (0.05)^2] = 0.1105.$$

Therefore

$$y_1 = y(1.1) = y_0 + k_2$$
$$= 0 + 0.1105 = 0.1105.$$

To find $y(1.2)$:
Now $h = 0.1$, $x_1 = x_0 + h = 1 + 0.1 = 1.1$, and $y_1 = 0.1105$.

$$k_1 = hf(x_1, y_1) = (0.1)f(1.1, 0.110$$
$$= (0.1)[(1.1)^2 + (0.1105)^2] = 0.12222,$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(1.15, 0.17161)$$
$$= (0.1)[(1.15)^2 + (0.17161)^2] = 0.135195.$$

Therefore

$$y_2 = y(1.2) = y_1 + k_2$$
$$= 0.1105 + 0.135195 = 0.245695.$$

Example 1.13. Given $\frac{dy}{dx} = 1 + y^2$ where $0 \leq x \leq 0.3$ with $y(0) = 0$ and $h = 0.1$ by using Ralston's method.

1.5.2 Runge-Kutta Method of Order Four

In the classical Runge-Kutta method of order four, the derivatives are evaluated at four points, once at each end and twice at the interval midpoints as given below:

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{1.11}$$

where

$$
\begin{aligned}
k_1 &= hf\left(x_i, y_i\right),\\
k_2 &= hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right),\\
k_3 &= hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right),\\
k_4 &= hf\left(x_i + h, y_i + k_3\right).
\end{aligned}
$$

The classical Runge-Kutta method of order four is illustrated schematically in Figure 1.4, and Figures 1.4(a) to (c) which show the determination of the slopes. Figure 1.4(a) shows the slope k_1 and how it is used to compute slope k_2. Figure 1.4(b) shows how slope k_2 is used to find the slope k_3. Figure 1.4(c) shows how the slope k_3 is used to find the slope k_4. Figure 1.4(d) shows the application of Equation (1.11) where the slope used for evaluating y_{i+1} is a weighted average of the slopes k_1, k_2, k_3 and k_4.

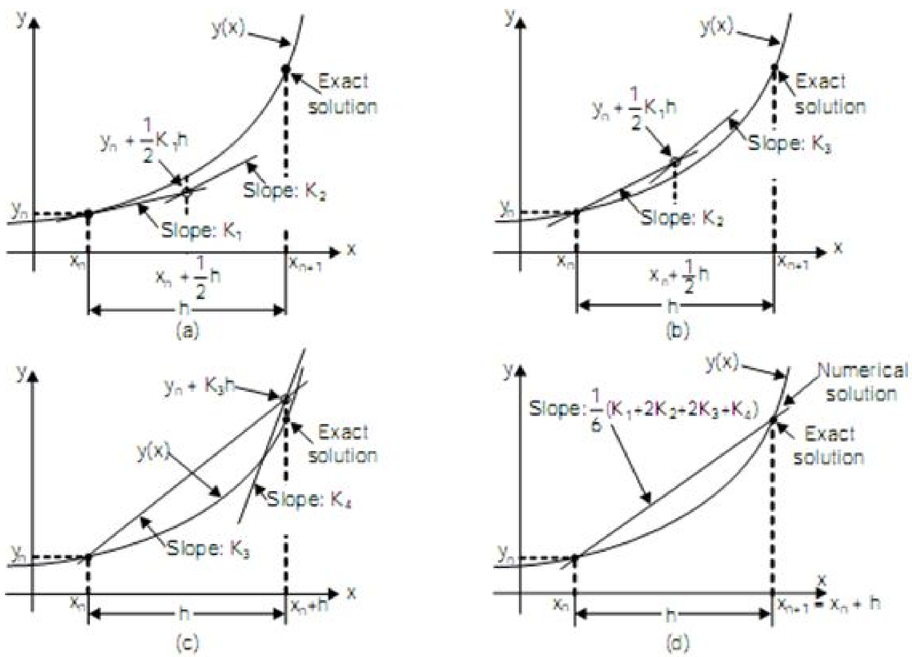

Figure 1.4: *classical fourth-order Runge Kutta method.*

The local truncation error in the classical Runge-Kutta method of order four is $O(h^5)$, and the global truncation error is $O(h^4)$. This method gives the most accurate solution compared to the other methods. Equation (1.11) is the most accurate formula available without extending outside the intervals $[x_i,\ x_{i+1}]$.

Example 1.14. Use the fourth-order Range-Kutta method with $h = 0.1$ to obtain an approximation of $y(1.5)$ for the solution of $\frac{dy}{dx} = 2x\ y$, $y(1) = 1$. The exact solution is given by $y = e^{x^2-1}$, and then determine the relative error and the percentage relative error.

Solution: For $n = 0$, from Equation (1.11), we have

$$\begin{aligned}
k_1 &= f(x_0, y_0) = x_0 y_0 = 2, \\
k_2 &= f\left(x_0 + \frac{1}{2}(0.1), y_0 + \frac{1}{2}(2)(0.1)\right) \\
&= 2\left[x_0 + \frac{1}{2}(0.1)\right]\left[y_0 + \frac{1}{2}(2)(0.1)\right] = 2.31, \\
k_3 &= f\left(x_0 + \frac{1}{2}0.1, y_0 + \frac{1}{2}(2.13)0.1\right) \\
&= 2\left[x_0 + \frac{1}{2}0.1\right]\left[y_0 + \frac{1}{2}(2.13)0.1\right] = 2.3426, \\
k_4 &= f(x_0 + 0.1, y_0 + (2.3426)0.1) = 2.7154.
\end{aligned}$$

Hence,

$$\begin{aligned}
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\, h \\
&= 1 + \frac{1}{6}(2 + 2(2.31) + 2(2.3426) + 2.7154)\, 0.1 = 1.2337.
\end{aligned}$$

Table (1.1) summaries the computations. In Table (1.1), exact value is computed from $y = e^{x^2-1}$. The absolute error = exact value - the value from the Runge-Kutta method.

$$\text{Percentage relative error} = \frac{|\text{error}|}{|\text{exact value}|}.$$

Table (1.1)

n	x_n	y_n	Exact Value	Absolute Error	Percentage Relative error
0	1	1	1	0	0
1	1.1	1.2337	1.2337	0	0
2	1.2	1.5527	1.5527	0	0
3	1.3	1.9937	1.9937	0	0
4	1.4	2.6116	2.6117	0.0001	0
5	1.5	3.4902	3.4904	0.0001	0

Example 1.15. Evaluate $y(1.1)$ and $y(1.2)$ using Runge-Kutta method of order four for the initial value problem $\frac{dy}{dx} = x^2 + y^2, \;\; y(0) = 1$.

Solution: Here $x_0 = 1$, $y_0 = 0$ and let $h = 0.1$.
To find $y(1.1)$:
We know that

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = (0.1)(1) = 0.1,\\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(1.05, 0.05)\\
&= (0.1)[(1.05)^2 + (0.05)^2] = 0.1105,\\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(1.05, 0.05525)\\
&= (0.1)[(1.05)^2 + (0.05525)^2] = 0.11055,\\
k_4 &= hf\left(x_0 + h, y_0 + k_3\right) = (0.1)f(1.05, 0.11055)\\
&= (0.1)[(1.05)^2 + (0.11055)^2] = 0.122222.
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad y(1.1) &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= 0 + \frac{1}{6}[0.1 + 2(0.1105) + 2(0.11055) + 0.122222]\\
&= 0.11072
\end{aligned}$$

To find $y(1.2)$

$$\begin{aligned}
k_1 &= hf(x_1, y_1) = (0.1)f(1.1, 0.11072) = 0.12226,\\
k_2 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(1.15, 0.17183)\\
&= (0.1)[(1.15)^2 + (0.17183)^2] = 0.135203,\\
k_3 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(1.15, 0.17832)\\
&= (0.1)[(1.15)^2 + (0.17832)^2] = 0.135430,\\
k_4 &= hf\left(x_1 + h, y_1 + k_3\right) = (0.1)f(1.2, 0.24615)\\
&= (0.1)[(1.2)^2 + (0.24615)^2] = 0.150059.
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad y(1.1) &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= 0.11072 + \frac{1}{6}[0.12226 + 2(0.135203) + 2(0.135430) + 0.150059]\\
&= 0.24631.
\end{aligned}$$

Example 1.16. Compute $y(0.1)$ by Runge-Kutta method of order four for the differential equation $\frac{dy}{dx} = xy + y^2, \;\; y(0) = 1$.

1.6 Solving Higher Order Ordinary Differential Equations

1.6.1 Euler's and Runge-Kutta Methods for Higher Order Ordinary Differential Equations

We have learned that Euler's and Runge-Kutta methods are used to solve first order ordinary differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \; y(0) = y_0.$$

What do we do to solve simultaneous (coupled) differential equations, or differential equations that are higher than first order? For example an n^{th} order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = f(x),$$

with $n-1$ initial conditions can be solved by assuming

$$\begin{aligned}
y &= z_1, \\
\frac{dy}{dx} &= \frac{dz_1}{dx} = z_2, \\
\frac{d^2y}{dx^2} &= \frac{dz_2}{dx} = z_3, \\
&\vdots \\
\frac{d^{n-1}y}{dx^{n-1}} &= \frac{dz_{n-1}}{dx} = z_n, \\
\frac{d^n y}{dx^n} &= \frac{dz_n}{dx} = \frac{1}{a_n}\left(-a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \ldots - a_1\frac{dy}{dx} - a_0 y + f(x)\right) \\
&= \frac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right).
\end{aligned}$$

The above Equations represent n first order differential equations as follows:

$$\begin{aligned}
\tfrac{dz_1}{dx} &= z_2 = f_1(z_1, z_2, \ldots, x), \\
\tfrac{dz_2}{dx} &= z_3 = f_2(z_1, z_2, \ldots, x), \\
&\vdots \\
\tfrac{dz_n}{dx} &= \tfrac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right).
\end{aligned}$$

Each of the n first order ordinary differential equations are accompanied by one initial condition. These first order ordinary differential equations are simultaneous in nature but can be solved by the methods used for solving first order ordinary differential equations that we have already learned.

Example 1.17. Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^{-x},\ y(0)=5,\ y'(0)=7.$$

Solution: The ordinary differential equation would be rewritten as follows: Assume

$$\frac{dy}{dx}=z,$$

then,

$$\frac{d^2y}{dx^2}=\frac{dz}{dx}.$$

Substituting this in the given second order ordinary differential equation gives:

$$\begin{aligned}3\frac{dz}{dx}+2z+5y&=e^{-x},\\ \frac{dz}{dx}&=\frac{1}{3}\left(e^{-x}-2z-5y\right).\end{aligned}$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is:

$$\begin{aligned}\frac{dy}{dx}&=z,\ y(0)=5,\\ \frac{dz}{dx}&=\frac{1}{3}\left(e^{-x}-2z-5y\right),\ z(0)=7.\end{aligned}$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 1.18. Given

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=e^{-t},\ y(0)=1,\ \frac{dy}{dt}(0)=2$$

, find by Euler's method

(a) $y(0.75)$,

(b) the absolute relative true error for part(a), if $y(0.75)|_{exact}=1.668$,

(c) $\frac{dy}{dt}(0.75)$,

Use a step size of $h=0.25$.

Solution: First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt}=z,$$

then,

$$\frac{dz}{dt} + 2z + y = e^{-t},$$
$$\frac{dz}{dt} = e^{-t} - 2z - y.$$

So the two simultaneous first order differential equations are:

$$\frac{dy}{dt} = z = f_1(t, y, z), \ y(0) = 1, \tag{1.12}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), \ \ z(0) = 2. \tag{1.13}$$

Using Euler's method on Equations (1.12) and (1.13), we get:

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)\, h, \tag{1.14}$$
$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)\, h. \tag{1.15}$$

(a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the values of $y(0.75)$. For $i = 0,\ t_0 = 0,\ y_0 = 1,\ z_0 = 2$, and from Equation (1.14) we have:

$$\begin{aligned} y_1 &= y_0 + f_1(t_0, y_0, z_0)\, h \\ &= 1 + f_1(0, 1, 2)(0.25) = 1 + 2(0.25) = 1.5, \end{aligned}$$

y_1 is the approximate value of y at $t = t_1 = t_0 + h = 0 + 0.25 = 0.25$,

$$y_1 = y(0.25) \approx 1.5.$$

From Equation (1.15) we have:

$$\begin{aligned} z_1 &= z_0 + f_2(t_0, y_0, z_0)\, h \\ &= 2 + f_2(0, 1, 2)(0.25) = 2 + \left(e^{-0} - 2(2) - 1\right)(0.25) = 1, \end{aligned}$$

$z_1$is the approximate value of z(same as $\frac{dy}{dt}$) at $t = 0.25$; $\ z_1 = z(0.25) \approx 1$.
For $i = 1,\ t_1 = 0.25,\ y_1 = 1.5,\ z_1 = 1$, from Equation (1.14) we have:

$$\begin{aligned} y_2 &= y_1 + f_1(t_1, y_1, z_1)\, h \\ &= 1.5 + f_1(0.25, 1.5, 1)(0.25) = 1.5 + (1)(0.25) = 1.75, \end{aligned}$$

y_2 is the approximate value of y at $t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$, $\ y_2 = y(0.5) \approx 1.75$, from Equation (1.15) we have:

$$\begin{aligned} z_2 &= z_1 + f_2(t_1, y_1, z_1)\, h \\ &= 1 + f_2(0.25, 1.5, 1)(0.25) \\ &= 1 + \left(e^{-0.25} - 2(1) - 1.5\right)(0.25) = 1 + (-2.7211)(0.25) = 0.3197, \end{aligned}$$

$z_2$is the approximate value of $z$at $t = t_2 = 0.5$; $z_2 = z\,(0.5) \approx 0.31970$.
For $i = 2,\ t_2 = 0.5,\ y_2 = 1.75,\ z_2 = 0.31970$, from Equation (1.14) we have:

$$\begin{aligned} y_3 &= y_2 + f_1\,(t_2, y_2, z_2)\,h \\ &= 1.75 + f_1\,(0.50, 1.75, 0.31970)\,(0.25) \\ &= 1.75 + (0.31970)\,(0.25) = 1.8299, \end{aligned}$$

y_3 is the approximate value of y at $t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$,

$$y_3 = y\,(0.75) \approx 1.8299.$$

From Equation (1.15) we have:

$$\begin{aligned} z_3 &= z_2 + f_2\,(t_2, y_2, z_2)\,h \\ &= 0.31972 + f_2\,(0.50, 1.75, 0.31970)\,(0.25) \\ &= 0.31972 + \left(e^{-0.50} - 2\,(0.31970) - 1.75\right)(0.25) \\ &= 0.31972 + (-1.7829)\,(0.25) = -.0126, \end{aligned}$$

z_3 is the approximate value of z at $t = t_3 = 0.75$,

$$z_3 = z\,(0.75) \approx -0.12601,$$
$$y\,(0.75) \approx y_3 = 1.8299.$$

b) The exact value of $y\,(0.75)$ is $y\,(0.75)|_{exact} = 1.668$.
The absolute relative true error in the result from part (a) is:

$$|\in_t| = \left|\frac{1.668 - 1.8299}{1.668}\right| \times 100 = 9.7062\%.$$

c) $\frac{dy}{dx}\,(0.75)= z_3 \approx -0.12601$.

1.7 EXERCISES

1. Use Taylor's method for two steps to compute y(0.2) and y(0.4) of $y' = 1 - 2\,x\,y,\ y(0) = 0$.

2. Find y at $x = 0.1$ and 0.2 of $y' + y + x\,y^2 = 0,\ y(0) = 1$, and find truncation error, using Runge-Kutta fourth order method.

3. Use Taylor's method to find the value of y at $x = 0.1$ and $x = 0.2$, and truncation error of $y' - 2y = 3e^x$where $y(0) = 0$.

4. Given $y' = x^3 + y\,,\ y(0) = 2$, compute $y(0.2)$ and $y(0.4)$ using the Runge Kutta method of fourth order.

5. Use Taylor's method to compute $y(1.1)$ and $y(1.2)$ of $y' = x\,y^{\frac{1}{3}},\; y(1) = 1$.

6. Use Rnage-Kutta fourth order to find the approximate solution $y(0.2)$ and $y(0.4)$ of $y' = \frac{y^2-x^2}{y^2+x^2},\;\; y(0) = 1$.

7. Use Taylor method to solve $\frac{dy}{dx} = 2\,y + 3e^x,\;\; y(0) = 0$ for $x = 0.1$, $x = 0.2$.

8. Use Euler and Modified Euler to find approximate solution of y at $x = 0.2$ for $y' = 2 + \sqrt{x\,y}$, $y(1) = 1$, only two steps.

9. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = x\,e^x + x$, about $x_0 = 0$, and then find a bound for the error on the interval $[0, 1]$.

10. Find the fourth order Taylor series method for the function $y' + 4y = x^2$, with $y(0) = 1$, and then determine $y(0.4)$.

11. Use Euler's method to approximate the solution of the following initial value problem.

$$y' = x\,e^{3x} - 2y,\; 0 \le x \le 1,\; y(0) = 0, \quad \text{with} \;\; h = 0.5.$$

12. Use Runge-Kutta second order to find the approximate solution $0 \le t \le 0.5,\; h = 0.1$ of $y' = t^2 - y + 1,\;\; y(0) = 1$.

1.6 Solving Higher Order Ordinary Differential Equations

1.6.1 Euler's and Runge-Kutta Methods for Higher Order Ordinary Differential Equations

We have learned that Euler's and Runge-Kutta methods are used to solve first order ordinary differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \; y(0) = y_0.$$

What do we do to solve simultaneous (coupled) differential equations, or differential equations that are higher than first order? For example an n^{th} order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = f(x),$$

with $n-1$ initial conditions can be solved by assuming

$$\begin{aligned}
y &= z_1, \\
\frac{dy}{dx} &= \frac{dz_1}{dx} = z_2, \\
\frac{d^2 y}{dx^2} &= \frac{dz_2}{dx} = z_3, \\
&\vdots \\
\frac{d^{n-1} y}{dx^{n-1}} &= \frac{dz_{n-1}}{dx} = z_n, \\
\frac{d^n y}{dx^n} &= \frac{dz_n}{dx} = \frac{1}{a_n}\left(-a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} \ldots - a_1 \frac{dy}{dx} - a_0 y + f(x)\right) \\
&= \frac{1}{a_n}\left(-a_{n-1} z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right).
\end{aligned}$$

The above Equations represent n first order differential equations as follows:

$$\begin{aligned}
\tfrac{dz_1}{dx} &= z_2 = f_1(z_1, z_2, \ldots, x), \\
\tfrac{dz_2}{dx} &= z_3 = f_2(z_1, z_2, \ldots, x), \\
&\vdots \\
\tfrac{dz_n}{dx} &= \tfrac{1}{a_n}\left(-a_{n-1} z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right).
\end{aligned}$$

Each of the n first order ordinary differential equations are accompanied by one initial condition. These first order ordinary differential equations are simultaneous in nature but can be solved by the methods used for solving first order ordinary differential equations that we have already learned.

Example 1.17. Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^{-x},\ y\left(0\right)=5,\ y'\left(0\right)=7.$$

Example 1.18. Given

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=e^{-t},\ y\left(0\right)=1,\ \frac{dy}{dt}\left(0\right)=2$$

, find by Euler's method

(a) $y\left(0.75\right)$,

(b) the absolute relative true error for part(a), if $y\left(0.75\right)|_{exact}=1.668$,

(c) $\frac{dy}{dt}\left(0.75\right)$,

Use a step size of $h=0.25$.

Boundary Value Problems

2.1-Finite-difference method

The finite-difference method for the solution of a two-point boundary value problem consists in replacing the derivatives occurring in the differential equation (and in the boundary conditions as well) by means of their finite-difference approximations and then solving the resulting linear system of equations by a standard procedure.

To obtain the appropriate finite-difference approximations to the derivatives, we proceed as follows.

Expanding $y(x+h)$ in Taylor's series, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{6} y'''(x) + \cdots \quad \textbf{(1)}$$

from which we obtain

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2} y''(x) - \cdots$$

Thus we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \quad \textbf{(2)}$$

which is the forward difference approximation for $y'(x)$. Similarly, expansion of $y(x-h)$ in Taylor's series gives

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2} y''(x) - \frac{h^3}{6} y'''(x) + \cdots \quad \textbf{(3)}$$

from which we obtain

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \quad \textbf{(4)}$$

which is the backward difference approximation for $y'(x)$.

It is clear that Eq. (1) is a better approximation to $y'(x)$ than either Eq. (2) or Eq. (4). Again, adding Eqs. (1) and (3), we get an approximation for $y''(x)$

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2). \qquad (5)$$

In a similar manner, it is possible to derive finite-difference approximations to higher derivatives.

To solve the boundary-value problem

we divide the range $[x_0, x_n]$ into n equal subintervals of width h so that

$$x_i = x_0 + ih, \qquad i = 1, 2, \ldots, n.$$

The corresponding values of y at these points are denoted by

$$y(x_i) = y_i = y(x_0 + ih), \qquad i = 0, 1, 2, \ldots, n.$$

From Eqs. (4) and (5), values of $y'(x)$ and $y''(x)$ at the point $x = x_i$ can now be written as

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$$

and

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2).$$

Some simple examples of two-point linear boundary-value problem is :

$$y''(x) + f(x)\,y'(x) + g(x)\,y(x) = r(x)$$

with the boundary conditions

$$y(x_0) = a \quad \text{and} \quad y(x_n) = b$$

Satisfying the differential equation at the point $x = x_i$, we get

$$y_i'' + f_i y_i' + g_i y_i = r_i$$

Substituting the expressions for y_i' and y_i'', this gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + f_i \frac{y_{i+1} - y_{i-1}}{2h} + g_i y_i = r_i, \qquad i = 1, 2, \ldots, n-1,$$

where $y_i = y(x_i)$, $g_i = g(x_i)$, etc.

***Example* 1:** A boundary-value problem is defined by

$$y'' + y + 1 = 0, \qquad 0 \le x \le 1$$

where

$$y(0) = 0 \quad \text{and} \quad y(1) = 0.$$

With $h = 0.5$, use the finite-difference method to determine the value of $y(0.5)$.

***Example* 2:** Solve the boundary-value problem

$$\frac{d^2y}{dx^2} - y = 0$$

with

$$y(0) = 0 \quad \text{and} \quad y(2) = 3.62686.$$

2.2-**Shooting method**

Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} = f(x, y, dy/dx)$$

with y(0)=A and y(1)=B, we use a shooting method. First, we formulate the ordinary differential equation as an initial value problem. We have

$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = f(x, y, z)$$

The initial condition y(0)=A is known, but the second initial condition z(0)=b is unknown. Our goal is to determine b such that y(1)=B.

We define the function F=F(b) such that

$$F(b) = y(1) - B$$

Example: Solve the boundary value problem

$$\frac{d^2y}{dx^2} - 2y = 8x(x-9)$$

with boundary conditions y(0)=0, y(9)=0 .

Example: Solve the boundary value problem $\frac{d^2y}{dx^2} + \frac{dy}{xdx} - \frac{y}{x^2} = 0$

With boundary condition $y(5) = 0.0038, y(8) = 0.003$

CS 450 – Numerical Analysis

Chapter 10: Boundary Value Problems for Ordinary Differential Equations †

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Boundary Value Problems

Boundary Value Problems

- Side conditions prescribing solution or derivative values at specified points are required to make solution of ODE unique
- For initial value problem, all side conditions are specified at single point, say t_0
- For *boundary value problem* (BVP), side conditions are specified at more than one point
- kth order ODE, or equivalent first-order system, requires k side conditions
- For ODEs, side conditions are typically specified at endpoints of interval $[a, b]$, so we have *two-point boundary value problem* with boundary conditions (BC) at a and b.

Boundary Value Problems, continued

- General *first-order two-point BVP* has form

  $$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \qquad a < t < b$$

  with BC

  $$\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \boldsymbol{0}$$

  where $\boldsymbol{f}: \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\boldsymbol{g}: \mathbb{R}^{2n} \to \mathbb{R}^n$

- Boundary conditions are *separated* if any given component of $\boldsymbol{g}$ involves solution values only at a or at b, but not both

- Boundary conditions are *linear* if they are of form

  $$\boldsymbol{B}_a\, \boldsymbol{y}(a) + \boldsymbol{B}_b\, \boldsymbol{y}(b) = \boldsymbol{c}$$

  where $\boldsymbol{B}_a, \boldsymbol{B}_b \in \mathbb{R}^{n \times n}$ and $\boldsymbol{c} \in \mathbb{R}^n$

- BVP is *linear* if ODE and BC are both linear

Example: Separated Linear Boundary Conditions

- Two-point BVP for second-order scalar ODE

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

is equivalent to first-order system of ODEs

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ f(t, y_1, y_2) \end{bmatrix}, \qquad a < t < b$$

with separated linear BC

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(b) \\ y_2(b) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Existence and Uniqueness

- Unlike IVP, with BVP we cannot begin at initial point and continue solution step by step to nearby points
- Instead, solution is determined everywhere simultaneously, so existence and/or uniqueness may not hold
- For example,

$$u'' = -u, \qquad 0 < t < b$$

with BC

$$u(0) = 0, \qquad u(b) = \beta$$

with b integer multiple of π, has infinitely many solutions if $\beta = 0$, but no solution if $\beta \neq 0$

Existence and Uniqueness, continued

- In general, solvability of BVP

  $$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \qquad a < t < b$$

  with BC

  $$\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \boldsymbol{0}$$

  depends on solvability of algebraic equation

  $$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}(b; \boldsymbol{x})) = \boldsymbol{0}$$

  where $\boldsymbol{y}(t; \boldsymbol{x})$ denotes solution to ODE with initial condition $\boldsymbol{y}(a) = \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$

- Solvability of latter system is difficult to establish if $\boldsymbol{g}$ is nonlinear

Existence and Uniqueness, continued

- For *linear* BVP, existence and uniqueness are more tractable
- Consider linear BVP

$$\boldsymbol{y}' = \boldsymbol{A}(t)\,\boldsymbol{y} + \boldsymbol{b}(t), \qquad a < t < b$$

  where $\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$ are continuous, with BC

$$\boldsymbol{B}_a\,\boldsymbol{y}(a) + \boldsymbol{B}_b\,\boldsymbol{y}(b) = \boldsymbol{c}$$

- Let $\boldsymbol{Y}(t)$ denote matrix whose ith column, $\boldsymbol{y}_i(t)$, called ith *mode*, is solution to $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$ with initial condition $\boldsymbol{y}(a) = \boldsymbol{e}_i$, ith column of identity matrix
- Then BVP has unique solution if, and only if, matrix

$$\boldsymbol{Q} \equiv \boldsymbol{B}_a\boldsymbol{Y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b)$$

  is nonsingular

Existence and Uniqueness, continued

- Assuming $\boldsymbol{Q}$ is nonsingular, define

$$\boldsymbol{\Phi}(t) = \boldsymbol{Y}(t)\,\boldsymbol{Q}^{-1}$$

and *Green's function*

$$\boldsymbol{G}(t,s) = \begin{cases} \boldsymbol{\Phi}(t)\boldsymbol{B}_a\boldsymbol{\Phi}(a)\boldsymbol{\Phi}^{-1}(s), & a \le s \le t \\ -\boldsymbol{\Phi}(t)\boldsymbol{B}_b\,\boldsymbol{\Phi}(b)\boldsymbol{\Phi}^{-1}(s), & t < s \le b \end{cases}$$

- Then solution to BVP given by

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\,\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\,\boldsymbol{b}(s)\,ds$$

- This result also gives absolute condition number for BVP

$$\kappa = \max\{\|\boldsymbol{\Phi}\|_\infty,\ \|\boldsymbol{G}\|_\infty\}$$

Conditioning and Stability

- Conditioning or stability of BVP depends on interplay between growth of solution modes and boundary conditions
- For IVP, instability is associated with modes that grow exponentially as time increases
- For BVP, solution is determined everywhere simultaneously, so there is no notion of "direction" of integration in interval $[a, b]$
- Growth of modes increasing with time is limited by boundary conditions at b, and "growth" (in reverse) of decaying modes is limited by boundary conditions at a
- For BVP to be well-conditioned, growing and decaying modes must be controlled appropriately by boundary conditions imposed

Numerical Methods for BVPs

Numerical Methods for BVPs

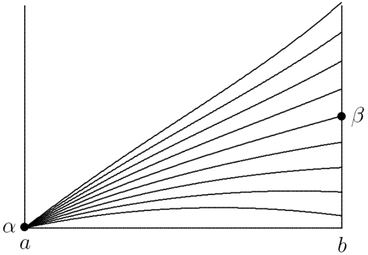
- For IVP, initial data supply all information necessary to begin numerical solution method at initial point and step forward from there

- For BVP, we have insufficient information to begin step-by-step numerical method, so numerical methods for solving BVPs are more complicated than those for solving IVPs

- We will consider four types of numerical methods for two-point BVPs
  - Shooting
  - Finite difference
  - Collocation
  - Galerkin

Shooting Method

- In statement of two-point BVP, we are given value of $u(a)$
- If we also knew value of $u'(a)$, then we would have IVP that we could solve by methods discussed previously
- Lacking that information, we try sequence of increasingly accurate guesses until we find value for $u'(a)$ such that when we solve resulting IVP, approximate solution value at $t = b$ matches desired boundary value, $u(b) = \beta$



Shooting Method, continued

- For given γ, value at b of solution $u(b)$ to IVP

$$u'' = f(t, u, u')$$

with initial conditions

$$u(a) = \alpha, \qquad u'(a) = \gamma$$

can be considered as function of γ, say $g(\gamma)$

- Then BVP becomes problem of solving equation $g(\gamma) = \beta$
- One-dimensional zero finder can be used to solve this scalar equation

Example: Shooting Method

- Consider two-point BVP for second-order ODE

$$u'' = 6t, \qquad 0 < t < 1$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- For each guess for $u'(0)$, we will integrate resulting IVP using classical fourth-order Runge-Kutta method to determine how close we come to hitting desired solution value at $t = 1$

- For simplicity of illustration, we will use step size $h = 0.5$ to integrate IVP from $t = 0$ to $t = 1$ in only two steps

- First, we transform second-order ODE into system of two first-order ODEs

$$\boldsymbol{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$$

Example, continued

- We first try guess for initial slope of $y_2(0) = 1$

$$\boldsymbol{y}^{(1)} = \boldsymbol{y}^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} 1.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix}\right) = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 2.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 2.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- So we have hit $y_1(1) = 2$ instead of desired value $y_1(1) = 1$

Example, continued

- We try again, this time with initial slope $y_2(0) = -1$

$$\boldsymbol{y}^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} -0.625 \\ 1.500 \end{bmatrix} + \begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 0.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 0.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- So we have hit $y_1(1) = 0$ instead of desired value $y_1(1) = 1$, but we now have initial slope bracketed between -1 and 1

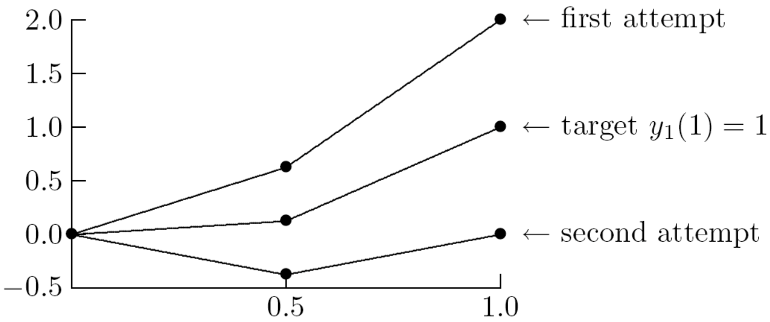
Example, continued

- We omit further iterations necessary to identify correct initial slope, which turns out to be $y_2(0) = 0$

$$
\begin{aligned}
\boldsymbol{y}^{(1)} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{0.5}{6} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0.0 \\ 1.5 \end{bmatrix} + 2 \begin{bmatrix} 0.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix} \\
\boldsymbol{y}^{(2)} &= \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix} + \frac{0.5}{6} \left(\begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix} + 2 \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} + 2 \begin{bmatrix} 1.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\end{aligned}
$$

- So we have indeed hit target solution value $y_1(1) = 1$

Example, continued



⟨ interactive example ⟩

Multiple Shooting

- Simple shooting method inherits stability (or instability) of associated IVP, which may be unstable even when BVP is stable
- Such ill-conditioning may make it difficult to achieve convergence of iterative method for solving nonlinear equation
- Potential remedy is *multiple shooting*, in which interval $[a, b]$ is divided into subintervals, and shooting is carried out on each
- Requiring continuity at internal mesh points provides BC for individual subproblems
- Multiple shooting results in larger *system* of nonlinear equations to solve

Finite Difference Method

Finite Difference Method

- *Finite difference method* converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximations

- For example, to solve two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

we introduce mesh points $t_i = a + ih$, $i = 0, 1, \ldots, n+1$, where $h = (b-a)/(n+1)$

- We already have $y_0 = u(a) = \alpha$ and $y_{n+1} = u(b) = \beta$ from BC, and we seek approximate solution value $y_i \approx u(t_i)$ at each interior mesh point t_i, $i = 1, \ldots, n$

Finite Difference Method, continued

- We replace derivatives by finite difference approximations such as

$$u'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

- This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

  to be solved for unknowns y_i, $i = 1, \ldots, n$

- System of equations may be linear or nonlinear, depending on whether f is linear or nonlinear

Finite Difference Method, continued

- For these particular finite difference formulas, system to be solved is tridiagonal, which saves on both work and storage compared to general system of equations
- This is generally true of finite difference methods: they yield sparse systems because each equation involves few variables

Example: Finite Difference Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- To keep computation to minimum, we compute approximate solution at one interior mesh point, $t = 0.5$, in interval $[0, 1]$
- Including boundary points, we have three mesh points, $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$
- From BC, we know that $y_0 = u(t_0) = 0$ and $y_2 = u(t_2) = 1$, and we seek approximate solution $y_1 \approx u(t_1)$

Example, continued

- Replacing derivatives by standard finite difference approximations at t_1 gives equation

$$\frac{y_2 - 2y_1 + y_0}{h^2} = f\left(t_1, y_1, \frac{y_2 - y_0}{2h}\right)$$

- Substituting boundary data, mesh size, and right-hand side for this example we obtain

$$\frac{1 - 2y_1 + 0}{(0.5)^2} = 6t_1$$

  or

$$4 - 8y_1 = 6(0.5) = 3$$

  so that

$$y(0.5) \approx y_1 = 1/8 = 0.125$$

Example, continued

- In a practical problem, much smaller step size and many more mesh points would be required to achieve acceptable accuracy
- We would therefore obtain *system* of equations to solve for approximate solution values at mesh points, rather than single equation as in this example

⟨ interactive example ⟩

Collocation Method

Collocation Method

- *Collocation method* approximates solution to BVP by finite linear combination of basis functions
- For two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

we seek approximate solution of form

$$u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$$

where ϕ_i are basis functions defined on $[a, b]$ and $\boldsymbol{x}$ is n-vector of parameters to be determined

Collocation Method

- Popular choices of basis functions include polynomials, B-splines, and trigonometric functions
- Basis functions with global support, such as polynomials or trigonometric functions, yield *spectral method*
- Basis functions with highly localized support, such as B-splines, yield *finite element method*

Collocation Method, continued

- To determine vector of parameters $\boldsymbol{x}$, define set of n *collocation points*, $a = t_1 < \cdots < t_n = b$, at which approximate solution $v(t, \boldsymbol{x})$ is forced to satisfy ODE and boundary conditions
- Common choices of collocation points include equally-spaced points or Chebyshev points
- Suitably smooth basis functions can be differentiated analytically, so that approximate solution and its derivatives can be substituted into ODE and BC to obtain system of algebraic equations for unknown parameters $\boldsymbol{x}$

Example: Collocation Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- To keep computation to minimum, we use one interior collocation point, $t = 0.5$
- Including boundary points, we have three collocation points, $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$, so we will be able to determine three parameters
- As basis functions we use first three monomials, so approximate solution has form

$$v(t, \boldsymbol{x}) = x_1 + x_2 t + x_3 t^2$$

Example, continued

- Derivatives of approximate solution function with respect to t are given by

$$v'(t, \boldsymbol{x}) = x_2 + 2x_3 t, \qquad v''(t, \boldsymbol{x}) = 2x_3$$

- Requiring ODE to be satisfied at interior collocation point $t_2 = 0.5$ gives equation

$$v''(t_2, \boldsymbol{x}) = f(t_2, v(t_2, \boldsymbol{x}), v'(t_2, \boldsymbol{x}))$$

or

$$2x_3 = 6t_2 = 6(0.5) = 3$$

- Boundary condition at $t_1 = 0$ gives equation

$$x_1 + x_2 t_1 + x_3 t_1^2 = x_1 = 0$$

- Boundary condition at $t_3 = 1$ gives equation

$$x_1 + x_2 t_3 + x_3 t_3^2 = x_1 + x_2 + x_3 = 1$$

Example, continued

- Solving this system of three equations in three unknowns gives

$$x_1 = 0, \qquad x_2 = -0.5, \qquad x_3 = 1.5$$

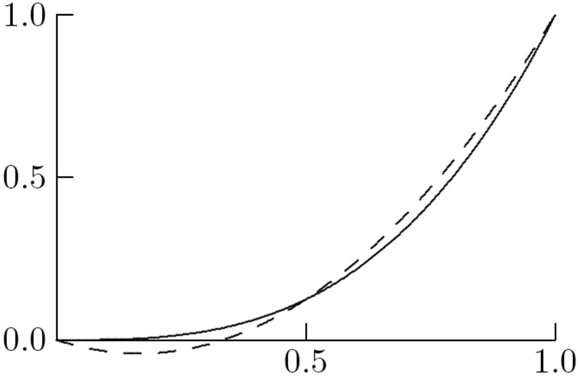
so approximate solution function is quadratic polynomial

$$u(t) \approx v(t, \boldsymbol{x}) = -0.5t + 1.5t^2$$

- At interior collocation point, $t_2 = 0.5$, we have approximate solution value

$$u(0.5) \approx v(0.5, \boldsymbol{x}) = 0.125$$

Example, continued



⟨ interactive example ⟩

Galerkin Method

Galerkin Method

- Rather than forcing residual to be zero at finite number of points, as in collocation, we could instead minimize residual over entire interval of integration

- For example, for *Poisson equation* in one dimension,

$$u'' = f(t), \qquad a < t < b,$$

with homogeneous BC $\quad u(a) = 0, \qquad u(b) = 0,$

subsitute approx solution $\quad u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$

into ODE and define residual

$$r(t, \boldsymbol{x}) = v''(t, \boldsymbol{x}) - f(t) = \sum_{i=1}^{n} x_i \phi_i''(t) - f(t)$$

Galerkin Method, continued

- Using *least squares method*, we can minimize

$$F(\boldsymbol{x}) = \tfrac{1}{2} \int_a^b r(t, \boldsymbol{x})^2 \, dt$$

  by setting each component of its gradient to zero

- This yields symmetric system of linear algebraic equations $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$a_{ij} = \int_a^b \phi_j''(t)\phi_i''(t) \, dt, \qquad b_i = \int_a^b f(t)\phi_i''(t) \, dt$$

  whose solution gives vector of parameters $\boldsymbol{x}$

Galerkin Method, continued

- More generally, *weighted residual method* forces residual to be orthogonal to each of set of *weight functions* or *test functions* w_i,

$$\int_a^b r(t, \boldsymbol{x}) w_i(t)\, dt = 0, \quad i = 1, \ldots, n$$

- This yields linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where now

$$a_{ij} = \int_a^b \phi_j''(t) w_i(t)\, dt, \qquad b_i = \int_a^b f(t) w_i(t)\, dt$$

whose solution gives vector of parameters $\boldsymbol{x}$

Galerkin Method, continued

- Matrix resulting from weighted residual method is generally not symmetric, and its entries involve second derivatives of basis functions
- Both drawbacks are overcome by *Galerkin method*, in which weight functions are chosen to be *same* as basis functions, i.e., $w_i = \phi_i$, $i = 1, \ldots, n$
- Orthogonality condition then becomes

$$\int_a^b r(t, \boldsymbol{x})\phi_i(t)\,dt = 0, \quad i = 1, \ldots, n$$

or

$$\int_a^b v''(t, \boldsymbol{x})\phi_i(t)\,dt = \int_a^b f(t)\phi_i(t)\,dt, \quad i = 1, \ldots, n$$

Galerkin Method, continued

- Degree of differentiability required can be reduced using integration by parts, which gives

$$\begin{aligned}\int_a^b v''(t,\boldsymbol{x})\phi_i(t)\,dt &= v'(t)\phi_i(t)|_a^b - \int_a^b v'(t)\phi_i'(t)\,dt \\ &= v'(b)\phi_i(b) - v'(a)\phi_i(a) - \int_a^b v'(t)\phi_i'(t)\,dt\end{aligned}$$

- Assuming basis functions ϕ_i satisfy homogeneous boundary conditions, so $\phi_i(0) = \phi_i(1) = 0$, orthogonality condition then becomes

$$-\int_a^b v'(t)\phi_i'(t)\,dt = \int_a^b f(t)\phi_i(t)\,dt, \quad i = 1, \ldots, n$$

Galerkin Method, continued

- This yields system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, with

$$a_{ij} = -\int_a^b \phi_j'(t)\phi_i'(t)\,dt, \qquad b_i = \int_a^b f(t)\phi_i(t)\,dt$$

  whose solution gives vector of parameters $\boldsymbol{x}$

- $\boldsymbol{A}$ is symmetric and involves only first derivatives of basis functions

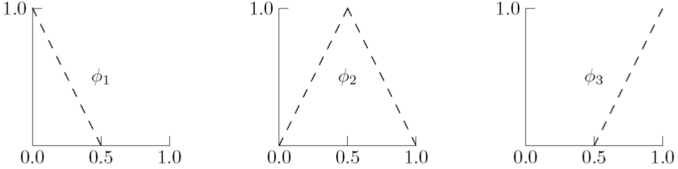
Example: Galerkin Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$

with BC $\quad u(0) = 0, \qquad u(1) = 1$

- We will approximate solution by piecewise linear polynomial, for which B-splines of degree 1 ("hat" functions) form suitable set of basis functions



- To keep computation to minimum, we again use same three mesh points, but now they become knots in piecewise linear polynomial approximation

Example, continued

- Thus, we seek approximate solution of form

$$u(t) \approx v(t, \boldsymbol{x}) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t)$$

- From BC, we must have $x_1 = 0$ and $x_3 = 1$

- To determine remaining parameter x_2, we impose Galerkin orthogonality condition on interior basis function ϕ_2 and obtain equation

$$-\sum_{j=1}^{3} \left(\int_0^1 \phi_j'(t)\phi_2'(t)\,dt \right) x_j = \int_0^1 6t\phi_2(t)\,dt$$

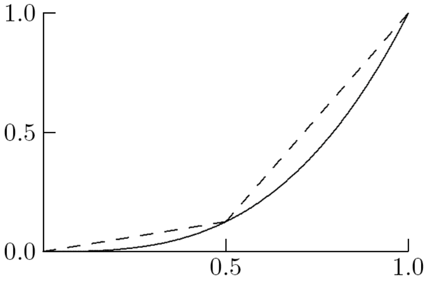
  or, upon evaluating these simple integrals analytically

$$2x_1 - 4x_2 + 2x_3 = 3/2$$

Example, continued

- Substituting known values for x_1 and x_3 then gives $x_2 = 1/8$ for remaining unknown parameter, so piecewise linear approximate solution is

$$u(t) \approx v(t, \boldsymbol{x}) = 0.125\phi_2(t) + \phi_3(t)$$



- We note that $v(0.5, \boldsymbol{x}) = 0.125$

Example, continued

- More realistic problem would have many more interior mesh points and basis functions, and correspondingly many parameters to be determined
- Resulting system of equations would be much larger but still sparse, and therefore relatively easy to solve, provided local basis functions, such as "hat" functions, are used
- Resulting approximate solution function is less smooth than true solution, but nevertheless becomes more accurate as more mesh points are used

⟨ interactive example ⟩

Eigenvalue Problems

Eigenvalue Problems

- Standard eigenvalue problem for second-order ODE has form

$$u'' = \lambda f(t, u, u'), \qquad a < t < b$$

  with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

  where we seek not only solution u but also parameter λ

- Scalar λ (possibly complex) is *eigenvalue* and solution u is corresponding *eigenfunction* for this two-point BVP

- Discretization of eigenvalue problem for ODE results in algebraic eigenvalue problem whose solution approximates that of original problem

Example: Eigenvalue Problem

- Consider linear two-point BVP

$$u'' = \lambda g(t) u, \qquad a < t < b$$

with BC

$$u(a) = 0, \qquad u(b) = 0$$

- Introduce discrete mesh points t_i in interval $[a, b]$, with mesh spacing h and use standard finite difference approximation for second derivative to obtain algebraic system

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda g_i y_i, \quad i = 1, \ldots, n$$

where $y_i = u(t_i)$ and $g_i = g(t_i)$, and from BC $y_0 = u(a) = 0$ and $y_{n+1} = u(b) = 0$

Example, continued

- Assuming $g_i \neq 0$, divide equation i by g_i for $i = 1, \ldots, n$, to obtain linear system

$$\boldsymbol{Ay} = \lambda \boldsymbol{y}$$

where $n \times n$ matrix $\boldsymbol{A}$ has tridiagonal form

$$\boldsymbol{A} = \frac{1}{h^2} \begin{bmatrix} -2/g_1 & 1/g_1 & 0 & \cdots & 0 \\ 1/g_2 & -2/g_2 & 1/g_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1/g_{n-1} & -2/g_{n-1} & 1/g_{n-1} \\ 0 & \cdots & 0 & 1/g_n & -2/g_n \end{bmatrix}$$

- This standard algebraic eigenvalue problem can be solved by methods discussed previously

Summary – ODE Boundary Value Problems

- Two-point BVP for ODE specifies BC at both endpoints of interval
- *Shooting* method replaces BVP by sequence of IVPs, with missing initial conditions determined by nonlinear equation solver
- *Finite difference* method replaces derivatives in ODE by finite differences defined on mesh of points, resulting in system of linear algebraic equations to solve for sample values of ODE solution
- *Collocation* method approximates ODE solution by linear combination of suitably smooth basis functions, with coefficients determined by requiring approximate solution to satisfy ODE at discrete set of collocation points
- *Galerkin* method approximates ODE solution by linear combination of basis functions, with coefficients determined by requiring residual to be orthogonal to each basis function