زانكۆی سەلاحەدین – هەولێر

Salahaddin University-Erbil

College of Basic Education

Mathematic Department

[Some direct methods for solving linear system]

Research project

Submitted to the department of (mathematices) in partial fulfillment of the requirements for the degree of Baccalaureus in Mathematics

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Supervisor Cetificate

This research project has been written under my supervision and has been submitted for the award of the degree of Baccalaureus in Mathematic.

Signature

Name: Mr Ghazi S. Ahmed

Date: / /

I confirm that all requirments have been completed.

Signature:

Name:

Head of the Department of ……

Date: / /

Dedication

In the name of Allah, Most Merciful, Most Compassionate.

We dedicated this humble work

To our supervisor Mr. Ghazi S. Ahmed

And to our beloved Department, Mathematic Department

In Salahaddin University, College of Basic Education

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Abstract

Solving systems of linear equations (or linear systems or, also, simultaneous equations) is a common situation in many scientific and technological problems. Many methods, either analytical or numerical, have been developed to solve them. A general method most used in Linear Algebra is the Gaussian Elimination, or variations of this. Sometimes they are referred to as “direct methods”. Basically, it is an algorithm that transforms the system into an equivalent one but with a triangular matrix, thus allowing a simpler resolution. In many cases, though, whenever the matrix of the system has a specific structure or is sparse and the like, other methods can be more effective.

Introduction

Our Research is about Some direct method for solving linear system, it consists of two chapters. first chapter We're talking about two ways for solving linear system, the first way is Gaussian Elimination Method and the second way is Doolittle factorization Method. The second chapter is analysing two ways with the program.

Chapter 1

System of linear equations and their solutions

Basic Definitions

A system of equations is a collection of M linear equations in N unknown quantities expressible in the form.

$$a_{11}\ x_1 + a_{12}x_2 + \ldots\ldots\ a_{1n}x_n = b_1$$

$$a_{21}\ x_1 + a_{22}x_2 + \ldots\ldots\ a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}\ x_1 + a_{m2}x_2 + \ldots\ldots\ a_{mn}x_n = b_n$$

where a_{ij}, b_i, and x_j, $1 \leq i \leq m$, $1 \leq j \leq n$ are real.

for example, $3X_1 + 2X_2 = 7$ is linear, but neither $3X_1 + 2\cos(X_2) = 7$ nor $3X_1 + 2X_2 = \frac{1}{X_1}$ is linear.

Solution of system of linear equations

A solution of the system of linear equations in unknown $x_1, x_2, \ldots\ldots, x_n$ is an ordered list of n numbers

$s_1, s_2 \ldots\ldots\ s_n$ such that when we put s_1 for x_1, s_2 for x_2 ……. and s_n for x_n then each equation of the system becomes true simultaneously.

For example, $x_1 = 1, x_2 = 2$ satisfy the system.

$$2x_1 + 3x_2 = 8$$

$$3x_1 - x_2 = 1$$

we write $x_1 = 1, x_2 = 2$ is a solution or (1,2) is a solution.

The set of all possible solutions of a system of equations is called its solution set**.** A solution set can be a finite or an empty set.

Elementary Row Operation

There are four elementary row operations for producing equivalent matrices:

1. **Row Swap**: interchange any two rows.
2. **Row+**: Row addition−add a row to any other row.
3. $*$ $\boldsymbol{Row}$**:** scalar multiplication− multiply (or divide) all the elements of a row by the same nonzero real number.
4. $*$ $\boldsymbol{Row+}$ **:** multiply all the entries of a row (pivot row) by a nonzero real number and add each resulting product to the corresponding entry of another specified row (target row)

Inverse Of a Square Matrix

Let A be an $n \times n$ Matrix and let I_n be the n×n identity matrix. If there exists a

Matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

Then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

1- Gaussian Elimination Method

The Gaussian elimination method is one of the most popular and widely used direct methods for solving linear systems of algebraic equations. No method of solving linear systems requires fewer operations than the Gaussian procedure. The goal of the Gaussian elimination method for solving linear systems is to convert the original system into the equivalent upper-triangular system from which each unknown is determined by backward substitution.

The Gaussian elimination procedure starts with forward elimination, in which the first equation in the linear system is used to eliminate the first variable from the rest of the (n - 1) equations. Then the new second equation is used to eliminate the second variable from the rest of the (n - 2) equations, and so on. If (n - 1) such elimination is performed, then the resulting system will be the triangular form.

Once this forward elimination is completed, we can determine whether the system is overdetermined or underdetermined or has a unique solution. If it has a unique solution, then backward substitution is used to solve the triangular system easily and one can find the unknown variables involved in the system.

Now we shall describe the method in detail for a system of n linear equations.

Consider the following system of n linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n.
\end{aligned}
\tag{3.13}
$$

a- Forward Elimination

Consider the first equation of the given system (3.13)

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1 \qquad (3.14)$$

As the first pivotal equation with first pivot element a_{11} .

Then the first equation times multiples $m_{i1}=(\frac{a_{i1}}{a_{11}})$ and $i = 2,3,\ldots,n$, is subtracted from the ith.

Equation to eliminate the first the variable x_1, producing an equivalent system.

$$\begin{aligned}
a_{11}x_1+\quad & a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1\\
& a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\ldots+a_{2n}^{(1)}x_n=b_2^{(1)}\\
& a_{32}^{(1)}x_2+a_{33}^{(1)}x_3+\ldots+a_{3n}^{(1)}x_n=b_3^{(1)}\\
& \vdots\\
& a_{n2}^{(1)}x_2+a_{n3}^{(1)}x_3+\ldots+a_{nn}^{(1)}x_n=b_n^{(1)}.
\end{aligned} \qquad (3.15)$$

Now consider the second equation of system (3.15), which is

$$a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\ldots+a_{2n}^{(1)}x_n=b_2^{(1)}. \qquad (3.16)$$

The second pivotal equation with the second pivot element, $a_{22}^{(1)}$. Then the second equation times multiples $m_{i2}=(a_{i2}^{(1)}/a_{22}^{(1)})$ and i=3,…,n, is subtracted from the ith equation to eliminate the second variable x_2, producing an equivalent system

$$\begin{aligned}
a_{11}x_1+\quad & a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1\\
& a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\ldots+a_{2n}^{(1)}x_n=b_2^{(1)}\\
& a_{33}^{(2)}x_3+\ldots+a_{3n}^{(2)}x_n=b_3^{(2)}\\
& \vdots\\
& a_{n3}^{(2)}x_3+\ldots+a_{nn}^{(2)}x_n=b_n^{(2)}
\end{aligned} \qquad (3.17)$$

Now consider a third equation of system(3.17), which is

$$a_{33}^{(2)}x_3+\dots\ +a_{3n}^{(2)}x_n = b_3^{(2)}$$

The third pivotal equation with the third pivot element $a_{33}^{(2)}$. Then the third equation times multiples $m_{i3}=(a_{i3}^{(2)}/a_{33}^{(2)})$ and $i=4,\dots,n$, is subtracted from the ith equation to eliminate the third variable x_3. Similarly, after $(n-1)\ th$ steps, we have the nth pivotal equation, which has only one unknown variable x_n; that is,

$$\begin{aligned} a_{11}\,x_1 \ + \ a_{12}x_2+a_{13}x_3+\dots+a_{1n}x_n &= b_1 \\ + \ a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\dots\ +a_{2n}^{(1)}x_n &= b_2^{(1)} \\ +a_{33}^{(2)}x_3+\dots\ +a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \vdots \qquad & \quad \vdots \\ a_{nn}^{(n-1)x_n} &= b_n^{(n-1)} \end{aligned} \qquad (3.19)$$

With the nth pivotal element $a_{nn}^{(n-1)}$. After getting the upper-triangular system, which is equivalent to the original system, forward elimination is completed.

Example

Solve the system of linear equations by gauss Elimination method.

$$x+y+2z=8$$
$$-x-2y+3z=1$$
$$3x-7y+4z=10$$

Solution

Augmented matrix is

$$[A:B]=\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

$R_2 \rightarrow R_1+R_2$, $R_3 \rightarrow -3R_1+R_3$

$$=\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

$R_3 \rightarrow -10R_2+R_3$

$$=\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

$R_2 \rightarrow -R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

$R_3 \rightarrow -R_3/52$ $= \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

The equivalent system of equations form is:

$$x + y + 2z = 8$$
$$y - 5z = -9$$
$$z = 2$$

b- Back substitution:

$z = 2$

$y = 5z - 9 \ = 5(2) - 9 \ = 10 - 9 \ = 1$

$x = \ -y - 2z + 8 \ = \ -1 - 2(2) + 8 \ = \ -1 - 4 + 8 = 3$

The solution is $x = 3,\ y = 1,\ z = 2.$

Hence the set of solutions is a unique solution.

2- Doolittle factorization Method

To solve $A_{n\times_n}x_{n\times 1}$=$B_{n\times 1}$…… (1) by Doolittle factorization method, multiplying A by a lower triangular matrix M_1:

$$M_1A = A_1 \; where \; M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & 0 & \cdots & 0 \\ -m_{41} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ -m_{n1} & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$; m_{i1} = \frac{a_{i1}}{a_{11}}; i = 2,3, \dots, n. \; where \; a_{i1} \in A \; for \; i = 1,2, \dots, n.$$

Multiplying A_1 by a lower triangular matrix M_2:

$$M_2A_1 = M_2M_1A = A_2 \; where \; M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -m_{32} & 1 & 0 & \cdots & 0 \\ 0 & -m_{42} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & -m_{n2} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$m_{i2} = \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}; i = 3,4, \dots, n$$

$$where \; a_{i2}^{(1)} \in A_1 \; for \; i = 2,3, \dots, n.$$

And so, on finally we get

$$M_{n-1}M_{n-2} \cdots M_2M_1A = A_{n-1} = U \; (upper \; traingular \; matrix)$$

$$Let \; M = M_{n-1}M_{n-2} \cdots M_2M_1$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -m_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\ -m_{31} & -m_{32} & 1 & 0 & \cdots & 0 & 0 \\ -m_{41} & -m_{42} & -m_{43} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ -m_{n1} & -m_{n2} & -m_{n3} & -m_{n4} & \cdots & -m_{n,n-1} & 1 \end{bmatrix} n \times n$$

Since $|M| = 1 \neq 0 \Rightarrow M^{-1} = M_1^{-1}M_2^{-1} \cdots M_{n-2}^{-1}M_{n-1}^{-1} \; exists \; where$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ m_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 & \cdots & 0 & 0 \\ m_{41} & m_{42} & m_{43} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ m_{n1} & m_{n2} & m_{n3} & m_{n4} & \cdots & m_{n,n-1} & 1 \end{bmatrix} n \times n$$

$$\therefore A = M^{-1}.A_{n-1}$$

Then we solve two triangular systems

M^{-1}Y=B for Y And A_{n-1}X=Y for X(solution) Where X=$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $and\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Example: Solve $Ax = B$ where $\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $and\ B = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$ by Doolittle factorization method.

Solution:

$A = A_0$

M_1=$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$ $M_{21} = \frac{a_{21}^{(0)}}{a_{11}^{(0)}} = \frac{-1}{1} = -1$

$$M_{31} = \frac{a_{31}^{(0)}}{a_{11}^{(0)}} = \frac{3}{1} = 3$$

$$A_1 = M_1.A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{pmatrix} A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & -10 & -2 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{pmatrix} M_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-10}{-1} = 10$$

$$A_2 = M_2.A_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & -10 & -2 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{pmatrix}$$

$$M_1^{-1} = M_1.I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$M_2^{-1} = M_2.\mathrm{I}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}$$

$$M^{-1} = M_1^{-1}.M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{pmatrix}$$

$$M^{-1}y = B \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

$$y_1 = 8$$

$$-8 + y_2 = 1$$
$$y_2 = 9$$

$$24 + 90 + y_3 = 10$$
$$y_3 = -104$$

$$y = \begin{pmatrix} 8 \\ 9 \\ -104 \end{pmatrix}$$

$$A_2\, X = y \quad \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ -104 \end{pmatrix}$$

$$-52x_3 = -104 \;\rightarrow\; x_3 = 2$$
$$-x_2 + 10 = 9 \rightarrow\; x_2 = 1$$
$$x_1 + 1 + 4 = 8 \rightarrow\; x_1 = 3$$
$$X = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Chapter 2

MATLAB Programming for methods for solving linear system equations.

1- Gaussian Elimination Method

```
format rat
disp('Gauss Elimination Method')
A=[1 1 2 8;-1 -2 3 1;3 -7 4 10]
[n,m]=size(A);
for k=1:n-1
    if A(k,k)==0;
        for j=k:n
            R=A(k,:); A(k,:)=A(j,:); A(j,:)=R;
        end
    end
    for i=k+1:n
        mm=A(i,k)/A(k,k);
        for j=1:m
            A(i,j)=A(i,j)-mm*A(k,j);
        end
    end
end
A
x=zeros(1,n);
for i=n:-1:1
    xx=0;
    for j=i:n
        xx=xx+A(i,j)*x(j);
    end
    x(i)=(1/A(i,i))*(A(i,m)-xx);
end
x
```

A =

1	1	2	8
-1	-2	3	1
3	-7	4	10

A =

1	1	2	8
0	-1	5	9
0	0	-52	-104

x =

3	**1**	**2**

2- Doolittle factorization Method

```
format rat
A=[1 1 2;-1 -2 3;3 -7 4]
B=[8 1 10]
[n,m]=size(A); I=ones(n);BB=eye(n);
for k=1:n-1
    M=eye(n);
    for i=k+1:n
        mm=-A(i,k)/A(k,k); M(i,k)=mm*I(i,k);
    end
    M; BB=BB*inv(M); A=M*A;
end
BB;
y=zeros(1,n);
for i=1:n
    yy=0;
    for j=1:n
        yy=yy+BB(i,j)*y(j);
    end
    y(i)=(1/BB(i,i))*(B(1,i)-yy);
end
y
B=y;x=zeros(1,n);
for i=n:-1:1
    xx=0;
    for j=1:n
        xx=xx+A(i,j)*x(j);
    end
    x(i)=(1/A(i,i))*(B(1,i)-xx);
end
 x
```

A =

1	1	2
-1	-2	3
3	-7	4

B =

8	1	10

y =

8	9	-104

x =

3	**1**	**2**

Conclusion

As a result, we concluded that any linear system that analyzes it in any different direct way is the same result.

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