CH.3

Electric Flux Density, Gauss's Law, and Divergence

Electric Flux Density:

Electric Flux:

Another important concept in electrostatics is electric flux. If a unit test charge is placed near a point charge, it experiences a force. The direction of this force can be represented by the lines, radially coming outward from a positive charge. These lines are called streamlines or flux lines. Thus the electric field due to a charge can be imagined to be present around it in terms of a quantity called electric flux. The flux lines give the pictorial representation of distribution of electric flux around a charge. In the following sections we will explain the concept of electric flux, electric flux density, Gauss's law, applications of Gauss's law and the divergence theorem.

Faraday Experiment:

About 1837, Michael Faraday he was experimenting in his now-famous work on induced electromotive force, in his experiment, Faraday had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material or simply dielectric) that would occupy the entire volume between the concentric spheres.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge (+*Q*).
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was (-*Q*) equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres.

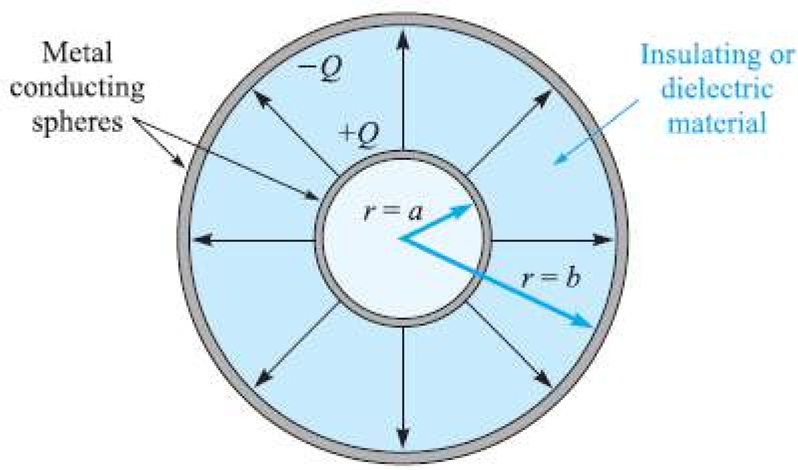


Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of **D** are not functions of the dielectric between the spheres.

He concluded that there was some sort of **"displacement"** from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as **displacement**, **displacement flux** or simply **electric flux** .

Faraday's experiments also showed, of course, that a larger **positive charge** on the inner sphere induced a correspondingly larger **negative charge** on the outer sphere, leading to a direct proportionality between the **electric flux** and the **charge** on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of

SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

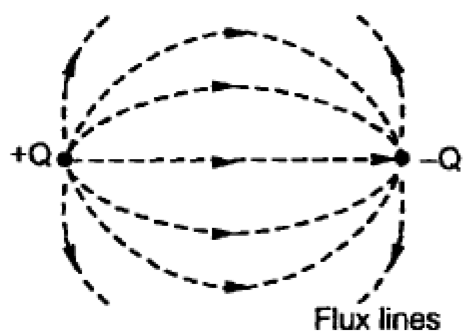
$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in **coulombs**. That by definition, one coulomb of electric charge gives rise to one coulomb of electric flux.

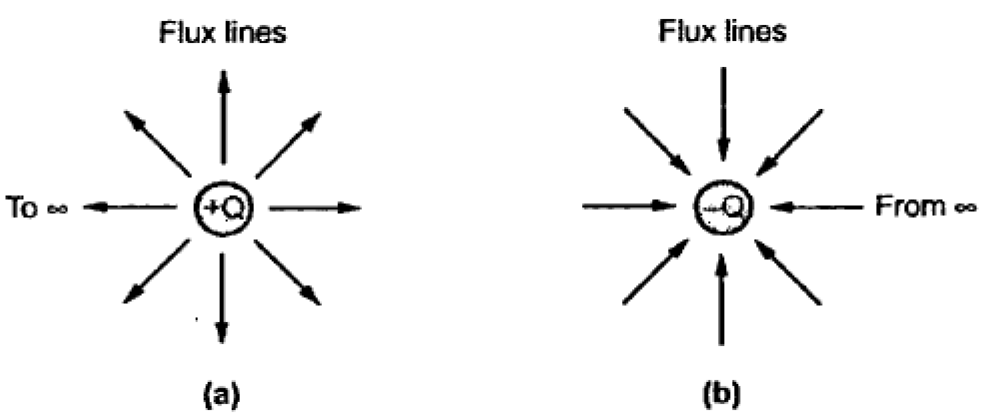
Key Point: *Thus the total number of lines of force in any particular electric field is called the electric flux. It is represented by the symbol Ψ . Similar to the charge, unit of electric flux is also coulomb C.*

Properties of electric flux lines

1- The flux lines start from positive charge and terminate on the negative charge



2- If the negative charge is absent, then the flux lines terminate at infinity. While in absence of positive charge, the electric flux terminates on the negative charge from infinity.



3. There are more number of lines i.e. crowding of lines if electric field is stronger.
4. These lines are parallel and never cross each other.
5. The lines are independent of the medium in which charges are placed.
6. The lines always enter or leave the charged surface, normally.
7. If the charge on a body is ± Q coulombs, then the total number of lines originating or terminating on it is also Q. But the total number of lines is nothing but a flux.

$$\therefore \quad \boxed{\text{Electric flux } \psi = Q \text{ coulombs (numerically)}}$$

This is according to SI units. Hence if Q is large, flux ψ is more surrounding the charge and vice versa.
The **flux** is a **scalar field**. Let us define now a **vector field** associated with the flux called **electric flux density**.

Electric Flux Density (D):

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, ψ coulombs of electric flux are produced by the charge Q (= Ψ) Coulombs distributed uniformly over a surface having an area of $(4\pi a^2)$ m^2.
The **density of the flux** at this surface is $\Psi/(4\pi a^2)$ or $Q/(4\pi a^2)$ C/m^2, and this is an important new quantity.

Electric flux density; measured in **coulombs per square meter** (sometimes described as " **lines per square meter,**" for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of ***displacement flux density*** or ***displacement density***. **Electric flux density** is more descriptive, however, and we will use the term consistently.

The electric flux density D is a **vector field** and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric field intensity **E**.

- The ***direction*** of **D** at a point is the direction of the **flux lines** at that point.
- The ***magnitude*** is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad (inner\ sphere)$$

$$\mathbf{D}|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad (outer\ sphere)$$

And at any radial distance r , where $a \leq r \leq b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Electric flux density from a point charge:

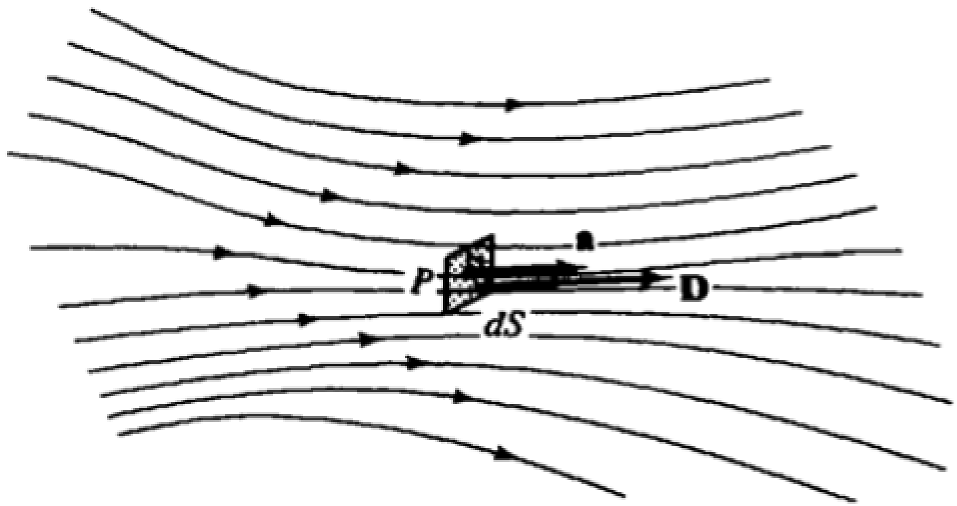
If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a **point charge** in the limit, but the **electric flux density** at a point r meters from the point charge is still given by

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r} \qquad (1)$$

for Q lines of flux are symmetrically directed outward from the point and pass through an *imaginary spherical surface* of area ($4\pi r^2$).

If in the neighborhood of point *P* the lines of flux have the direction of the unit vector **a** (see the figure below) and if an amount of flux $d\Psi$ crosses the differential area dS, which is a normal to **a**, then the *electric flux density* at *P* is

$$\mathbf{D} = \frac{d\Psi}{dS}\mathbf{a} \qquad (C/m^2)$$



Relationship between **D** and **E**:

In the previous, it has been derived that the electric field intensity **E** at a distance of *r*, from a point charge *Q* in free space is given by,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r$$

Dividing the equations of **D** and **E** due to a point charge *Q* we get,

$$\frac{\mathbf{D}}{\mathbf{E}} = \frac{\frac{Q}{4\pi r^2}\mathbf{a}_r}{\frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r} = \epsilon_o$$

Therefore, $$\mathbf{D} = \epsilon_o \mathbf{E}$$ (free space only) (2)

This equation (2) is applicable only to a vacuum.

Thus **D** and **E** are related through the permittivity. If the medium in which charge is located is other than free space having relative permittivity ϵ_r then,

$$\mathbf{D} = \epsilon_o \epsilon_r \mathbf{E}$$

i.e.

$$\mathbf{D} = \epsilon \mathbf{E}$$

Equation (2) is not restricted solely to the field of a point charge. For a general volume charge distribution in free space,

$$\mathbf{E} = \int_{vol} \frac{\rho_v dv}{4\pi\epsilon_o R^2} \mathbf{a}_R \qquad (free\ space)$$

where this relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\mathbf{D} = \int_{vol} \frac{\rho_v dv}{4\pi R^2} \mathbf{a}_R$$

and (2) is therefore true for any free-space charge configuration; we shall consider (2) as defining **D** in free space.

Example:

We wish to find **D** in the region about a uniform line charge of 8 nC/m lying along the z axis in free space.

Solution. The **E** field is

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi(8.854 \times 10^{-12})\rho} \mathbf{a}_\rho = \frac{143.8}{\rho} \mathbf{a}_\rho \text{ V/m}$$

At $\rho = 3$ m, $\mathbf{E} = 47.9\mathbf{a}_\rho$ V/m.

Associated with the **E** field, we find

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho} \mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi\rho} \mathbf{a}_\rho = \frac{1.273 \times 10^{-9}}{\rho} \mathbf{a}_\rho \text{ C/m}^2$$

The value at $\rho = 3$ m is $\mathbf{D} = 0.424\mathbf{a}_\rho$ nC/m2

The total flux leaving a 5-m length of the line charge is equal to the total charge on that length, or $\Psi = 40$ nC.

Gauss's Law:

The generalizations of Faraday's experiment lead to the following statement, which is known as ***Gauss's law***:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

Let we obtain a mathematical form for this statement:
Let us imagine a distribution of charge, shown as a cloud of point charges in Figure 3.2, surrounded by a closed surface of any shape.

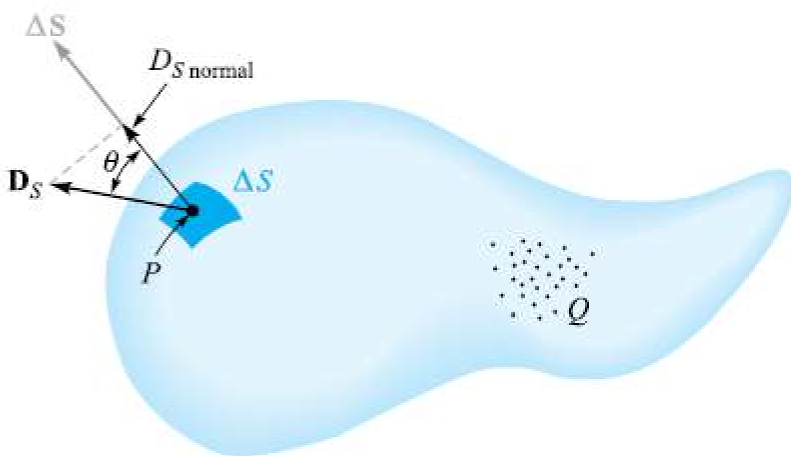


Figure 3.2 The electric flux density D_S at P arising from charge Q. The total flux passing through ΔS is $D_S \cdot \Delta S$.

- At every point on the surface the electric-flux-density vector **D** will have some value $\mathbf{D_S}$,
- where the subscript **S** merely reminds us that **D** must be evaluated at the surface,
- and $\mathbf{D_S}$ will in general vary in magnitude and direction from one point on the surface to another.

At any point P, consider an incremental element of surface $\Delta\mathbf{S}$ and let $\mathbf{D_S}$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure 3.2.

The flux crossing $\Delta\mathbf{S}$ is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = D_{S,norm}\Delta S = D_S \cos\theta\, \Delta S = \mathbf{D_S} \cdot \Delta\mathbf{S}$$

The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{closed \\ surface}} \mathbf{D_S} \cdot d\mathbf{S}$$

The vector surface element $d\mathbf{S}$ is taken to point out of S, so that $d\Psi$ is the amount of flux passing from the interior of S to the exterior of S through dS.

The resultant integral is a closed surface integral, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dxdy, \rho d\varphi d\rho,\ r^2 \sin\theta\, d\theta d\varphi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an **S** below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a closed surface. Such a surface is often called a ***gaussian surface***. We then have the mathematical formulation of **Gauss's law**,

$$\boxed{\Psi = \oint_S \mathbf{D_S} \cdot d\mathbf{S} = Q}$$

$$= charge\ enclosed$$

The charge enclosed might be several point charges, in which case

$$Q = \sum Q_n$$

or a line charge,

$$Q = \int \rho_L dL$$

or a surface charge, (not necessarily a closed surface),

$$Q = \int_S \rho_S dS$$

or a volume charge distribution,

$$Q = \int_{vol} \rho_v dv$$

The last form is usually used, and it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D_S} \cdot d\mathbf{S} = \int_{vol} \rho_v dv$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

Example: To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge *Q* at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius *a*.

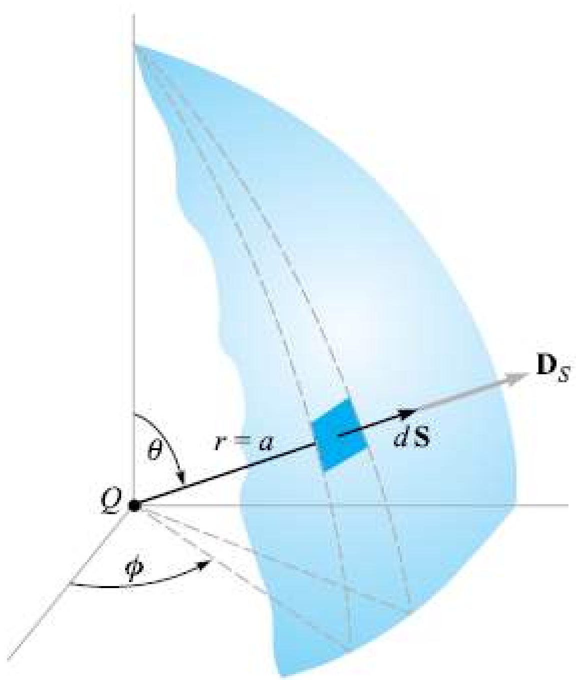


Figure 3.3 Applying Gauss's law to the field of a point charge Q on a spherical closed surface of radius *a*. The electric flux density D is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

Solution:

$$\Psi = \oint_S \mathbf{D_S} \cdot d\mathbf{S} = Q = charge\ enclosed \quad \ldots..Gauss'slaw \qquad \ldots\ldots\ldots(1)$$

We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r}$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a_r} \qquad \ldots\ldots\ldots \quad (\mathbf{2})$$

The differential element of area on a spherical surface is, in spherical coordinates,

$$d\mathbf{S}_r = r^2 sin\theta\ d\theta\ d\varphi\ \ \mathbf{a}_r$$

For our sphere,

$$d\mathbf{S} = a^2 sin\theta\ d\theta\ d\varphi\ \ \mathbf{a}_r \qquad \ldots\ldots\ldots \quad (3)$$

Substitute eq.(2) and eq.(3) into eq.(1);

$$\Psi = \oint_S \mathbf{D_S} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \left(\frac{Q}{4\pi a^2}\ \mathbf{a_r}\right) \cdot\ (a^2 sin\theta\ d\theta\ d\varphi\ \ \mathbf{a}_r)$$

$$= \frac{Q}{4\pi}\int_0^{\pi} sin\theta\ d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{Q}{4\pi}\ [-\cos\theta]_0^{\pi}\ \ [\varphi]_0^{2\pi}$$

$$= \frac{Q}{4\pi}\ [-\cos\pi + \cos 0]\ [2\pi]$$

$$= \frac{Q}{4\pi}\ [-(-1) + 1]\ [2\pi]$$

$$= \frac{Q}{4\pi}\ 4\pi$$

$$= Q$$

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

Application of Gauss's Law: Some Symmetrical Charge Distributions:

We now consider how we may use Gauss's law,

$$Q = \oint_S \mathbf{D_S} \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.
The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S\, dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, D_S = constant.

This allows us to replace the dot product with the product of the scalars D_S and dS and then to bring D_S outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface which $\mathbf{D}_S$ crosses normally, and this is simply the area of this section of that surface.

Homework:

1- Find **D** and **E** of a point charge $\boldsymbol{Q}$ at the origin of a spherical coordinate system using Gauss's law.

2- Find **D** and **E** for the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty\ to\ +\infty$.

3- Find **D** and **E** for two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent. Assume a charge distribution of ρ_S on the outer surface of the inner conductor.

Application of Gauss's Law:Differential volume element:

We are now going to apply the methods of **Gauss's law** to a slightly different type of problem - one that does not possess any symmetry at all.
At first glance, it might seem that our case is hopeless, for without symmetry, a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such

a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties and that is to choose such a very small closed surface that **D** is *almost* constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we will not obtain the value of **D** as our answer but will instead receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

Let us consider any point **P**, shown in Figure 3.6, located by a rectangular coordinate system. The value of **D** at the point **P** may be expressed in rectangular components, $\mathbf{D_o} = D_{xo}\mathbf{a}_x + D_{yo}\mathbf{a}_y + D_{zo}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at *P*, having sides of lengths $\Delta x, \Delta y, and\ \Delta z,$ and apply Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

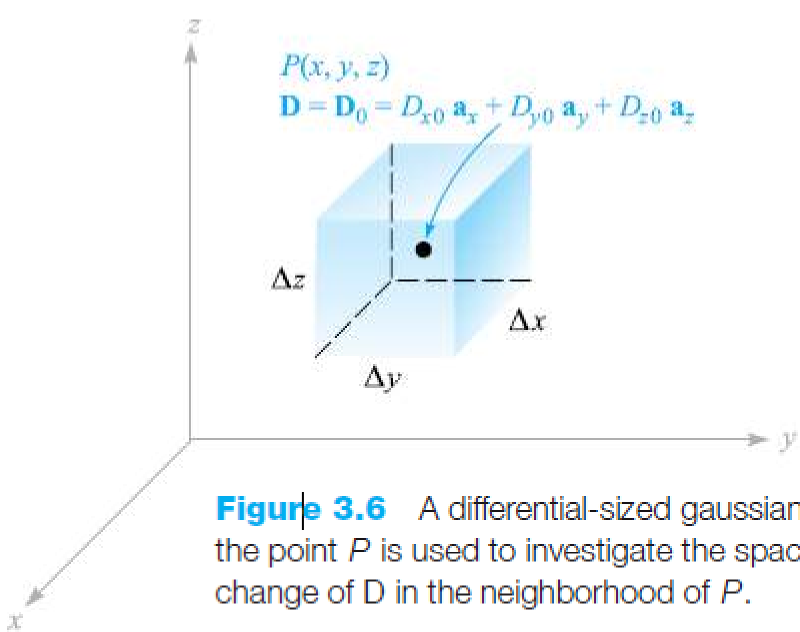


Figure 3.6 A differential-sized gaussian surface about the point *P* is used to investigate the space rate of change of D in the neighborhood of *P*.

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, **D** is essentially constant (over this portion of the entire closed surface) and

$$\begin{aligned}\int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x \\ &\doteq D_{\text{x,front}} \Delta y\, \Delta z\end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned}D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\end{aligned}$$

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y\, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned}\int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta\mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y\, \Delta z\, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y\, \Delta z\end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\Delta x\,\Delta y\,\Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y}\Delta x\,\Delta y\,\Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z}\Delta x\,\Delta y\,\Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D}\cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

or

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v \qquad (7)$$

The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \text{volume } \Delta v \qquad (8)$$

Example:

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y\, \mathbf{a}_x - e^{-x} \cos y\, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is 10^{-9} m^3, then we have enclosed about 2 nC.

Divergence and Maxwell's First Equation:

We will now obtain an exact relationship from (7), by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \qquad (9)$$

in which the charge density, ρ_v , is identified in the second equality.

The methods of the previous section could have been used on any vector **A** to find $\oint_S \mathbf{A} \cdot d\boldsymbol{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \qquad (10)$$

where **A** could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, **divergence**. The divergence of **A** is defined as

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \qquad (11)$$

and is usually abbreviated div **A**.

The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (11), where we shall consider **A** to be a member of the flux-density family of vectors in order to aid the physical interpretation:

*The divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

Writing (9) with our new term, we have

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \qquad \text{(rectangular)} \qquad (12)$$

$$\text{div } \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)} \qquad (13)$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)} \qquad (14)$$

Divergence merely tells us how much flux is leaving a small volume on a per-unit-volume basis; no direction is associated with it.

Example:

Find div **D** at the origin if $\mathbf{D} = e^{-x} \sin y \, \mathbf{a}_x - e^{-x} \cos y \, \mathbf{a}_y + 2z\mathbf{a}_z$.

Solution. We use (12) to obtain

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$
$$= -e^{-x} \sin y + e^{-x} \sin y + 2 = 2$$

The value is the constant 2, regardless of location.
If the units of **D** are C/m^2, then the units of div **D** are C/m^3. This is a volume charge density, a concept discussed in the next section.

Maxwell's First Equation

As mentioned previously,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \qquad (9)$$

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \quad \text{(rectangular)} \qquad (12)$$

we can combine Eqs. (9) and (12) and form the relation between electric flux density and charge density:

$$\text{div } \mathbf{D} = \rho_v \qquad (15)$$
$$\nabla \cdot \mathbf{D} = \rho_v$$

.. Maxwell's 1st Eq.
point form of Gauss's law

- This is the ***first of Maxwell's four equations*** as they apply to electrostatics and steady magnetic fields,
- and it states that **the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.**
- This equation is aptly called the **point form of Gauss's law**. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point.

Because the divergence may be expressed as the sum of three partial derivatives, **Maxwell's first equation** is also described as the **differential-equation form of Gauss's law**, and conversely, **Gauss's law is recognized as the integral form of Maxwell's first equation**.

As a specific illustration, let us consider the divergence of **D** in the region about a point charge Q located at the origin. We have the field

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

and use (14), the expression for divergence in spherical coordinates:

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

Because D_θ and D_ϕ are zero, we have

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{Q}{4\pi r^2}\right) = 0 \qquad (\text{if } r \neq 0)$$

Thus, $\rho_v = 0$ everywhere except at the origin, where it is infinite.

Divergence Theorem:

From Gauss's law, we have

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q \qquad (1)$$

While the charge enclosed in a volume is given by,

$$Q = \int_{\text{vol}} \rho_v dv \qquad (2)$$

But according to Gauss's law in the point form,

$$\nabla\cdot\mathbf{D} = \rho_v \qquad (3)$$

Using in (2),

$$Q = \int_{\text{vol}} \nabla\cdot\mathbf{D}\,dv \qquad (4)$$

Equating (1) and (4) constitute the ***divergence theorem***,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

- The divergence theorem is true for any vector field.
- It relates a triple integration throughout some volume to a double integration over the surface of that volume.
- i.e with the help of the divergence theorem, the surface integral can be converted into a volume integral, provided that the closed surface encloses certain volume.
- Thus volume integral on right hand side of the theorem must be calculated over a volume which must be enclosed by the closed surface on left hand side. The theorem is applicable only under this condition.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Figure 3.7, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell enters, or converges on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

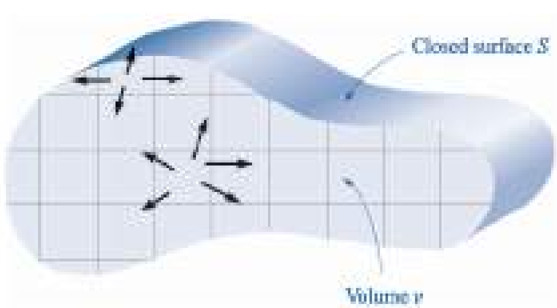


FIGURE 3.7
The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Example:

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that $\mathbf{D}$ is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$

$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3 \int_0^2 (D_x)_{x=0} dy\,dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy\,dz$$

$$- \int_0^3 \int_0^1 (D_y)_{y=0} dx\,dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx\,dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\,dz = \int_0^3 \int_0^2 2y\,dy\,dz$$

$$= \int_0^3 4\,dz = 12$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\,dv = \int_0^3 \int_0^2 \int_0^1 2y\,dx\,dy\,dz = \int_0^3 \int_0^2 2y\,dy\,dz$$

$$= \int_0^3 4\,dz = 12$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

Example: (Q.4.17. in Schaum's outlines book)

Given that $\mathbf{A} = 30e^{-r}\mathbf{a}_r - 2z\,\mathbf{a}_z$ in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $r = 2,\ z = 0,\ and\ \ z = 5$ (Fig. 4.7).

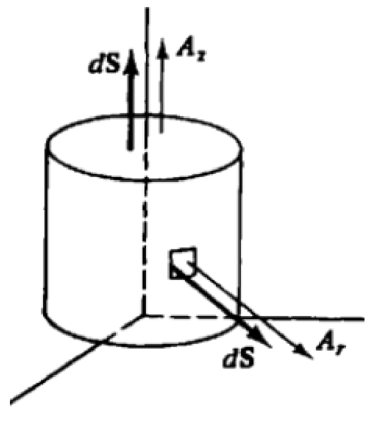


Fig. 4-7

<u>The Solution:</u>

The divergence theorem states that

$$\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_{vol} \nabla\cdot\mathbf{A}\, dv$$

Now

$$\oint_S \mathbf{A}\cdot d\mathbf{S} = \left[\oint_{side} + \oint_{tope} + \oint_{bottom}\right]\mathbf{A}\cdot d\mathbf{S}$$

Consider $d\mathbf{S}$ normal to $\mathbf{a}_r$ direction which is for the side surface.

$\therefore \quad d\mathbf{S} = rd\phi dz\,\mathbf{a_r}$

$\therefore \quad \mathbf{A}\cdot d\mathbf{S} = (30e^{-r}\mathbf{a}_r - 2z\,\mathbf{a}_z)\cdot rd\phi dz\,\mathbf{a_r}$

$$= 30re^{-r}(\mathbf{a}_r\cdot\mathbf{a}_r)d\phi dz$$

$$= 30re^{-r}d\phi dz$$

$$\therefore \quad \oint_{Side} \mathbf{A}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} 30re^{-r}d\phi dz \qquad with \quad r = 2$$

$$= 30\times 2\times e^{-2}\times[\phi]_0^{2\pi}\times[z]_0^5 = 255.1$$

The $d\mathbf{S}$ on top has direction $\mathbf{a}_r$ hence for top surface,

$d\mathbf{S} = rdrd\phi\,\mathbf{a_z}$

$\therefore \quad \mathbf{A}\cdot d\mathbf{S} = (30e^{-r}\mathbf{a}_r - 2z\,\mathbf{a}_z)\cdot rdrd\phi\,\mathbf{a_z}$

$$= -2\,\mathrm{z}\,r\,dr\,d\phi$$

$$\therefore \quad \oint_{top} \mathbf{A}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{r=0}^{2} -2\,\mathrm{z}\,r\,dr\,d\phi \qquad with \quad z = 5$$

$$= -2\times 5\times\left[\frac{r^2}{2}\right]_0^2\times[\phi]_0^{2\pi} = -40\pi$$

While $d\mathbf{S}$ for bottom has direction $-\mathbf{a_z}$ hence for bottom surface,

$d\mathbf{S} = rdrd\phi\,(-\mathbf{a_z})$

$\therefore \quad \mathbf{A}\cdot d\mathbf{S} = (30e^{-r}\mathbf{a}_r - 2z\,\mathbf{a}_z)\cdot rdrd\phi\,(-\mathbf{a_z}) = 2\,\mathrm{z}\,r\,dr\,d\phi$

But z=0 for the bottom surface, as shown in the fig. 4.7.

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = 255.1 - 40\pi + 0$$
$$= 129.4363$$

This is the left hand side of divergence theorem.

Now evaluate $\int_v (\nabla \cdot \mathbf{A}) dv$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

And

$A_r = 30e^{-r}, \quad A_\phi = 0, \quad A_z = -2z$

$$\therefore \quad \nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(30\, re^{-r}) + 0 + \frac{\partial}{\partial z}(-2z)$$

$$= \frac{1}{r}\{30\, r(-e^{-r}) + 30e^{-r}\,(1)\} + (-2)$$
$$= -30e^{-r} + \frac{30}{r}e^{-r} - 2$$

$$\therefore \quad \int_v (\nabla \cdot \mathbf{A}) dv = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=0}^{2}\left(-30e^{-r} + \frac{30}{r}e^{-r} - 2\right) r\, dr\, d\phi\, dz$$

$$= \int_{z=0}^{5}\int_{\phi=0}^{\pi}\int_{r=0}^{2}(-30re^{-r} + 30e^{-r} - 2r)\; dr\, d\phi\, dz$$

$$= \left\{-30r\left[\frac{e^{-r}}{-1}\right] - \int(-30)\left[\frac{e^{-r}}{-1}\right]dr + 30\left[\frac{e^{-r}}{-1}\right] - \left[2\frac{r^2}{2}\right]\right\}[z]_0^5[\phi]_0^{2\pi}$$

Obtained using integration by parts.

$$= [30\, r\, e^{-r} + 30\, e^{-r} - 30\, e^{-r} - r^2]_0^2\, [5]\, [2\pi]$$
$$= [60\;\, e^{-2} - 2^2]\, [10\pi] = 129.437$$

This is same as obtained from the left hand side.

Problems

1. A point charge of 15 nC is located at the origin, find the total electric flux leaving: (a) The surface of a sphere of radius 5 m centered at the point (1, -1, 2). (b) The top z=0.5 face of a cube, 1 m on a side centered at the origin, edges parallel to the coordinate axes. (c) that portion of a right circular cylinder r = 5, for which z ≥ 0.

2. Calculate the total electric flux leaving the cylindrical surface $r = 4.5$ and $z = \pm 3.5$ if the charge distribution is: (a) 2 C point charges on the x- axis at x = 0, ± 1, ±2, ±3, ….. .(b) a line charge on the x-axis, $\rho_l = 2\cos 0.1x$ C/m. (c) a surface charge $\rho_s = 0.1\, r^2$ C/m^2 on the plane z = 0.

3. Find the total charge lying within the sphere $r = 2$ if **D** equals (a) $\frac{1}{r^2}\mathbf{a}_r$. (b) $\frac{1}{r}\mathbf{a}_r$.

4. Using Gauss's law to find the electric flux density for (a) a point charge, (b) a uniform line charge distribution ρ_l lying along the z – axis and extending from -∞ to ∞.

5. Determine an expression for the volume charge density that gives rise to the field (a) $\mathbf{D} = e^{4x}e^{-5y}e^{-2z}\left(2\,\mathbf{a}_x - 2.5\,\mathbf{a}_y - \mathbf{a}_z\right)$,

(b) $\mathbf{D} = e^{-2z}\left(2r\phi\,\mathbf{a}_r + r\,\mathbf{a}_\phi - 2r^2\phi\,\mathbf{a}_z\right)$

6. Given that $\mathbf{D} = (5r^2/4)\,\mathbf{a}_r \;\; (c/m^2)$ in spherical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $r = 4\,m$ and $\theta = \pi/4$.