1.4. given points $A(8,-5,4)$ and $B(-2,3,2)$, find:

a) the distance from A to B.

$$|\mathbf{B}-\mathbf{A}| = |(-10,8,-2)| = \underline{12.96}$$

b) a unit vector directed from A towards B. This is found through

$$\mathbf{a}_{AB} = \frac{\mathbf{B}-\mathbf{A}}{|\mathbf{B}-\mathbf{A}|} = \underline{(-0.77,0.62,-0.15)}$$

c) a unit vector directed from the origin to the midpoint of the line AB.

$$\mathbf{a}_{0M} = \frac{(\mathbf{A}+\mathbf{B})/2}{|(\mathbf{A}+\mathbf{B})/2|} = \frac{(3,-1,3)}{\sqrt{19}} = \underline{(0.69,-0.23,0.69)}$$

d) the coordinates of the point on the line connecting A to B at which the line intersects the plane $z=3$. Note that the midpoint, $(3,-1,3)$, as determined from part c happens to have z coordinate of 3. This is the point we are looking for.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2+2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1,2,-1)$ and $Q(-2,1,3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1,2,-1) = \underline{(48,36,18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2,1,3) = (-48,72,162)$, so

$$\mathbf{a}_G = \frac{(-48,72,162)}{|(-48,72,162)|} = \underline{(-0.26,0.39,0.88)}$$

c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P}-\mathbf{Q}}{|\mathbf{P}-\mathbf{Q}|} = \frac{(3,-1,4)}{\sqrt{26}} = \underline{(0.59,0.20,-0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2+2), 18z^2)|$, or $10 = |(4xy, 2x^2+4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2 \int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \; + \int_3^{4.5} \int_{100°}^{130°} 3 \, d\phi \, dz$$
$$+ \int_3^{4.5} \int_{100°}^{130°} 5 \, d\phi \, dz \; + \; 2 \int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \; + \; 4 \times 2 \; + \; 2 \times \left[\frac{30°}{360°} \times 2\pi \times 3 \; + \; \frac{30°}{360°} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100°$, $z = 3$) and B($\rho = 5$, $\phi = 130°$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. At point $P(-3, 4, 5)$, express the vector that extends from P to $Q(2, 0, -1)$ in:

a) rectangular coordinates.

$$\mathbf{R}_{PQ} = \mathbf{Q} - \mathbf{P} = \underline{5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z}$$

Then $|\mathbf{R}_{PQ}| = \sqrt{25 + 16 + 36} = 8.8$

b) cylindrical coordinates. At P, $\rho = 5$, $\phi = \tan^{-1}(4/-3) = -53.1°$, and $z = 5$. Now,

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\rho = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\rho = 5\cos\phi - 4\sin\phi = 6.20$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\phi = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\phi = -5\sin\phi - 4\cos\phi = 1.60$$

Thus

$$\mathbf{R}_{PQ} = \underline{6.20\mathbf{a}_\rho + 1.60\mathbf{a}_\phi - 6\mathbf{a}_z}$$

and $|\mathbf{R}_{PQ}| = \sqrt{6.20^2 + 1.60^2 + 6^2} = 8.8$

c) spherical coordinates. At P, $r = \sqrt{9 + 16 + 25} = \sqrt{50} = 7.07$, $\theta = \cos^{-1}(5/7.07) = 45°$, and $\phi = \tan^{-1}(4/-3) = -53.1°$.

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_r = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_r = 5\sin\theta\cos\phi - 4\sin\theta\sin\phi - 6\cos\theta = 0.14$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\theta = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi - 4\cos\theta\sin\phi - (-6)\sin\theta = 8.62$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\phi = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\phi = -5\sin\phi - 4\cos\phi = 1.60$$

1.24. (continued)

Thus

$$\mathbf{R}_{PQ} = \underline{0.14\mathbf{a}_r + 8.62\mathbf{a}_\theta + 1.60\mathbf{a}_\phi}$$

and $|\mathbf{R}_{PQ}| = \sqrt{0.14^2 + 8.62^2 + 1.60^2} = 8.8$

d) Show that each of these vectors has the same magnitude. Each does, as shown above.

1.25. Given point $P(r = 0.8,\ \theta = 30^\circ,\ \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2}\left(\cos\phi\,\mathbf{a}_r + \frac{\sin\phi}{\sin\theta}\,\mathbf{a}_\phi\right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.

b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.

c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. a) Determine an expression for $\mathbf{a}_y$ in spherical coordinates at $P(r = 4,\ \theta = 0.2\pi,\ \phi = 0.8\pi)$: Use $\mathbf{a}_y \cdot \mathbf{a}_r = \sin\theta\sin\phi = 0.35$, $\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta\sin\phi = 0.48$, and $\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos\phi = -0.81$ to obtain

$$\mathbf{a}_y = \underline{0.35\mathbf{a}_r + 0.48\mathbf{a}_\theta - 0.81\mathbf{a}_\phi}$$

b) Express $\mathbf{a}_r$ in cartesian components at P: Find $x = r\sin\theta\cos\phi = -1.90$, $y = r\sin\theta\sin\phi = 1.38$, and $z = r\cos\theta = -3.24$. Then use $\mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta\cos\phi = -0.48$, $\mathbf{a}_r \cdot \mathbf{a}_y = \sin\theta\sin\phi = 0.35$, and $\mathbf{a}_r \cdot \mathbf{a}_z = \cos\theta = 0.81$ to obtain

$$\mathbf{a}_r = \underline{-0.48\mathbf{a}_x + 0.35\mathbf{a}_y + 0.81\mathbf{a}_z}$$

1.27. The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and 50°, and $\phi = 20^\circ$ and 60° identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}\int_2^4 r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}(4^2 + 2^2)\sin\theta\,d\theta\,d\phi + \int_2^4\int_{20^\circ}^{60^\circ} r(\sin 30^\circ + \sin 50^\circ)\,dr\,d\phi$$
$$+\, 2\int_{30^\circ}^{50^\circ}\int_2^4 r\,dr\,d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_2^4 dr \ +\ 2\int_{30^\circ}^{50^\circ}(4+2)\,d\theta + \int_{20^\circ}^{60^\circ}(4\sin 50^\circ + 4\sin 30^\circ + 2\sin 50^\circ + 2\sin 30^\circ)\,d\phi$$
$$= \underline{17.49}$$

The cylindrical coordinate system will be used, with the charge in the $z = 0$ plane as shown in Fig. 2-17.

$$d\mathbf{E} = \frac{\rho_s r\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)}\left(\frac{-r\mathbf{a}_r + z\mathbf{a}_z}{\sqrt{r^2+x^2}}\right)$$

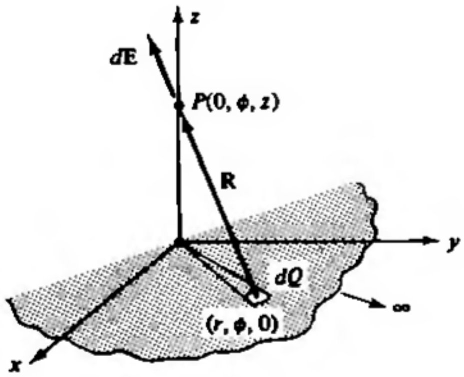

Fig. 2-17

Symmetry about the z axis results in cancellation of the radial components.

$$\mathbf{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_s r z\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)^{3/2}}\mathbf{a}_z$$

$$= \frac{\rho_s z}{2\epsilon_0}\left[\frac{-1}{\sqrt{r^2+z^2}}\right]_0^{\infty}\mathbf{a}_z = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_z$$

This result is for points above the xy plane. Below the xy plane the unit vector changes to $-\mathbf{a}_z$. The generalized form may be written using $\mathbf{a}_n$, the unit normal vector:

$$E = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_n$$

The electric field is everywhere normal to the plane of the charge and its magnitude is independent of the distance from the plane.

2.13. As shown in Fig. 2-18, the plane $y = 3$ m contains a uniform charge distribution of density $\rho_s = (10^{-8}/6\pi)$ C/m^2. Determine $\mathbf{E}$ at all points.

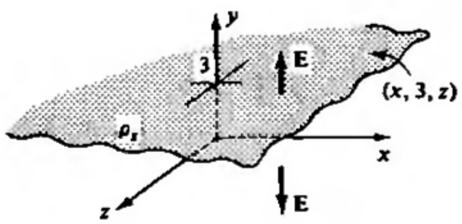

Fig. 2-18

For $y > 3$ m,

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_n$$

$$= 30\mathbf{a}_y \text{ V/m}$$

and for $y < 3$ m,

$$\mathbf{E} = -30\mathbf{a}_y \text{ V/m}$$

2.14. Two infinite uniform sheets of charge, each with density ρ_s, are located at $x = \pm 1$ (Fig. 2-19). Determine **E** in all regions.

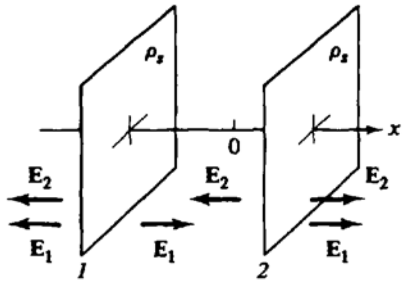

Fig. 2-19

Only parts of the two sheets of charge are shown in Fig. 2-19. Both sheets result in **E** fields that are directed along x, independent of the distance. Then

$$\mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} -(\rho_s/\epsilon_0)\mathbf{a}_x & x < -1 \\ 0 & -1 < x < 1 \\ (\rho_s/\epsilon_0)\mathbf{a}_x & x > 1 \end{cases}$$

2.15. Repeat Problem 2.14 with ρ_s on $x = -1$ and $-\rho_s$ on $x = 1$.

$$\mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} 0 & x < -1 \\ (\rho_s/\epsilon_0)\mathbf{a}_x & -1 < x < 1 \\ 0 & x > 1 \end{cases}$$

2.16. A uniform sheet charge with $\rho_s = (1/3\pi)\,\text{nC/m}^2$ is located at $z = 5$ m and a uniform line charge with $\rho_\ell = (-25/9)\,\text{nC/m}$ at $z = -3$ m, $y = 3$ m. Find **E** at $(x, -1, 0)$ m.

The two charge configurations are parallel to the x axis. Hence the view in Fig. 2-20 is taken looking at the yz plane from positive x. Due to the sheet charge,

$$\mathbf{E}_s = \frac{\rho_2}{2\epsilon_0}\mathbf{a}_n$$

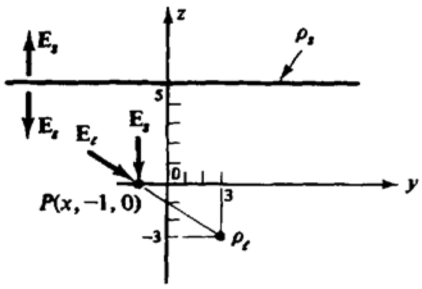

Fig. 2-20

At P, $\mathbf{a}_n = -\mathbf{a}_z$ and

$$\mathbf{E}_s = -6\mathbf{a}_z \text{ V/m}$$

Due to the line charge,

$$\mathbf{E}_\ell = \frac{\rho_\ell}{2\pi\epsilon_0 r}\mathbf{a}_r$$

and at P

$$\mathbf{E}_\ell = 8\mathbf{a}_y - 6\mathbf{a}_z \quad \text{V/m}$$

The total electric field is the sum, $\mathbf{E} = \mathbf{E}_\ell + \mathbf{E}_s = 8\mathbf{a}_y - 12\mathbf{a}_z$ V/m.

2.17. Determine $\mathbf{E}$ at $(2, 0, 2)$ m due to three standard charge distributions as follows: a uniform sheet at $x = 0$ m with $\rho_{s1} = (1/3\pi)$ nC/m^2, a uniform sheet at $x = 4$ m with $\rho_{s2} = (-1/3\pi)$ nC/m^2, and a uniform line at $x = 6$ m, $y = 0$ m with $\rho_\ell = -2$ nC/m.

Since the three charge configurations are parallel with $\mathbf{a}_z$, there will be no z component of the field. Point $(2, 0, 2)$ will have the same field as any point $(2, 0, z)$. In Fig. 2-21, P is located between the two sheet charges, where the fields add due to the difference in sign.

$$\begin{aligned}\mathbf{E} &= \frac{\rho_{s1}}{2\epsilon_0}\mathbf{a}_n + \frac{\rho_{s2}}{2\epsilon_0}\mathbf{a}_n + \frac{\rho_\ell}{2\pi\epsilon_0 r}\mathbf{a}_r \\ &= 6\mathbf{a}_x + 6\mathbf{a}_x + 9\mathbf{a}_x \\ &= 21\mathbf{a}_x \text{ V/m}\end{aligned}$$

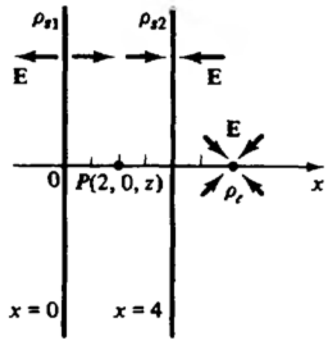

Fig. 2-21

2.18. As shown in Fig. 2-22, charge is distributed along the z axis between $z = \pm 5$ m with a uniform density $\rho_\ell = 20$ nC/m. Determine $\mathbf{E}$ at $(2, 0, 0)$ m in cartesian coordinates. Also express the answer in cylindrical coordinates.

$$d\mathbf{E} = \frac{20 \times 10^{-9}\,dz}{4\pi(10^{-9}/36\pi)(4 + z^2)}\left(\frac{2\mathbf{a}_x - z\mathbf{a}_z}{\sqrt{4 + z^2}}\right) \quad \text{(V/m)}$$

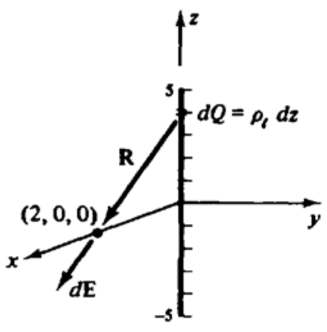

Fig. 2-22

Symmetry with respect to the $z = 0$ plane eliminates any z component in the result.

$$\mathbf{E} = 180\int_{-5}^{5} \frac{2\,dz}{(4 + z^2)^{3/2}}\mathbf{a}_x = 167\mathbf{a}_x \text{ V/m}$$

In cylindrical coordinates, $\mathbf{E} = 167\mathbf{a}_r$ V/m.

4.5. Given $\mathbf{A} = x^2\mathbf{a}_x + yz\mathbf{a}_y + xy\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$.

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xy) = 2x + z$$

4.6. Given $\mathbf{A} = (x^2 + y^2)^{-1/2}\mathbf{a}_x$, find $\nabla \cdot \mathbf{A}$ at $(2, 2, 0)$.

$$\nabla \cdot \mathbf{A} = -\tfrac{1}{2}(x^2 + y^2)^{-3/2}(2x) \qquad \text{and} \qquad \nabla \cdot \mathbf{A}\big|_{(2,2,0)} = -8.84 \times 10^{-2}$$

4.7. Given $\mathbf{A} = r \sin\phi\,\mathbf{a}_r + 2r\cos\phi\,\mathbf{a}_\phi + 2z^2\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$.

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r^2 \sin\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(2r\cos\phi) + \frac{\partial}{\partial z}(2z^2)$$
$$= 2\sin\phi - 2\sin\phi + 4z = 4z$$

4.8. Given $\mathbf{A} = 10\sin^2\phi\,\mathbf{a}_r + r\mathbf{a}_\phi + [(z^2/r)\cos^2\phi]\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at $(2, \phi, 5)$.

$$\nabla \cdot \mathbf{A} = \frac{10\sin^2\phi + 2z\cos^2\phi}{r} \qquad \text{and} \qquad \nabla \cdot \mathbf{A}\big|_{(2,\phi,5)} = 5$$

4.9. Given $\mathbf{A} = (5/r^2)\mathbf{a}_r + (10/\sin\theta)\mathbf{a}_\theta - r^2\phi\sin\theta\,\mathbf{a}_\phi$, find $\nabla \cdot \mathbf{A}$.

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(5) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(10) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\phi\sin\theta) = -r$$

4.10. Given $\mathbf{A} = 5\sin\theta\,\mathbf{a}_\theta + 5\sin\phi\,\mathbf{a}_\phi$, find $\nabla \cdot \mathbf{A}$ at $(0.5, \pi/4, \pi/4)$.

$$\nabla \cdot \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(5\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(5\sin\phi) = 10\frac{\cos\theta}{r} + 5\frac{\cos\phi}{r\sin\theta}$$

and

$$\nabla \cdot \mathbf{A}\big|_{(0.5,\pi/4,\pi/4)} = 24.14$$

4.11. Given that $\mathbf{D} = \rho_0 z\mathbf{a}_z$ in the region $-1 \le z \le 1$ in cartesian coordinates and $\mathbf{D} = (\rho_0 z/|z|)\mathbf{a}_z$ elsewhere, find the charge density.

$$\nabla \cdot \mathbf{D} = \rho$$

For $-1 \le z \le 1$,

$$\rho = \frac{\partial}{\partial z}(\rho_0 z) = \rho_0$$

and for $z < -1$ or $z > 1$,

$$\rho = \frac{\partial}{\partial z}(\mp\rho_0) = 0$$

The charge distribution is shown in Fig. 4-5.

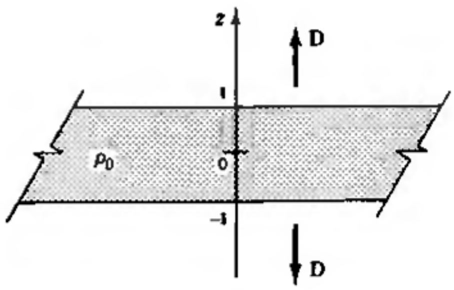

Fig. 4-5

4.12. Given that $\mathbf{D} = (10r^3/4)\mathbf{a}_r$ (C/m²) in the region $0 < r \le 3$ m in cylindrical coordinates and $\mathbf{D} = (810/4r)\mathbf{a}_r$ (C/m²) elsewhere, find the charge density.

For $0 < r \le 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{10r^4}{4}\right) = 10r^2 \quad (\text{C/m}^3)$$

and for $r > 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}(810/4) = 0$$

4.13. Given that

$$\mathbf{D} = \frac{Q}{\pi r^2}(1 - \cos 3r)\mathbf{a}_r$$

in spherical coordinates, find the charge density.

$$\rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{Q}{\pi r^2}(1 - \cos 3r)\right] = \frac{3Q}{\pi r^2}\sin 3r$$

4.14. In the region $0 < r \le 1$ m, $\mathbf{D} = (-2 \times 10^{-4}/r)\mathbf{a}_r$ (C/m²) and for $r > 1$ m, $\mathbf{D} = (-4 \times 10^{-4}/r^2)\mathbf{a}_r$ (C/m²), in spherical coordinates. Find the charge density in both regions.

For $0 < r \le 1$ m,

$$\rho = \frac{1}{r^2}\frac{\partial}{\partial r}(-2 \times 10^{-4} r) = \frac{-2 \times 10^{-4}}{r^2} \quad (\text{C/m}^3)$$

and for $r > 1$ m,

$$\rho = \frac{1}{r^2}\frac{\partial}{\partial r}(-4 \times 10^{-4}) = 0$$

4.15. In the region $r \le 2$, $\mathbf{D} = (5r^2/4)\mathbf{a}_r$ and for $r > 2$, $\mathbf{D} = (20/r^2)\mathbf{a}_r$, in spherical coordinates. Find the charge density.

For $r \le 2$,

$$\rho = \frac{1}{r^2}\frac{\partial}{\partial r}(5r^4/4) = 5r$$

and for $r > 2$,

$$\rho = \frac{1}{r^2}\frac{\partial}{\partial r}(20) = 0$$

4.16. Given that $\mathbf{D} = (10x^3/3)\mathbf{a}_x$ (C/m²), evaluate both sides of the divergence theorem for the volume of a cube, 2 m on an edge, centered at the origin and with edges parallel to the axes.

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{D})\, dv$$

Since $\mathbf{D}$ has only an x component, $\mathbf{D} \cdot d\mathbf{S}$ is zero on all but the faces at $x = 1$ m and $x = -1$ m (see Fig. 4-6).

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} \frac{10(1)}{3}\mathbf{a}_x \cdot dy\, dz\, \mathbf{a}_x + \int_{-1}^{1}\int_{-1}^{1} \frac{10(-1)}{3}\mathbf{a}_x \cdot dy\, dz\, (-\mathbf{a}_x)$$
$$= \frac{40}{3} + \frac{40}{3} = \frac{80}{3}\ \text{C}$$

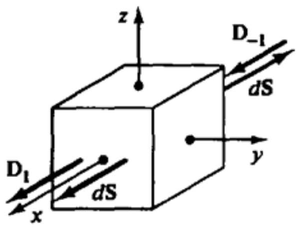

Fig. 4-6

Now for the right side of the divergence theorem. Since $\nabla \cdot \mathbf{D} = 10x^2$,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{D})\, dv = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (10x^2)\, dx\, dy\, dz = \int_{-1}^{1}\int_{-1}^{1} \left[10\frac{x^3}{3}\right]_{-1}^{2} dy\, dz = \frac{80}{3}\,\text{C}$$

4.17. Given that $\mathbf{A} = 30e^{-r}\mathbf{a}_r - 2z\mathbf{a}_z$ in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $r = 2$, $z = 0$, and $z = 5$ (Fig. 4-7).

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{A})\, dv$$

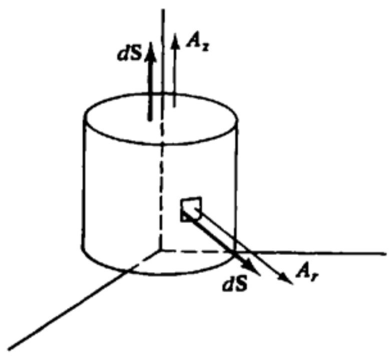

Fig. 4-7

It is noted that $A_z = 0$ for $z = 0$ and hence $\mathbf{A} \cdot d\mathbf{S}$ is zero over that part of the surface.

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int_0^5\int_0^{2\pi} 30e^{-2}\mathbf{a}_r \cdot 2\, d\phi\, dz\, \mathbf{a}_r + \int_0^{2\pi}\int_0^2 -2(5)\mathbf{a}_z \cdot r\, dr\, d\phi\, \mathbf{a}_z$$
$$= 60e^{-2}(2\pi)(5) - 10(2\pi)(2) = 129.4$$

For the right side of the divergence theorem:

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(30re^{-r}) + \frac{\partial}{\partial z}(-2z) = \frac{30e^{-r}}{r} - 30e^{-r} - 2$$

and $$\int (\nabla \cdot \mathbf{A})\, dv = \int_0^5\int_0^{2\pi}\int_0^2 \left(\frac{30e^{-r}}{r} - 30e^{-r} - 2\right) r\, dr\, d\phi\, dz = 129.4$$

4.18 Given that $\mathbf{D} = (10r^3/4)\mathbf{a}_r$ (C/m²) in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $r = 1$ m, $r = 2$ m, $z = 0$ and $z = 10$ m (see Fig. 4-8).

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{D})\, dv$$

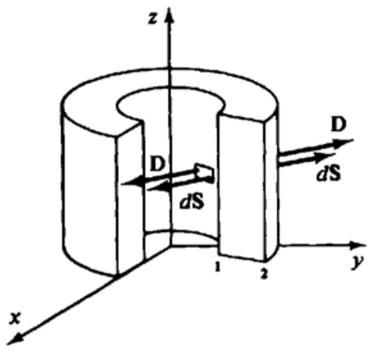

Fig. 4-8

Since **D** has no z component, $\mathbf{D}\cdot d\mathbf{S}$ is zero for the top and bottom. On the inner cylindrical surface $d\mathbf{S}$ is in the direction $-\mathbf{a}_r$.

$$\oint \mathbf{D}\cdot d\mathbf{S} = \int_0^{10}\int_0^{2\pi} \frac{10}{4}(1)^3\mathbf{a}_r\cdot(1)\,d\phi\,dz(-\mathbf{a}_r) + \int_0^{10}\int_0^{2\pi}\frac{10}{4}(2)^3\mathbf{a}_r\cdot(2)\,d\phi\,dz\mathbf{a}_r = \frac{-200\pi}{4} + 16\frac{200\pi}{4} = 750\pi\ \text{C}$$

From the right side of the divergence theorem:

$$\nabla\cdot\mathbf{D} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{10r^4}{4}\right) = 10r^2$$

and

$$\int(\nabla\cdot\mathbf{D})\,dv = \int_0^{10}\int_0^{2\pi}\int_1^2 (10r^2)r\,dr\,d\phi\,dz = 750\pi\ \text{C}$$

4.19. Given that $\mathbf{D} = (5r^2/4)\mathbf{a}_r$ (C/m²) in spherical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $r = 4$ m and $\theta = \pi/4$ (see Fig. 4-9).

Mathcad

$$\oint \mathbf{D}\cdot d\mathbf{S} = \int(\nabla\cdot\mathbf{D})\,dv$$

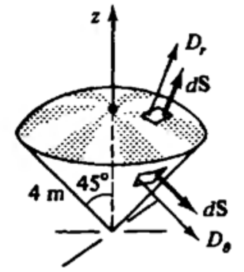

Fig. 4-9

Since **D** has only a radial component, $\mathbf{D}\cdot d\mathbf{S}$ has a nonzero value only on the surface $r = 4$ m.

$$\oint \mathbf{D}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/4}\frac{5(4)^2}{4}\mathbf{a}_r\cdot(4)^2\sin\theta\,d\theta\,d\phi\mathbf{a}_r = 589.1\ \text{C}$$

For the right side of the divergence theorem:

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{5r^4}{4}\right) = 5r$$

and

$$\int (\nabla \cdot \mathbf{D})\, dv = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 (5r)r^2 \sin\theta\, dr\, d\theta\, d\phi = 589.1\ \mathrm{C}$$

Supplementary Problems

4.20. Develop the divergence in spherical coordinates. Use the delta-volume with edges Δr, $r\,\Delta\theta$, and $r\sin\theta\,\Delta\phi$.

4.21. Show that $\nabla \cdot \mathbf{E}$ is zero for the field of a uniform sheet charge.

4.22. The field of an electric dipole with the charges at $\pm d/2$ on the z axis is

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Show that the divergence of this field is zero.

4.23. Given $\mathbf{A} = e^{5x}\mathbf{a}_x + 2\cos y\,\mathbf{a}_y + 2\sin z\,\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at the origin. *Ans.* 7.0

4.24. Given $\mathbf{A} = (3x + y^2)\mathbf{a}_x + (x - y^2)\mathbf{a}_y$, find $\nabla \cdot \mathbf{A}$. *Ans.* $3 - 2y$

4.25. Given $\mathbf{A} = 2xy\mathbf{a}_x + z\mathbf{a}_y + yz^2\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at $(2, -1, 3)$. *Ans.* -8.0

4.26. Given $\mathbf{A} = 4xy\mathbf{a}_x - xy^2\mathbf{a}_y + 5\sin z\,\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at $(2, 2, 0)$. *Ans.* 5.0

4.27. Given $\mathbf{A} = 2r\cos^2\phi\,\mathbf{a}_r + 3r^2\sin z\,\mathbf{a}_\phi + 4z\sin^2\phi\,\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$. *Ans.* 4.0

4.28. Given $\mathbf{A} = (10/r^2)\mathbf{a}_r + 5e^{-2z}\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at $(2, \phi, 1)$. *Ans.* -2.60

4.29. Given $\mathbf{A} = 5\cos r\,\mathbf{a}_r + (3ze^{-2r}/r)\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$ at (π, ϕ, z). *Ans.* -1.59

4.30. Given $\mathbf{A} = 10\mathbf{a}_r + 5\sin\theta\,\mathbf{a}_\theta$, find $\nabla \cdot \mathbf{A}$. *Ans.* $(2 + \cos\theta)(10/r)$

4.31. Given $\mathbf{A} = r\mathbf{a}_r - r^2\cot\theta\,\mathbf{a}_\theta$, find $\nabla \cdot \mathbf{A}$. *Ans.* $3 - r$

4.32. Given $\mathbf{A} = [(10\sin^2\theta)/r]\mathbf{a}_r$ (N/m), find $\nabla \cdot \mathbf{A}$ at (2 m, $\pi/4$ rad, $\pi/2$ rad). *Ans.* 1.25 N/m^2.

4.33. Given $\mathbf{A} = r^2\sin\theta\,\mathbf{a}_r + 13\phi\mathbf{a}_\theta + 2r\mathbf{a}_\phi$, find $\nabla \cdot \mathbf{A}$. *Ans.* $4r\sin\theta + \left(\frac{13\phi}{r}\right)\cot\theta$

4.34. Show that the divergence of $\mathbf{E}$ is zero if $\mathbf{E} = (100/r)\mathbf{a}_\phi + 40\mathbf{a}_z$.

4.35. In the region $a \le r \le b$ (cylindrical coordinates),

$$\mathbf{D} = \rho_0\left(\frac{r^2 - a^2}{2r}\right)\mathbf{a}_r$$

2.23. Given the surface charge density, $\rho_s = 2\,\mu\text{C/m}^2$, in the region $\rho < 0.2$ m, $z = 0$, and is zero elsewhere, find **E** at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of **E** will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of **E** will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.4}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1\text{ kV/m}}$.

b) With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1\text{ kV/m}}$.

2.24. Surface charge density is positioned in free space as follows: $20\,\text{nC/m}^2$ at $x = -3$, $-30\,\text{nC/m}^2$ at $y = 4$, and $40\,\text{nC/m}^2$ at $z = 2$. Find the magnitude of **E** at the three points, $(4, 3, -2)$, $(-2, 5, -1)$, and $(0, 0, 0)$. Since all three sheets are infinite, the field magnitude associated with each one will be $\rho_s/(2\epsilon_0)$, which is position-independent. For this reason, the *net* field magnitude will be the same everywhere, whereas the field direction will depend on which side of a given sheet one is positioned. We take the first point, for example, and find

$$\mathbf{E}_A = \frac{20\times10^{-9}}{2\epsilon_0}\mathbf{a}_x + \frac{30\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - \frac{40\times10^{-9}}{2\epsilon_0}\mathbf{a}_z = 1130\mathbf{a}_x + 1695\mathbf{a}_y - 2260\mathbf{a}_z\text{ V/m}$$

The magnitude of $\mathbf{E}_A$ is thus $\underline{3.04\,\text{kV/m}}$. This will be the magnitude at the other two points as well.

2.25. Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2, 0, 6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, $0.2\,\text{nC/m}^2$ at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12\times10^{-9})}{4\pi\epsilon_0}\frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4+36)^{1.5}}\right] + \left[\frac{(3\times10^{-9})}{2\pi\epsilon_0}\frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4+9)}\right] - \left[\frac{(0.2\times10^{-9})\mathbf{a}_x}{2\epsilon_0}\right]$$
$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z\text{ V/m}}$$

2.26. A uniform line charge density of 5 nC/m is at $y = 0$, $z = 2$ m in free space, while -5 nC/m is located at $y = 0$, $z = -2$ m. A uniform surface charge density of $0.3\,\text{nC/m}^2$ is at $y = 0.2$ m, and $-0.3\,\text{nC/m}^2$ is at $y = -0.2$ m. Find $|\mathbf{E}|$ at the origin: Since each pair consists of equal and opposite charges, the effect at the origin is to double the field produce by one of each type. Taking the sum of the fields at the origin from the surface and line charges, respectively, we find:

$$\mathbf{E}(0,0,0) = -2\times\frac{0.3\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - 2\times\frac{5\times10^{-9}}{2\pi\epsilon_0(2)}\mathbf{a}_z = -33.9\mathbf{a}_y - 89.9\mathbf{a}_z$$

so that $|\mathbf{E}| = \underline{96.1\text{ V/m}}$.