CH.2

Electrostatics

Coulomb's Law and Electric Field Intensity

Introduction:

Electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics can be defined as the study of electric charges which are static i.e. are at rest. An electric charge has its effect in a region or a space around it. This region is called an electric field of that charge. Such an electric field produced due to stationary electric charge dose not varies with time. It is time invariant and called **static electric field**. The study of such time invariant electric fields in a space or vacuum, produced by various types of static charge distributions is called **electrostatics**.

A very common example of such a field is a field used in cathode ray tube for focusing and deflecting a beam. Electrostatics plays a very important role in our day to day life. A variety of machines such as X-ray machine and medical instruments used for electrocardiograms, scanning etc. use the principle of electrostatics. Many other applications in fields of medicin , engineering, agricultural activities, and electronic components based on electrostatics.

In this chapter, we introduce Coulomb's electrostatic force law. We will restrict the study to fields in *vacuum* or *free space*; this would apply to media such as air and other gases.

Coulomb's Law and Electric Field Intensity

<u>Coulomb's Law:</u> The force (*F*) between two point charges Q_1 and Q_2 is proportional to the product of the charges and inversely proportional to the square of the distance (*R*) between them. Or

$$F = k\frac{Q_1 Q_2}{R^2} \quad (N) \qquad \ldots\ldots\ldots(1)$$

Where k is the constant of proportionality,

$$k = 1/4\pi\epsilon_o \text{ of unit } (m/F)$$

and ϵ_o is permittivity of free space,

$$\epsilon_o = 8.854 \times 10^{-12} = \frac{1}{36\pi} \times 10^{-9} \quad (F/m) \;\; or \;\; (C^2/N.m^2)$$

Coulomb's Law now

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_o R^2} \quad (N) \qquad \ldots\ldots\ldots(2)$$

Note: Actually,

$$\mathrm{k} = 1/4\pi\epsilon$$

Where ϵ = Permittivity of the medium in which charge are located, in general ϵ is expressed as,

$$\epsilon = \epsilon_o \epsilon_r$$

Where ϵ_o = Permittivity of free space or vacuum.
ϵ_r = Relative permittivity or dielectric constant of the medium with respect to free space
ϵ = Absolute permittivity

For the free space or vacuum, the relatve permittivity ϵ_r =1, hence $\epsilon = \epsilon_o$

Then $$k = 1/4\pi\epsilon_o$$

F is a vector (has both manitude and direction). For example, the directon of the force $\mathbf{F_2}$ exerted on Q_2 by Q_1 is the direction that Q_2 would move if it were free to move and Q_1 were fixed in spce.

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{12}^2}\,\mathbf{a}_{12} \quad (N) \quad \dots\dots\dots \quad (3)$$

The unit vector $\mathbf{a}_{12}$ is directed along the line joining the two charges.

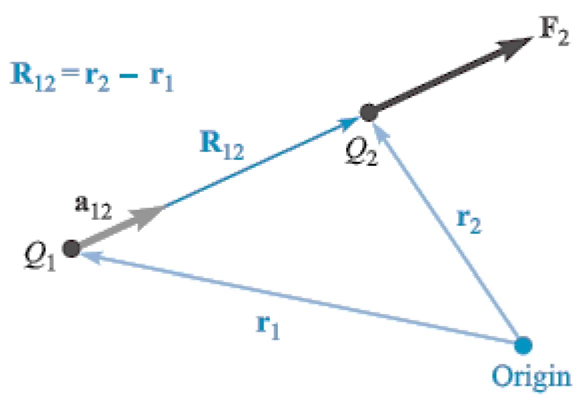


Figure 2.1 If Q_1 and Q_2 have like signs, the vector force F_2 on Q_2 is in the same direction as the vector R_{12}.

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{12}^2}\,\mathbf{a}_{21}$$

For a group of point charges Q_1, Q_2 , Q_3, ………, Q_n therefore the force effected on a charge Q_o as a result of these charges is

$$\mathbf{F} = \frac{Q_o}{4\pi\epsilon_o}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\,\mathbf{a}_{io} \quad (N) \quad \dots\dots\dots\dots \quad (4)$$

To feeling of how large the force in Eq.(1) is, we compare it with the gravitational force that is also an inverse square law with distance.

$$\frac{\mathbf{F}_e}{\mathbf{F}_g} = -\frac{e^2/(4\pi\epsilon_o R^2)}{G m_e^2/R^2} = -\frac{e^2}{m_e^2}\frac{1}{4\pi\epsilon_o G} \approx -4.16 \times 10^{42}$$

i.e. $\mathbf{F}_e \gg \mathbf{F}_g$

where $G = 6.67 \times 10^{-11}\ [m^3 s^{-2} kg^{-1}]$ is the gravitational constant. This ratio is so huge that it exemplifies why electrical forces often dominates physical phenomena. What the minus sign is mean?

<u>Example:</u> A charge $Q_1 = 3\times10^{-4}$ C at M (1,2,3) and a charge of $Q_2 = -10^{-4}$ C at N (2,0,5) are located in a vacuum. Determine the force exerted on Q_2 by Q_1.

Solution:

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$$
$$= (2\mathbf{a}_x + 0\mathbf{a}_y + 5\mathbf{a}_z) - (1\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)$$
$$= 1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{1}{3}(1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$$

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R_{12}^2}\mathbf{a}_{12}$$

$$= \frac{1}{4\pi(8.854\times10^{-12})}\frac{(3\times10^{-4})(-10^{-4})}{3^2}\frac{1}{3}(1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$$

$$\cong \underline{\underline{-10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z \text{ N}}}$$

Electric field intensity:

If a very small positive charge Q_t (test charge) is located at a distance (R) from second charge Q_1 , hence the force exerted on the test charge is

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_o R_{1t}^2}\,\mathbf{a}_{1t} \quad (N)$$

and the force $\mathbf{F}_t$ per test charge is called the electric field intensity,

$$\mathbf{E_1} = \frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_o R_{1t}^2}\mathbf{a}_{1t} \quad (N/C) \;\; \boldsymbol{or} \;\; (V/m)$$

$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1 , that acts on a unit positive test charge. More generally, we write the defining expression:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}}$$

in which **E**, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity (meaning the electric field arising from the test charge itself is not included in **E**).

The electric field of a single point charge becomes:

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2}\mathbf{a}_R}$$

that R is the magnitude of the vector **R**, the directed line segment from the point at which the point charge Q is located to the point at which **E** is desired, and $\mathbf{a}_R$ is a unit vector in the **R** direction.

Let us arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r (the radius of sphere). Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_o r^2}\mathbf{a}_r$$

or

$$E_r = \frac{Q_1}{4\pi\epsilon_o r^2}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

If we consider a charge that is not at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$ as illustrated in Figure 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as r – r′, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

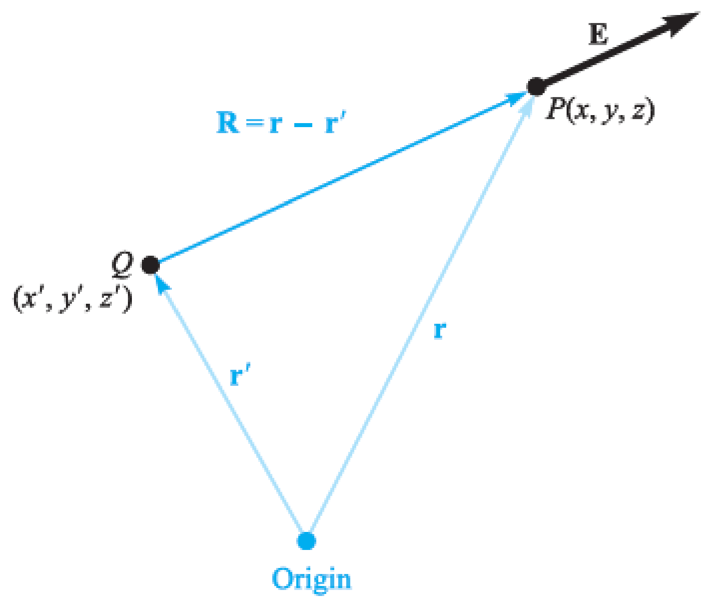


Figure 2.2 The vector r′ locates the point charge Q, the vector r identifies the general point in space P(x, y, z), and the vector R from Q to P(x, y, z) is then R = r – r′.

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Because the coulomb forces are linear, the **electric field intensity** arising from **two point charges**, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, $\mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Figure 2.3.

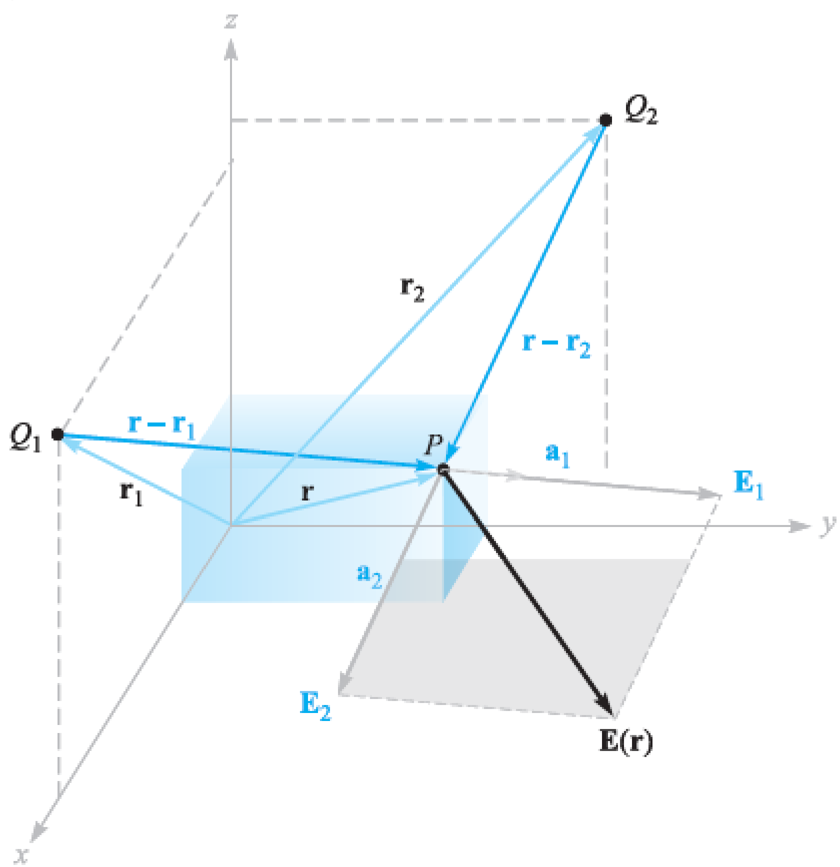


Figure 2.3 The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

If we add more charges at other positions, the **field** due to ***n*** **point charges** is

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n}\frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

Example:

Find **E** at $P(1, 1, 1)$ caused by four identical 3 nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

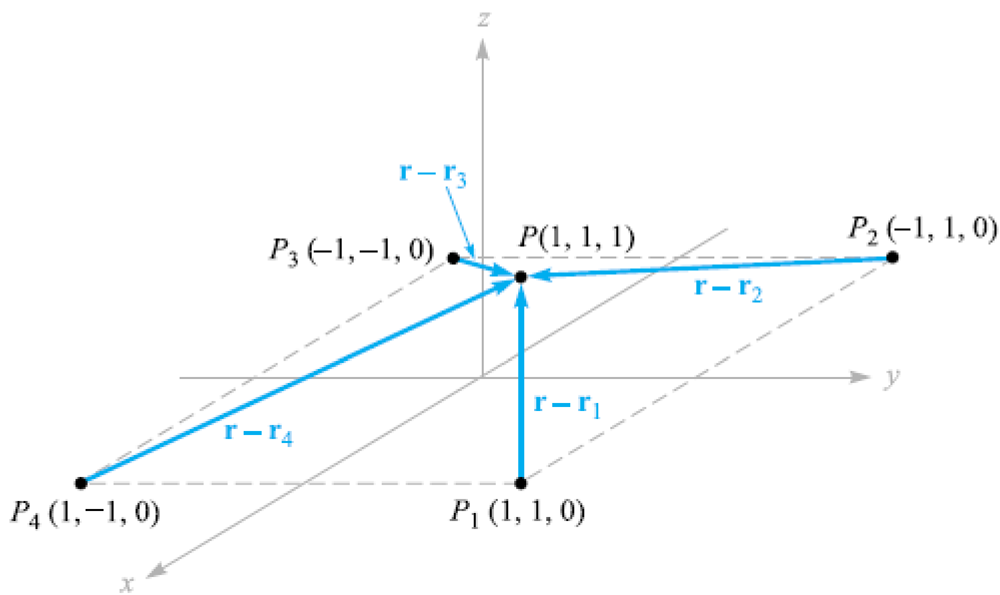


Figure 2.4 A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\ \mathrm{V \cdot m}$, we may now use (11) to obtain

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_m|^2} \mathbf{a}_m \qquad (11)$$

$$\mathbf{E} = 26.96 \left[\frac{\mathbf{a}_z}{1} \frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3} \frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} \right]$$

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

Field due to Continuous Charge Distribution

For a point charge Q,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

For an element charge dQ,

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

For continuous charge distribution

$$E = \int \frac{dQ}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

$$\rightarrow dQ = \begin{cases} \rho_v\, dv \\ \rho_s\, dS \\ \rho_l\, dl \end{cases}$$

1- **Field due to Volume Charge Distribution**

ρ_v = volume charge density , ($\rho_v = \frac{Q}{v}$),

having the units of coulombs per cubic meter ; (C/m^3).

The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v$$

and we may define ρ_v mathematically by using a limiting process on above eq.,

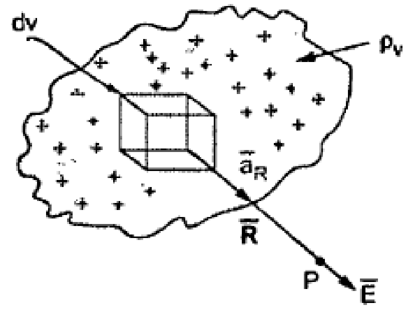
$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv} \qquad => \quad dQ = \rho_v dv$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{vol} \rho_v dv$$

Only one integral sign is customarily indicated, but the differential dv signifies integration throughout a volume, and hence a triple integration.

$dQ = \rho_v dv$



$$\therefore \quad \mathbf{E} = \int \frac{\rho_v dv}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

2- Field due to Surface Charge Distribution

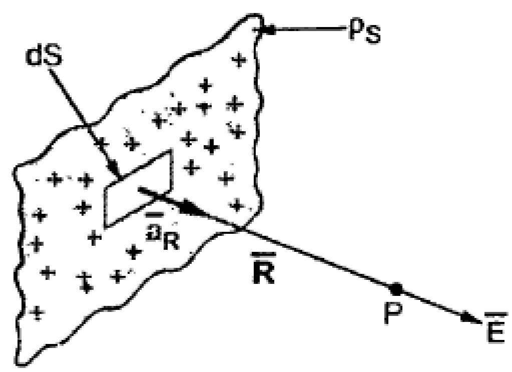
ρ_s= Surface charge density , $\left(\rho_s = \frac{Q}{s}\right)$

For small element

$$\rho_s = \frac{dQ}{dS}$$

$$dQ = \rho_s dS$$

$$\therefore \quad \mathbf{E} = \int \frac{\rho_s dS}{4\pi\epsilon_o R^2}\mathbf{a}_R$$



3- Field due to Line Charge Distribution

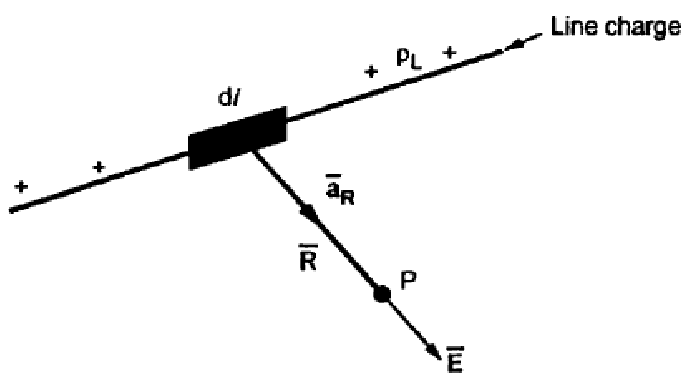
ρ_l = line charge density , $\left(\rho_l = \frac{Q}{l}\right)$

For small element

$$\rho_l = \frac{dQ}{dl}$$

$$dQ = \rho_l dl$$

$$\therefore \quad \mathbf{E} = \int \frac{\rho_l dl}{4\pi\epsilon_o R^2}\mathbf{a}_R$$



Standard charge distribution:

- Electric field intensity due to a **point charge**:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

- Electric field intensity due to **infinite straight line charge**:

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon_o R}\mathbf{a}_R$$

- Electric field intensity due to **infinite plane sheet charge**

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o}\mathbf{a}_R$$