4.1. The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40^\circ,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E}\cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6\times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20\times 10^{-6}\,\text{C})(100\,\text{V/m})(6\times 10^{-6}\,\text{m}) = -12\times 10^{-9}\,\text{J} = \underline{-12\,\text{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6\times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20\times 10^{-6})(-200)(6\times 10^{-6}) = 2.4\times 10^{-8}\,\text{J} = \underline{24\,\text{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6\times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20\times 10^{-6})(300)(6\times 10^{-6}) = -3.6\times 10^{-8}\,\text{J} = \underline{-36\,\text{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$dW = -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times 10^{-6})$$
$$= \underline{-44.9\,\text{nJ}}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$dW = -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times 10^{-6})$$
$$= -(20\times 10^{-6})\big[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y)$$
$$+\ 222.9\big]\,(6\times 10^{-6})$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40^\circ) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40^\circ) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40^\circ) = -0.643$. Substituting these results in

$$dW = -(20\times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times 10^{-6}) = \underline{-41.8\,\text{nJ}}$$

4.2. Let $\mathbf{E} = 400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z$ in the neighborhood of point $P(6, 2, -3)$. Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by:

a) $\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$: We write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z) \cdot \frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}(10^{-3})$$
$$= -\frac{(4 \times 10^{-3})}{\sqrt{3}}(400 - 300 + 500) = \underline{-1.39\,\text{J}}$$

b) $-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$: The computation is similar to that of part a, but we change the direction:

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z) \cdot \frac{(-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z)}{\sqrt{14}}(10^{-3})$$
$$= -\frac{(4 \times 10^{-3})}{\sqrt{14}}(-800 - 900 - 500) = \underline{2.35\,\text{J}}$$

4.3. If $\mathbf{E} = 120\,\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50 μm charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -(50 \times 10^{-6})\left[120\mathbf{a}_\rho \cdot \frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}\right](2 \times 10^{-3})$$
$$= -(50 \times 10^{-6})(120)\left[(\mathbf{a}_\rho \cdot \mathbf{a}_x) - (\mathbf{a}_\rho \cdot \mathbf{a}_y)\right]\frac{1}{\sqrt{3}}(2 \times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4°$. Thus $(\mathbf{a}_\rho \cdot \mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho \cdot \mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\,\mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\,\mu\text{J}}$ as in part a. This is also found by going through the same procedure as in part a, but with the direction (roles of P and Q) reversed.

4.4. Find the amount of energy required to move a 6-C charge from the origin to $P(3, 1, -1)$ in the field $\mathbf{E} = 2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z$ V/m along the straight-line path $x = -3z$, $y = x + 2z$: We set up the computation as follows, and find the the result *does not depend on the path*.

$$W = -q\int \mathbf{E} \cdot d\mathbf{L} = -6\int(2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z) \cdot (dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$
$$= -6\int_0^3 2x\,dx \;+\; 6\int_0^1 3y^2\,dy \;-\; 6\int_0^{-1} 4\,dz = \underline{-24\,\text{J}}$$

4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E} \cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6 \times 10^{-9})}{\epsilon_0} \ln\left(\frac{5}{4}\right) = \underline{-3.026\text{ V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho} d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho} d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6 \times 10^{-9}) + (.06)(2 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\text{ V}}$$

4.18. A nonuniform linear charge density, $\rho_L = 8/(z^2 + 1)$ nC/m lies along the z axis. Find the potential at $P(\rho = 1, 0, 0)$ in free space if $V = 0$ at infinity: This last condition enables us to write the potential at P as a superposition of point charge potentials. The result is the integral:

$$V_P = \int_{-\infty}^{\infty} \frac{\rho_L dz}{4\pi\epsilon_0 R}$$

where $R = \sqrt{z^2 + 1}$ is the distance from a point z on the z axis to P. Substituting the given charge distribution and R into the integral gives us

$$V_P = \int_{-\infty}^{\infty} \frac{8 \times 10^{-9} dz}{4\pi\epsilon_0 (z^2 + 1)^{3/2}} = \frac{2 \times 10^{-9}}{\pi\epsilon_0} \frac{z}{\sqrt{z^2 + 1}}\Big|_{-\infty}^{\infty} = \underline{144\text{ V}}$$

4.19. The annular surface, 1 cm $< \rho <$ 3 cm, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho$ nC/m^2. Find V at $P(0, 0, 2\text{ cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\int \frac{\rho_s\, da}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi} \int_{.01}^{.03} \frac{(5 \times 10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\epsilon_0 \sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5 \times 10^{-9})}{2\epsilon_0}\right] \left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2} \ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\text{ V}}$$

4.27. Two point charges, 1 nC at (0, 0, 0.1) and −1 nC at (0, 0, −0.1), are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0|\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0|\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\text{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0(.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0(.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}\left[2.42\mathbf{a}_x + 1.41\mathbf{a}_z\right]\ \text{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\text{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9^\circ$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9^\circ)}{4\pi\epsilon_0(.5)^2} = \underline{5.76\,\text{V}}$$

4.28. A dipole located at the origin in free space has a moment $\mathbf{p}2 \times 10^{-9}\,\mathbf{a}_z$ C · m. At what points on the line $y = z$, $x = 0$ is:

a) $|E_\theta| = 1$ mV/m? We note that the line $y = z$ lies at $\theta = 45^\circ$. Begin with

$$\mathbf{E} = \frac{2\times10^{-9}}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) = \frac{10^{-9}}{2\sqrt{2}\pi\epsilon_0 r^3}(2\mathbf{a}_r + \mathbf{a}_\theta)\ \text{at}\ \theta = 45^\circ$$

from which

$$E_\theta = \frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad\Rightarrow\quad r^3 = 1.27\times10^{-4}\ \text{or}\ r = 23.3\,\text{m}$$

The y and z values are thus $y = z = \pm 23.3/\sqrt{2} = \underline{\pm 16.5\,\text{m}}$

b) $|E_r| = 1$ mV/m? From the above field expression, the radial component magnitude is twice that of the theta component. Using the same development, we then find

$$E_r = 2\frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad\Rightarrow\quad r^3 = 2(1.27\times10^{-4})\ \text{or}\ r = 29.4\,\text{m}$$

The y and z values are thus $y = z = \pm 29.4/\sqrt{2} = \underline{\pm 20.8\,\text{m}}$

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0 \mathbf{E} \cdot \mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20 \left[\frac{1}{x^2yz}\mathbf{a}_x + \frac{1}{xy^2z}\mathbf{a}_y + \frac{1}{xyz^2}\mathbf{a}_z \right] \text{ V/m}$$

The energy is now

$$W_E = 200\epsilon_0 \int_1^2 \int_1^2 \int_1^2 \left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4} \right] dx\,dy\,dz$$

The integral evaluates as follows:

$$\begin{aligned} W_E &= 200\epsilon_0 \int_1^2 \int_1^2 \left[-\left(\frac{1}{3}\right) \frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4} \right]_1^2 dy\,dz \\ &= 200\epsilon_0 \int_1^2 \int_1^2 \left[\left(\frac{7}{24}\right) \frac{1}{y^2z^2} + \left(\frac{1}{2}\right) \frac{1}{y^4z^2} + \left(\frac{1}{2}\right) \frac{1}{y^2z^4} \right] dy\,dz \\ &= 200\epsilon_0 \int_1^2 \left[-\left(\frac{7}{24}\right) \frac{1}{yz^2} - \left(\frac{1}{6}\right) \frac{1}{y^3z^2} - \left(\frac{1}{2}\right) \frac{1}{yz^4} \right]_1^2 dz \\ &= 200\epsilon_0 \int_1^2 \left[\left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{1}{4}\right) \frac{1}{z^4} \right] dz \\ &= 200\epsilon_0(3) \left[\frac{7}{96} \right] = \underline{387\text{ pJ}} \end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0(3) \left[\frac{1}{(1.5)^4(1.5)^2(1.5)^2} \right] = 2.07 \times 10^{-10} \text{ J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207\text{ pJ}}$.

4.32. In the region of free space where $2 < r < 3$, $0.4\pi < \theta < 0.6\pi$, $0 < \phi < \pi/2$, let $\mathbf{E} = k/r^2\,\mathbf{a}_r$.

a) Find a positive value for k so that the total energy stored is exactly 1 J: The energy is found through

$$\begin{aligned} W_E &= \int_v \frac{1}{2}\epsilon_0 E^2\,dv = \int_0^{\pi/2} \int_{0.4\pi}^{0.6\pi} \int_2^3 \frac{1}{2}\epsilon_0 \frac{k^2}{r^2} r^2 \sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{\pi}{2}(-\cos\theta)\Big|_{.4\pi}^{.6\pi} \left(\frac{1}{2}\right)\epsilon_0 k^2 \left(-\frac{1}{r}\right)\Big|_2^3 = \frac{0.616\pi}{24}\epsilon_0 k^2 = 1\text{ J} \end{aligned}$$

Solve for k to find $k = \underline{1.18 \times 10^6 \text{ V}\cdot\text{m}}$.

4.32b. Show that the surface $\theta = 0.6\pi$ is an equipotential surface: This will be the surface of a cone, centered at the origin, along which **E**, in the $\mathbf{a}_r$ direction, will exist. Therefore, the given surface *cannot* be an equipotential (the problem was ill-conceived). Only a surface of constant r could be an equipotential in this field.

c) Find V_{AB}, given points $A(2, \theta = \pi/2, \phi = \pi/3)$ and $B(3, \pi/2, \pi/4)$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_2^3 \frac{k}{r^2}\mathbf{a}_r \cdot \mathbf{a}_r \, dr = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6}$$

Using the result of part a, we find $V_{AB} = (1.18 \times 10^6)/6 = \underline{197 \text{ kV}}$.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of 5 μC in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5 \times 10^{-6}}{4\pi r^2}\mathbf{a}_r \text{ C/m}^2$$

b) Calculate the total energy stored in the electrostatic field: Use

$$W_E = \int_{vol} \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \, dv = \int_0^{2\pi} \int_0^{\pi} \int_{.04}^{\infty} \frac{1}{2} \frac{(5 \times 10^{-6})^2}{16\pi^2\epsilon_0 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= (4\pi)\left(\frac{1}{2}\right)\frac{(5 \times 10^{-6})^2}{16\pi^2\epsilon_0}\int_{.04}^{\infty} \frac{dr}{r^2} = \frac{25 \times 10^{-12}}{8\pi\epsilon_0}\frac{1}{.04} = \underline{2.81 \text{ J}}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5 \times 10^{-6})^2}{2(2.81)} = 4.45 \times 10^{-12} \text{ F} = \underline{4.45 \text{ pF}}$$

4.34. Given the potential field in free space, $V = 80\phi$ V (note that $\mathbf{a}_phi$ should not be present), find:

a) the energy stored in the region $2 < \rho < 4$ cm, $0 < \phi < 0.2\pi$, $0 < z < 1$ m: First we find

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{80}{\rho}\mathbf{a}_\phi \text{ V/m}$$

Then

$$W_E = \int_v w_E dv = \int_0^1 \int_0^{0.2\pi} \int_{.02}^{.04} \frac{1}{2}\epsilon_0 \frac{(80)^2}{\rho^2} \rho \, d\rho \, d\phi \, dz = 640\pi\epsilon_0 \ln\left(\frac{.04}{.02}\right) = \underline{12.3 \text{ nJ}}$$

b) the potential difference, V_{AB}, for $A(3 \text{ cm}, \phi = 0, z = 0)$ and $B(3\text{cm}, 0.2\pi, 1\text{m})$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{.2\pi}^0 -\frac{80}{\rho}\mathbf{a}_\phi \cdot \mathbf{a}_\phi \, \rho \, d\phi = -80(0.2\pi) = \underline{-16\pi \text{ V}}$$

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1\left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right]dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1 - e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part b, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.2. Let the current density be $\mathbf{J} = 2\phi\cos^2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$ A/m^2 within the region $2.1 < \rho < 2.5$, $0 < \phi < 0.1$ rad, $6 < z < 6.1$. Find the total current I crossing the surface:

a) $\rho = 2.2$, $0 < \phi < 0.1$, $6 < z < 6.1$ in the $\mathbf{a}_\rho$ direction: This is a surface of constant ρ, so only the radial component of $\mathbf{J}$ will contribute: At $\rho = 2.2$ we write:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_0^{0.1} 2(2)\cos^2\phi\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,2d\phi dz = 2(2.2)^2(0.1)\int_0^{0.1}\frac{1}{2}(1+\cos 2\phi)\,d\phi$$
$$= 0.2(2.2)^2\left[\frac{1}{2}(0.1) + \frac{1}{4}\sin 2\phi\Big|_0^{0.1}\right] = \underline{97\,\text{mA}}$$

b) $\phi = 0.05$, $2.2 < \rho < 2.5$, $6 < z < 6.1$ in the $\mathbf{a}_\phi$ direction: In this case only the ϕ component of $\mathbf{J}$ will contribute:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_{2.2}^{2.5} -\rho\sin 2\phi|_{\phi=0.05}\,\mathbf{a}_\phi\cdot\mathbf{a}_\phi\,d\rho\,dz = -(0.1)^2\frac{\rho^2}{2}\Big|_{2.2}^{2.5} = \underline{-7\,\text{mA}}$$

5.18. Let us assume a field $\mathbf{E} = 3y^2z^3\,\mathbf{a}_x + 6xyz^3\,\mathbf{a}_y + 9xy^2z^2\,\mathbf{a}_z$ V/m in free space, and also assume that point $P(2, 1, 0)$ lies on a conducting surface.

a) Find ρ_v just adjacent to the surface at P:

$$\rho_v = \nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = 6xz^3 + 18xy^2z \text{ C/m}^3$$

Then at P, $\rho_v = \underline{0}$, since $z = 0$.

b) Find ρ_s at P:

$$\rho_s = \mathbf{D} \cdot \mathbf{n}\Big|_P = \epsilon_0 \mathbf{E}\dot{\mathbf{n}}\Big|_P$$

Note however, that this computation involves evaluating $\mathbf{E}$ at the surface, yielding a value of 0. Therefore the surface charge density at P is $\underline{0}$.

c) Show that $V = -3xy^2z^3$ V: The simplest way to show this is just to take $-\nabla V$, which yields the given field: A more general method involves deriving the potential from the given field: We write

$$E_x = -\frac{\partial V}{\partial x} = 3y^2z^3 \;\Rightarrow\; V = -3xy^2z^3 + f(y, z)$$

$$E_y = -\frac{\partial V}{\partial y} = 6xyz^3 \;\Rightarrow\; V = -3xy^2z^3 + f(x, z)$$

$$E_z = -\frac{\partial V}{\partial z} = 9xy^2z^2 \;\Rightarrow\; V = -3xy^2z^3 + f(x, y)$$

where the integration "constants" are functions of all variables other than the integration variable. The general procedure is to adjust the functions, f, such that the result for V is the same in all three integrations. In this case we see that $f(x, y) = f(x, z) = f(y, z) = 0$ accomplishes this, and the potential function is $V = -3xy^2z^3$ as given.

d) Determine V_{PQ}, given $Q(1, 1, 1)$: Using the potential function of part c, we have

$$V_{PQ} = V_P - V_Q = 0 - (-3) = \underline{3\text{ V}}$$

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At 0 V}: \quad 2x^2y - z = 0$$

$$\text{At 60 V}: \quad 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

7.7. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$: Use Poisson's equation:

$$\rho_v = -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} \right)$$
$$= -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{-\cos 2\phi}{\rho} \right) - \frac{4}{\rho^2} \frac{\cos 2\phi}{\rho} \right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120^\circ)}{0.5^3} = -12\epsilon_0 = \underline{-106\,\text{pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30^\circ, 1)$: First, we find $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi$$
$$= \frac{\cos 2\phi}{\rho^2} \mathbf{a}_\rho + \frac{2 \sin 2\phi}{\rho^2} \mathbf{a}_\phi$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60^\circ}{4} \mathbf{a}_\rho + \frac{2 \sin 60^\circ}{4} \mathbf{a}_\phi = 0.125\, \mathbf{a}_\rho + 0.433\, \mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm|\mathbf{D}_B| = \pm\epsilon_0 |\mathbf{E}_B| = \pm 0.451\epsilon_0 = \underline{\pm 0.399\,\text{pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

7.21. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2\frac{dV}{dr}\right) = -\frac{200}{.6}r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2E_r \to 0$ when $r \to 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4}r^{-.4} + C_2$$

From our second boundary condition, $V \to 0$ as $r \to \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4}\text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi \int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0\frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3r^{-1.4}\text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4}dr' = \underline{833.3r^{-.4}\text{ V}}$$

8.7. Given points $C(5, -2, 3)$ and $P(4, -1, 2)$; a current element $I\,d\mathbf{L} = 10^{-4}(4, -3, 1)$ A · m at C produces a field $d\mathbf{H}$ at P.

a) Specify the direction of $d\mathbf{H}$ by a unit vector $\mathbf{a}_H$: Using the Biot-Savart law, we find

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_{CP}}{4\pi R_{CP}^2} = \frac{10^{-4}[4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z] \times [-\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z]}{4\pi 3^{3/2}} = \frac{[2\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{a}_z] \times 10^{-4}}{65.3}$$

from which

$$\mathbf{a}_H = \frac{2\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{a}_z}{\sqrt{14}} = \underline{0.53\mathbf{a}_x + 0.80\mathbf{a}_y + 0.27\mathbf{a}_z}$$

b) Find $|d\mathbf{H}|$.

$$|d\mathbf{H}| = \frac{\sqrt{14} \times 10^{-4}}{65.3} = 5.73 \times 10^{-6} \text{ A/m} \quad = \underline{5.73\ \mu\text{A/m}}$$

c) What direction $\mathbf{a}_l$ should $I\,d\mathbf{L}$ have at C so that $d\mathbf{H} = 0$? $I\,d\mathbf{L}$ should be collinear with $\mathbf{a}_{CP}$, thus rendering the cross product in the Biot-Savart law equal to zero. Thus the answer is $\mathbf{a}_l = \underline{\pm(-\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z)/\sqrt{3}}$