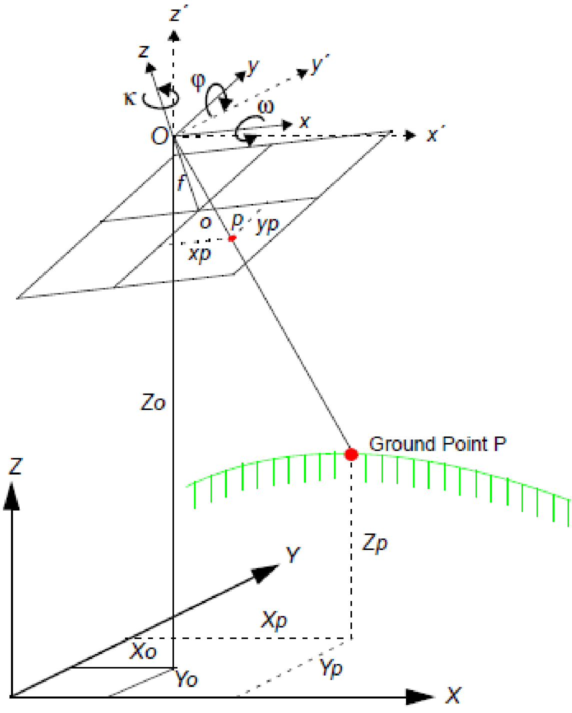
1. Collinearity Equation

- The functional model of the imaging system will be realized in the condition equation,
- Condition equations are related to image points, object points, and imaging system parameters.
- The condition equation is used for different purposes, depending on which variables are considered observables, known, or unknowns in the stochastic model.
- An example is the collinearity condition equation, which may be used for **space resection, space intersection, relative orientation, and other tasks.**
- The below figure is the geometry of a single point in a frame camera. As shown, the fundamental characteristic of frame imaging is that the perspective centre, the image point and the corresponding object point all lie on a line in space.



These image and object coordinate system will be related by three positional parameters and three orientation parameters, which can be expressed as follow:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = kM \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} ----(1-1)$$

The value of the z is constant in the negative principle of distance

$$\begin{bmatrix} x - xo \\ y - yo \\ -f \end{bmatrix} = kM \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} ----(1-2)$$

By expressing the Matrix **M** in its elements then, we can get:

$$\begin{bmatrix} x - xo \\ y - yo \\ -f \end{bmatrix} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} ----(1-3)$$

By multiplying the matrix by the vector and eliminating the **k** factor, it is possible to obtain a classical form of the collinearity equations:

$$x_{a1} - x_o = -f.\left[\frac{m_{11}(X_A - X_{o1}) + m_{12}(Y_A - Y_{o1}) + m_{13}(Z_A - Z_{o1})}{m_{31}(X_A - X_{o1}) + m_{32}(Y_A - Y_{o1}) + m_{33}(Z_A - Z_{o1})}\right] - - - - (1-4)$$

$$y_{a1} - y_o = -f.\left[\frac{m_{21}(X_A - X_{o1}) + m_{22}(Y_A - Y_{o1}) + m_{23}(Z_A - Z_{o1})}{m_{31}(X_A - X_{o1}) + m_{32}(Y_A - Y_{o1}) + m_{33}(Z_A - Z_{o1})}\right]$$

The above collinearity equations are used to determine the corresponding image coordinates when the interior, exterior orientation and object point are known.

Also can be written as

$$X - X_L = (Z - Z_L)\frac{m_{11}(x - x_o) + m_{21}(y - y_o) + m_{31}(-f)}{m_{13}(x - x_o) + m_{23}(y - y_o) + m_{33}(-f)} - - - - (1-5)$$

$$Y - Y_L = (Z - Z_L)\frac{m_{12}(x - x_o) + m_{22}(y - y_o) + m_{32}(-f)}{m_{13}(x - x_o) + m_{23}(y - y_o) + m_{33}(-f)}$$

The above collinearity equations are used to determine two components of object space (e.g. X and Y) given the interior, exterior orientations, and image coordinates in addition to the rest component of the object space.

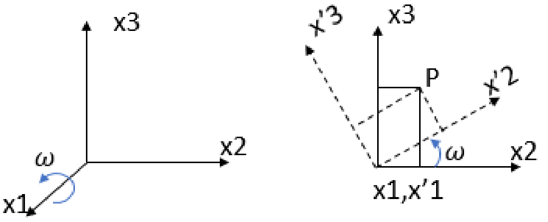
1.1. Derivation of the Rotation Matrix

Using the three rotation angles, the relationship between the image space coordinate system (*x, y,* and *z*) and ground space coordinate system (*X, Y,* and *Z;* or *x', y',* and *z'*) can be determined.

A 3 × 3 matrix defining the relationship between the two systems is used. This is referred to as the orientation or rotation matrix, *M*. The rotation matrix can be defined as follows:

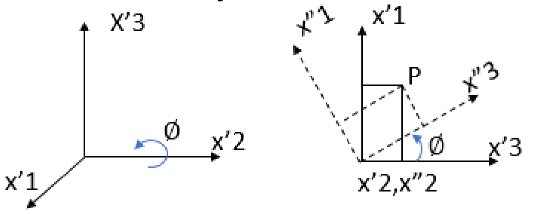
To derive the rotation matrix *M*, three rotations are performed sequentially: a primary rotation ω around the x-axis, followed by a secondary rotation ϕ around the y-axis, and a tertiary rotation κ around the z-axis.

Each of the three elementary rotations is represented in matrix form as follows:

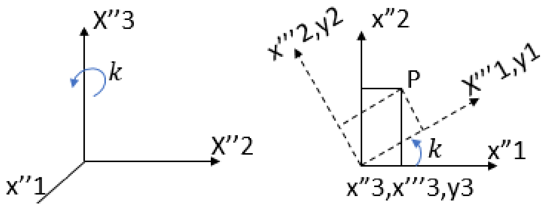


$$\begin{bmatrix} x'1 \\ x'2 \\ x'3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & sin\omega \\ 0 & -sin\omega & cos\omega \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = M\omega \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

Where x1,x2, and x3 are the coordinates before rotation and x'1,x'2 and x'3 are the coordinates after rotation. Similarly, a rotation of Ø about the x'2 axis and k about the x''3 axis are given by.



$$\begin{bmatrix} x''1 \\ x''2 \\ x''3 \end{bmatrix} = \begin{bmatrix} cosØ & 0 & -sinØ \\ 0 & 1 & 0 \\ sinØ & 0 & cosØ \end{bmatrix} \begin{bmatrix} x'1 \\ x'2 \\ x'3 \end{bmatrix} = MØ \begin{bmatrix} x'1 \\ x'2 \\ x'3 \end{bmatrix}$$



$$\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \begin{bmatrix} x'''1 \\ x'''2 \\ x'''3 \end{bmatrix} = \begin{bmatrix} cosk & sink & 0 \\ -sink & cos\,k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x''1 \\ x''2 \\ x''3 \end{bmatrix} = Mk \begin{bmatrix} x''1 \\ x''2 \\ x''3 \end{bmatrix}$$

By combining the above equations a relationship can be defined between the coordinates of an object point (P) relative to the (X, Y, Z) and $(X_{\omega\phi\kappa}, Y_{\omega\phi\kappa}, Z_{\omega\phi\kappa})$

$$\boldsymbol{P = M_{\omega} \times M_{\phi} \times M_{\kappa} \times P_{\omega\phi\kappa}}$$

By replacing $M_{\omega} \times M_{\phi} \times M_{\kappa}$ *by M*, M is a 3 × 3matrix

$$\boldsymbol{M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}}$$

Where each entry of the matrix can be computed by equations 8-6:

$$\boldsymbol{m_{11} = cos(Ø)\ cos\ (\kappa)}$$

$$\boldsymbol{m_{12} = sin(\omega\ )sin\ (\phi)cos(\kappa) + cos(\omega)sin(\kappa)}$$

$$\boldsymbol{m_{13} = -cos(\omega)sin(\phi)cos(\kappa) + sin(\omega)sin(\kappa)}$$

$$\boldsymbol{m_{21} = -cos(Ø)\ sin\ (\kappa)}$$

$$\boldsymbol{m_{22} = -sin(\omega)sin(Ø)\ sin\ (\kappa) + cos(\omega)cos(\kappa)}$$

$$\boldsymbol{m_{23} = cos(w)sin(\phi)sin(\kappa) + sin(\omega)cos(\kappa)}$$

$$\boldsymbol{m_{31} = sin(Ø)}$$

$$\boldsymbol{m_{32} = -sin(\omega)\ cos(\phi)}$$

$$\boldsymbol{m_{33} = cos(\omega)cos(\phi)}$$

Example 1 For a photograph with exterior orientation elements $(\omega, \phi, \kappa) = (2{,}5{,}15)$

Degrees and $(X_L, Y_L, Z_L) = (5000m, 10{,}000m, 2000m)$ and camera parameters (x_o, y_o, f) =(0.015, -0.0220, 152.4)mm, compute via the collinearity equations the coordinates of the ground point (X,Y,Z)=(5100, 9800, 100)in the fiducial-based image system.

Solution: Evaluating the rotation matrix, using equation s yields:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} 0.9622 & 0.2616 & -0.0751 \\ -0.2578 & 0.9645 & 0.0562 \\ 0.0871 & -0.0348 & 0.9956 \end{bmatrix}$$

Using equations (1-4) to obtain the image coordinates with respect to the [[principle point yields (x-xo, y-yo) =(15.159, -26.449)mm.

$$x_a - x_o = -152.4.\left[\frac{0.9622(5100-5000)+0.2616(9800-10000)+-0.0751(100-2000)}{0.0871(5100-5000)+-0.0348(9800-10000)+0.9956(100-2000)}\right]$$
$$= 15.158mm$$

xa=15.158+0.015=15.173mm

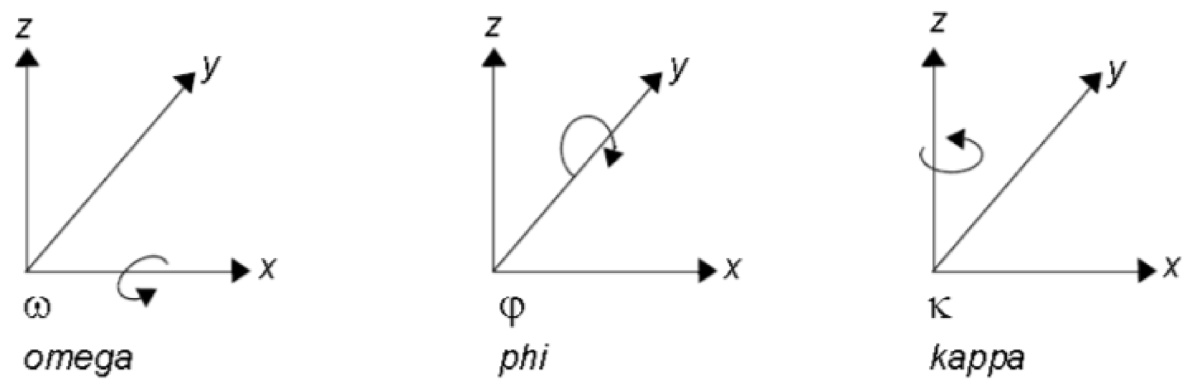
$$y_a - y_o = -152.4.\left[\frac{-0.2578(5100-5000)+.9645(9800-10000)+0.0562(100-2000)}{0.0871(5100-5000)+-0.0348(9800-10000)+0.9956(100-2000)}\right]$$
$$= -26.440mm$$

ya =-26.440+(-0.0220)=-26.462mm

The coordinates in the fiducial-based system are (xa,ya)=(15.174, -26.469)mm.

2. Exterior orientation

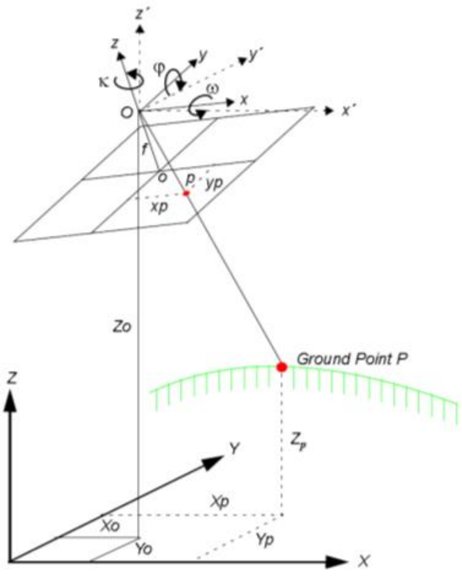
- Exterior orientation defines the camera's position and angular orientation at the time that the image is captured.
- The variables defining the position and orientation of an image are referred to as the elements of exterior orientation.
- The positional elements of exterior orientation include $\omega, \varphi,$ and, κ, and the position of the perspective centre (O) Xo,Yo,and Zo, with respect to the ground space coordinate system (X, Y, and Z).
- Zo is commonly referred to as the height of the camera above sea level, which is commonly defined by a datum.
- Three rotation angles are shown below.



- Omega is a rotation about the photographic x-axis
- Phi is a rotation about the photographic y-axis
- Kappa is a rotation about the photographic z-axis

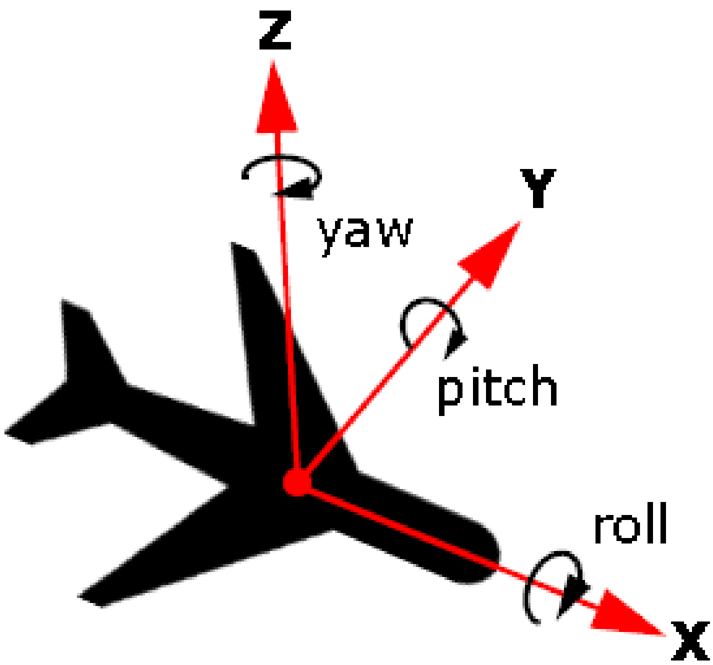
Rotations are positive if they are counterclockwise when viewed from the positive end of their respective axis.

The angular or rotational exterior orientation elements describe the relationship between the ground space coordinate system (*X*, *Y*, and *Z*) and the image space coordinate system (*x*, *y*, and *z*). The final element of the exterior orientation of the image is shown below:



As each frame is exposed precise information is captured (or calculated in post-processing) on the x, y, z and roll, pitch, yaw of the aircraft. By using GPS together with airborne surveys, which is attached with the inertial navigation system (INS).

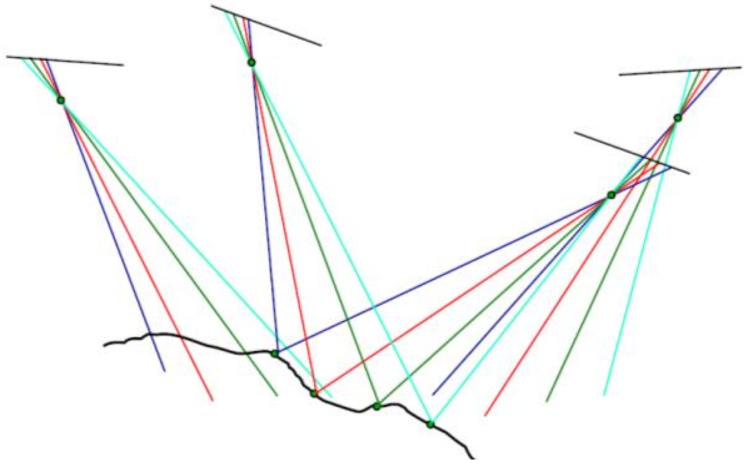
Each image in the aerial survey block has initial exterior orientation parameters, which minimize the need for control in a block triangulation process.



Exterior orientation can be divided into two types, relative and absolute orientations:

2.1. Relative (Exterior) Orientation

- It is the process of creating a stereo model using a pair of images taken for the same scene from different locations.
- Analytical relative orientation is the process of determining the relative angular attitude and positional displacement between the photographs that existed when photos were taken.



- It involves determining certain exterior orientation elements and calculating the remaining ones.
- In analytical relative orientation, it is common practice to fix the exterior orientation elements $\omega, \varphi, \kappa, XL \; and \; YL$ of the left photo of the stereo pair to zero values. The ZL value of the left value is set equal to f, and XL of the right photo is equal to the photo base b.
- The scale of the stereo model is approximately equal to the photo scale.

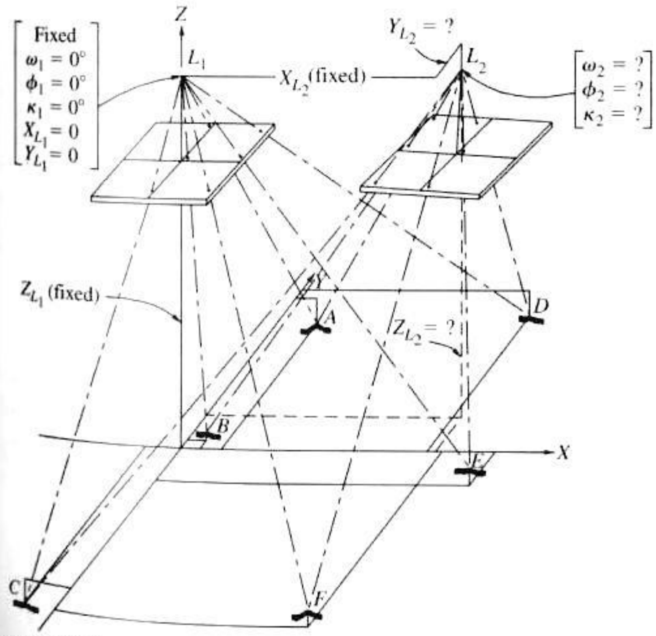


Figure 11-4
Analytical relative orientation of a stereopair

The coplanarity condition equation can be used for analytical relative orientation, but the collinearity condition is more commonly applied.

Example: A stereopair of near-vertical photograph is taken with a 152.114mm focal length camera. Photo coordinates of the images of six points in the overlap area are listed in the following table. Perform analytical relative orientation of the stereo pair.

Point	Left Photo Coordinates		Right Photo Coordinates	
	x, mm	y, mm	x, mm	y, mm
a	–4.870	1.992	–97.920	–2.910
b	89.296	2.706	–1.485	–1.836
c	0.256	84.138	–90.906	78.980
d	90.328	83.854	–1.568	79.482
e	–4.673	–86.815	–100.064	–95.733
f	88.591	–85.269	–0.973	–94.312

Solution

1. With an ASCII text editor, create the following data file with a ".dat" extension:

```
152.113
a  -4.870    1.992   -97.920   -2.910
b  89.296    2.706    -1.485   -1.836
c   0.256   84.138   -90.906   78.980
d  90.328   83.854    -1.568   79.482
e  -4.673  -86.815  -100.064  -95.733
f  88.591  -85.269    -0.973  -94.312
```

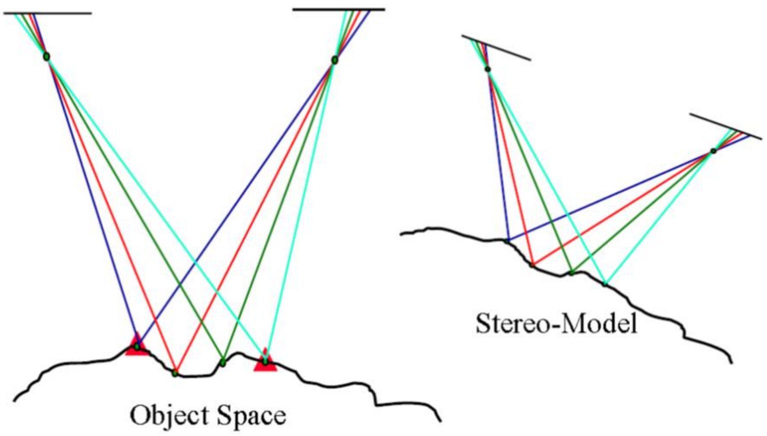
The first line of data in the input file is the camera focal length. The information on each of the following lines, from left to right, consists of the point identification, its x and y photo coordinates on the left photo, and its x and y photo coordinates on the right photo.

2. Run the relor program to produce the following results:

```
Exterior orientation parameters:
Parameter   Left pho  Right pho  SD right
Omega(deg)    0.0000    2.4099    0.0171
Phi(deg)      0.0000    0.5516    0.0181
Kappa(deg)    0.0000   -0.2067    0.0084
XL            0.0000   91.9740
YL            0.0000   -1.7346    0.0545
ZL          152.1130  148.3015    0.0196

Object space coordinates:
point      X        Y         Z      SD-X    SD-Y    SD-Z
  a   -4.8352   1.9730   1.0888  0.0127  0.0107  0.0975
  b   89.0970   2.7047   0.3391  0.0464  0.0109  0.0813
  c    0.2542  83.5234   1.1159  0.0117  0.0522  0.1001
  d   89.2672  82.8667   1.7862  0.0469  0.0488  0.0809
  e   -4.6333 -86.0755   1.2917  0.0126  0.0555  0.1032
  f   89.3101 -85.9635  -1.2348  0.0491  0.0528  0.0866

Photo coordinate residuals:
point   xl-res    yl-res    xr-res   yr-res
  a    -0.0001   -0.0048    0.0001   0.0047
  b     0.0001    0.0048   -0.0001  -0.0047
  c     0.0001    0.0026   -0.0001  -0.0027
  d    -0.0001   -0.0026    0.0001   0.0027
  e    -0.0000    0.0023    0.0000  -0.0022
  f     0.0000   -0.0023   -0.0000   0.0022
RMS     0.0001    0.0034    0.0001   0.0034

Standard error of unit weight: 0.0118
Degrees of freedom: 1
```

2.2. Absolute (exterior) orientation

It is the process of establishing the relationship between the model space and object space coordinates systems, so it can be ready for mapping and infer coordinates.



To achieve absolute orientations, seven parameters are involved,

- A uniform scale,
- Three translation, and,
- Three rotations.

In the analytical stereo plotter, the absolute orientation is achieved by adding:

- Two horizontal control points in the model provide the scale, the translation along the X and Y axes, and the k rotation around the Z-axis.
- Three elevation control points provide levelling information (the ωrotation around x and ϕ rotation around y axes, and Z transition.

more points provide redundancy, which enables a least-square solution.

Once the transformation parameters have been computed, they can be applied to the remaining stereo model points, including the X_L, Y_L, and Z_L coordinates of the left and right photographs. Thus operation gives the coordinates of all stereo model points in the ground system.

Example: Ground coordinates in a local vertical system for three control points are listed in the table below. For the result of the analytical relative orientation of the above example, perform analytical absolute orientation using a three-dimensions; conformal coordinate transformation.

Point	X, m	Y, m	Z, m
C	9,278.062	10,482.868	59.741
E	9,269.903	9,922.635	69.799
F	9,580.264	9,927.325	66.109

Solution

1. With an ASCII text editor, create the following data file with a ".dat" extension:

```
C    0.2542    83.5234    1.1159  9278.062  10482.868  59.741
E   -4.6333   -86.0755    1.2917  9269.903   9922.635  69.799
F   89.3101   -85.9635   -1.2348  9580.264   9927.325  66.109
#
A   -4.8352     1.9730    1.0888
B   89.0970     2.7047    0.3391
D   89.2672    82.8667    1.7862
Lpho  0  0  152.113
Rpho 91.9740 -1.7346 148.3015
#
```

For the above input file, the first three lines relate to the control points. From left to right the data include the point identification; its x, y, and z stereomodel coordinates; and its X, Y, and Z ground coordinates. The first # sign signifies that all control has been entered, and that the data following pertain to stereomodel points whose coordinates are to be transformed into the ground system. Each data line consists of the point identification, followed by its x, y, and z stereomodel coordinates. The second # sign completes the data.

2. Run the "3dconf" program to produce the following results:

```
Residuals:
Point   X res   Y res  Z res
    C   0.009   0.006 -0.000
    E   0.003  -0.023  0.000
    F  -0.012   0.017 -0.000

Standard Error of Unit Weight: 0.02335

Final Results:
Param       Value      Stan.Err.
scale      3.30297     0.00015
omega     -0.9819d     0.0033d
  phi     -0.8745d     0.0061d
kappa      0.8166d     0.0026d
   Tx    9281.220      0.015
   Ty   10206.994      0.015
   Tz      60.830      0.016

Transformed Points:
Point       X          Y          Z     SDev.X  SDev.Y  SDev.Z
    A   9265.105   10213.339   64.073   0.015   0.015   0.017
    B   9575.295   10220.215   66.213   0.017   0.017   0.028
    D   9572.011   10485.010   66.406   0.023   0.023   0.039
 Lpho   9273.552   10215.603  563.122   0.055   0.033   0.028
 Rpho   9577.546   10214.067  555.197   0.055   0.033   0.036
```

The values listed in the top table of the output are residuals in the X, Y, and Z control point coordinates. The center table lists the seven parameters of the three-dimensional conformal

Note: in the exterior orientation, the solution is found by using least-squares AX=L+V

A is the matrix of the coefficients, X is the unknown and the L is a matrix of the constant term. V is the matrix of the residuals.