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Salahaddin University-Erbil

Maximal chain of ideals of a ring

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. In Mathematics

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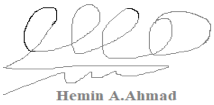
Certification of the Supervisor

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature: 

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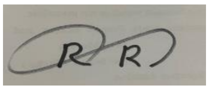
In view of the available recommendations, I forward this report for debate by the examining committee.



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Acknowledgment

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Abstract

In this project, we introduce a new method to find the wiener polynomial and wiener index of maximal ideal graphs $m(\mathbb{Z}_n)$ of rings $\mathbb{Z}_n$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k$.

Table of Contents

Certification of the Supervisor ii

Acknowledgment iii

Abstract iv

INTRODUCTION 1

CHAPTER ONE 2

Definitions and Backgrounds of ring theory 2

CHAPTER TWO 4

Definitions and Backgrounds of Graph Theory 4

CHAPTER THREE 7

References 25

پوختە .... a

الخلاصة .... a

INTRODUCTION

Let R be a ring. An ideal I_1 of R is maximal in an ideal I_2 of R if there is no ideal I_3 of R such that $I_1 \subset I_3 \subset I_2$ (Ahmad and Hummadi 2023). A chain of proper ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ of R is called maximal chain of ideals of R if I_{t-1} is maximal in I_t for each $t \in \mathbb{Z}^+$. The maximal ideal graph of R, denoted by $m(R)$, is the undirected graph with vertex set, the set of all ideals of R, where two vertices I and J are adjacent if and only if I maximal in J, or J maximal in I (Ahmad and Hummadi 2023). Let $d(u,v)$ denote the distance between vertices u and v in a graph G. The Wiener index of G is defined as $W(G) = \sum_{\{u,v\}} d(u,v)$ where the sum is over all unordered pairs $\{u,v\}$ of distinct vertices in G and the Wiener polynomial (with a parameter x) of G is $W(G;x) = \sum_{\{u,v\}} x^{d(u,v)}$ where the sum is taken over the same set of pairs (Sagan , Yeh and Zhang 1996). In the chapter three we introduce a new method to find diameter**,** Wiener index and the Wiener polynomial of maximal ideal graphs $m(\mathbb{Z}_n)$ where $n = p_1{}^{\alpha_1}\, p_2{}^{\alpha_2}\, p_3{}^{\alpha_3} \ldots p_k{}^{\alpha_k}$, p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$ and $1 \leq i \leq k$.

CHAPTER ONE

Definitions and Backgrounds of ring theory

Definition 1.1 (M and I 1969, 1)**.** A ring R is a set with two binary operations (addition and multiplication) such that

1) R is an abelian group with respect to addition (so that R has a zero element, denoted by 0, and every $x \in R$ has an (additive) inverse, $-x$).

2) Multiplication is associative ($(xy)z = x(yz)$) and distributive over addition ($x(y+z) = xy + xz, (y+z)x = yx + zx$).

We shall consider only rings which are commutative:

3) $xy = yx$ for all $x, y \in R$,

and have an identity element (denoted by 1):

4) $\exists 1 \in R$ such that $x1 = 1x = x$ for all $x \in R$.

Example 1.2 (Dummit and Foote 2004, 224)**.**

1. The ring of integers $\mathbb{Z}$, under the usual operations of addition and multiplication is a commutative ring with identity (the integer 1).
2. The quotient group $\mathbb{Z}/n\,\mathbb{Z}$ is a commutative ring with identity (the element 1) under the operations of addition and multiplication of residue classes.

Definition 1.3 (Dummit and Foote 2004, 228)**.** A subring of the ring R is a subgroup of R that is closed under multiplication.

Definition 1.4 (Dummit and Foote 2004, 242)**.** Let R be a ring, let I be a subset of R and let $r \in R$.

1) $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$.

2) A subset I of R is a left ideal of R if

a. I is a subring of R, and

b. I is closed under left multiplication by elements from R, i.e., $rI \subseteq I$ for all $r \subseteq R$.

Similarly I is a right ideal if (a) holds and in place of (b) one has

c. I is closed under right multiplication by elements from R, i.e., $Ir \subseteq I$ for all $r \in R$.

3) A subset I that is both a left ideal and a right ideal is called an ideal (or, for added emphasis, a two-sided ideal) of R.

Example 1.5. Consider the ring of all rational numbers $\mathbb{Q}$. Then $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but it is not an ideal of $\mathbb{Q}$.

Definition 1.6 (Dummit and Foote 2004, 255)**.** Assume R is commutative. An ideal P is called a prime ideal if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Definition 1.7 (Dummit and Foote 2004, 253)**.** An ideal M in an arbitrary ring R is called a maximal ideal if $M \neq R$ and the only ideals containing M are M and R.

CHAPTER TWO

Definitions and Backgrounds of Graph Theory

Definition 2.1 (Gross, Yellen and Zhang 2014, 2)**.** A graph $G = (V, E)$ consists of two sets $V$and E.

1) The elements of V are called vertices (or nodes).
2) The elements of E are called edges.
3) Each edge has a set of one or two vertices associated to it, which are called its endpoints. An edge is said to join its endpoints.

Definition 2.2 (Naduvath 2017, 23)**.** A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a closed walk.

Definition 2.3 (Naduvath 2017, 23)**.** A trail is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A tour is a trail that begins and ends on the same vertex.

Definition 2.4 (Naduvath 2017, 23)**.** A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A cycle or a circuit is a path that begins and ends on the same vertex.

Definition 2.5 (Naduvath 2017, 23)**.** The length of a walk or circuit or path or cycle is the number of edges in it.

Definition 2.6 (Naduvath 2017, 24)**.** The distance between two vertices u and v in a graph G, denoted by $d_G(u; v)$ or simply $d(u; v)$, is the length (number of edges) of a shortest path (also called a graph geodesic) connecting them. This distance is also known as the geodesic distance.

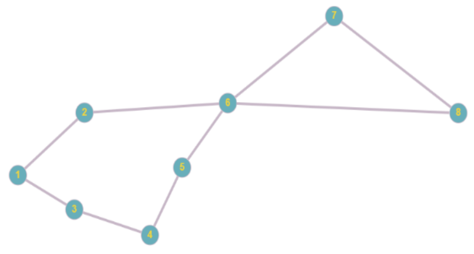
Definition 2.7 (Naduvath 2017, 24)**.** The eccentricity of a vertex v, denoted by $\varepsilon(v)$, is the greatest geodesic distance between v and any other vertex. It can be thought of as how far a vertex is from the vertex most distant from it in the graph.

Definition 2.8 (Naduvath, 2017, p. 24). The radius r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = \min_{v \in V(G)} \varepsilon(v)$.

Definition 2.9 (Naduvath 2017, 24)**.**The diameter of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = \max_{v \in V(G)} \varepsilon(v)$.

Example 2.10 The following figure illustrates a graph with eight vertices

$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and nine edges $E = \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8)\}$.



1. The distance between elements are as follows:

$d(1,2) = 1, d(1,3) = 1, d(1,4) = 2, d(1,5) = 3, d(1,6) = 2, d(1,7) = 3,$
$d(1,8) = 3, d(2,3) = 2, d(2,4) = 3, d(2,5) = 2, d(2,6) = 1, d(2,7) = 2,$
$d(2,8) = 2, d(3,4) = 1, d(3,5) = 2, d(3,6) = 3, d(3,7) = 4, d(3,8) = 4,$
$d(4,5) = 1, d(4,6) = 2, d(4,7) = 3, d(4,8) = 3, d(5,6) = 1, d(5,7) =$
$2, d(5,8) = 2, d(6,7) = 1, d(6,8) = 1, d(7,8) = 1.$

2. The eccentricity of vertices are as follows:

$$\varepsilon(1) = Max\{d(1,2), d(1,3), d(1,4), d(1,5), d(1,6), d(1,7), d(1,8)\}$$

$$= Max\{1, 2, 3\} = 3.$$

$$\varepsilon(2) = Max\{d(2,1), d(2,3), d(2,4), d(2,5), d(2,6), d(2,7), d(2,8)\}$$

$$= Max\{1, 2, 3\} = 3.$$

$$\varepsilon(3) = Max\{d(3,1), d(3,2), d(3,4), d(3,5), d(3,6), d(3,7), d(3,8)\}$$

$$= Max\{1, 2, 3,4\} = 4.$$

$$\varepsilon(4) = Max\{d(4,1), d(4,2), d(4,3), d(4,5), d(4,6), d(4,7), d(4,8)\}$$

$$= Max\{1, 2, 3\} = 3.$$

$$\varepsilon(5) = Max\{d(5,1), d(5,2), d(5,3), d(5,4), d(5,6), d(5,7), d(5,8)\}$$

$$= Max\{1, 2, 3\} = 3.$$

$$\varepsilon(6) = Max\{d(6,1), d(6,2), d(6,3), d(6,4), d(6,5), d(6,7), d(6,8)\}$$

$$= Max\{1, 2, 3\} = 3.$$

$$\varepsilon(7) = Max\{d(7,1), d(7,2), d(7,3), d(7,4), d(7,5), d(7,6), d(7,8)\}$$

$$= Max\{1, 2, 3,4\} = 4.$$

$$\varepsilon(8) = Max\{d(8,1), d(8,2), d(8,3), d(8,4), d(8,5), d(8,6), d(8,7)\}$$

$$= Max\{1, 2, 3,4\} = 4.$$

3. The radius of a graph G is

$$rad(G) = min\{\varepsilon(1), \varepsilon(2), \varepsilon(3), \varepsilon(4), \varepsilon(5), \varepsilon(6), \varepsilon(7), \varepsilon(8)\}$$
$= min\{3, 3, 4, 3, 3, 3, 4, 4\}$. So that $red(G) = 3.$

4. The diameter of a graph G is

$$Diam(G) = max\{\varepsilon(1), \varepsilon(2), \varepsilon(3), \varepsilon(4), \varepsilon(5), \varepsilon(6), \varepsilon(7), \varepsilon(8)\}$$
$= max\{3, 3, 4, 3, 3, 3, 4, 4\}$. So that $Diam(G) = 4.$

CHAPTER THREE

In this chapter, we study maximal chain of ideals of rings $\mathbb{Z}_{\mathrm{n}}$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k \neq 1$. Then we find the maximal ideal graph $m(\mathbb{Z}_n)$ of the ring $\mathbb{Z}_n$ for some $n \in \mathbb{Z}^+$. Finally the Wiener index, Wiener polynomial, dimeter and radical of the maximal ideal graphs $m(\mathbb{Z}_n)$ are investigated.

Definition 3.1 (Ahmad and Hummadi 2023)**.** An ideal H_1 of a ring R is maximal in an ideal H_2 of R if there is no ideal H_3 of R such that $H_1 \subset H_3 \subset H_2$.

Example 3.2 Consider the ring of integers $\mathbb{Z}$. Then

1. The ideals of $\mathbb{Z}$ are the form $n\mathbb{Z}$ where $n \in \mathbb{Z}^+ \cup \{0\}$**.**
2. The nonzero prime (resp. maximal) ideals of $\mathbb{Z}$ are the form $n\mathbb{Z}$ where n is a prime number. Furthermore, the zero ideal is prime but it is not maximal.
3. For each prime number p, if $n = pm$, then $n\mathbb{Z}$ is maximal in $m\mathbb{Z}$.
4. In the ring of integers $\mathbb{Z}$, the zero ideal is not maximal in any another ideal.

Definition 3.3 (Ahmad and Hummadi 2023)**.** A chain of proper ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ of a ring R is called maximal chain of ideals of R if I_{t-1} is maximal in I_t for each $t \in \mathbb{Z}^+$. If $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ is a finite chain, then I_0 is said to be the initial ideal and I_h is the terminal ideal of the chain. An ideal K_0 of M is called a maximal ideal of length m with respect to the maximal chain of ideals $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset M$. The length of K_0 is said to be ∞, if there is no such finite maximal chain of ideals with initial ideal K_0.

Definition 3.4. Let R be a commutative ring with identity. The maximal ideal graph of R, denoted by $m(R)$, is the undirected graph with vertex set, the set of all ideals of R, where two vertices I and J are adjacent if and only if I maximal in J, or J maximal in I.

Remark 3.5. Let R be a ring and $m(R)$ is the maximal ideal graph of R. Then

1. The length of the maximal chain $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ of R is h and the length of the path $I_0\ e_1\ I_1\ \ e_2\ I_2\ e_3$ ... $e_h\ I_h$ of $m(R)$ is h.
2. $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ is a shortest maximal chain of ideals of R with the initial ideal I_0 and terminal ideal I_h if and only if $I_0\ e_1\ I_1\ \ e_2\ I_2\ e_3$ ... $e_h\ I_h$ is a shortest path of $m(R)$ with the initial vertex I_0 and terminal vertex I_h where $e_i = (I_{i-1},\ I_i)$.

Remark 3.6. Let R be a commutative ring with identity. If $|V(m(R))| > 2$, then the $m(R)$ graph is not complete.

Proof. Suppose R has at least three ideals $I =< 0 >, J$ and K. Without loss of generality if I is a maximal in both J and K, then neither J maximal in K nor K maximal in J. So that two vertices J and K are not adjacent.

Theorem 3.7. If R is an Artinian ring, then the graph $mG(R)$ is connected.

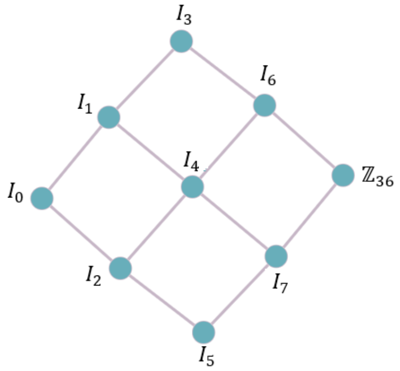
Proof. By (Ahmad and Hummadi 2023, Theorem, 2.12), the result is obtained.

Example 3.8. Consider the ring $\mathbb{Z}_{36} = \{0, 1,\ 2, \dots, 35\}$. The ring $\mathbb{Z}_{36}$ has the following proper ideals: $I_0 =< 0 >$, $I_1 =< 18 >= \{0, 18\}$, $I_2 =< 12 >$ = {0, 12, 24}, $I_3 =< 9 >$={0, 9, 18, 27}, $I_4 =< 6 >$ ={0, 6, 12, 18, 24, 30}, $I_5 =< 4 >$={0, 4, 8, 12, 16, 20, 24, 28, 32}, $I_6 =< 3 >$={0, 3, 6, 12, 15,18, 21, 24, 27, 30, 33}, $I_7 =< 2 >$={0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34}.

The following diagram illustrates the maximal chain of ideals of the ring $\mathbb{Z}_{36}$.

$$I_0 \subset \begin{cases} I_1 \subset \begin{cases} I_3 \subset I_6 \subset \mathbb{Z}_{36} \\ I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{36} \\ I_7 \subset \mathbb{Z}_{36} \end{cases} \end{cases} \\ I_2 \subset \begin{cases} I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{36} \\ I_7 \subset \mathbb{Z}_{36} \end{cases} \\ I_5 \subset I_7 \subset \mathbb{Z}_{36} \end{cases} \end{cases}$$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{36})$



Definition 3.9 (Sagan , Yeh and Zhang 1996)**.** Let $d(u, v)$ denote the distance between vertices u and v in a graph G. The Wiener index of G is defined as $W(G) = \sum_{\{u,v\}} d(u, v)$ where the sum is over all unordered pairs $\{u, v\}$ of distinct vertices in G. If x is a parameter, then the Wiener polynomial of G is $W(G; x) = \sum_{\{u,v\}} x^{d(u,v)}$ where the sum is taken over the same set of pairs.

Theorem 3.10. Let G be a graph and $W(G)$, $W(G; x)$ be the Wiener index and Wiener polynomial of G respectively. Then

1. $deg(W(G;\ q))$ equals the diameter of G.
2. $W(G) = f'(1)$

Proof.

1. By (Sagan , Yeh and Zhang 1996, 960 , Theorem 1.1), the result is obtained.
2. By (Sagan , Yeh and Zhang 1996, 960, theorem 1.1(5)), the result is obtained.

The following proposition is easy to prove

Proposition 3.11. If $\boldsymbol{R}$ is a field, then

1. $W(m(\boldsymbol{R})) = 1$ and $W(m(\boldsymbol{R}); x) = x$.
2. $rad(m(\boldsymbol{R})) = diam(m(\boldsymbol{R})) = 1$.

Theorem 3.12. Let P_n be a path with n vertices for some $n \in \mathbb{Z}^+$. Then

1. $W(P_n) = \binom{n+1}{3} = \frac{(n+1)!}{(n-2)!3!}$;
2. $W(P_n; x) = (n-1)x + (n-2)x^2 + (n-3)x^3 + \cdots + 2x^{n-2} + x^{n-1}$.
3. $diam(m(P_n)) = n - 1$

Proof.

1. By (Sagan , Yeh and Zhang 1996, Theorem 1.3(5)), the result is obtained.
2. By (Sagan , Yeh and Zhang 1996, Theorem 1.2(5)), the result is obtained.
3. By **Theorem 3.10(1),** the result is obtained.

Theorem 3.13. Consider the ring $\mathbb{Z}_{p^n}$ where p is a prime number and $n \in \mathbb{Z}^+$. Let $I_i =< p^i >$ for $0 \leq i \leq n$. Then

1. For any two ideals I_r, I_s of $\mathbb{Z}_{p^n}$, $d(I_r, I_s) = |r - s|$.
2. $W\left(m(\mathbb{Z}_{p^n})\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!}$
3. $W\left(m(\mathbb{Z}_{p^n}); x\right) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n$
4. $diam\left(m(\mathbb{Z}_{p^n})\right) = n$.
5. $rad\left(m(\mathbb{Z}_{p^n})\right) = \begin{cases} \frac{n}{2} & if\ n\ is\ an\ even\ number \\ \frac{n+1}{2} & if\ n\ is\ an\ add\ number \end{cases}$

Proof. It is clear that the ideals of $\mathbb{Z}_{p^n}$ are of the form $I_i =< p^i >=$ for $0 \leq i \leq n$. That is there are $n + 1$ ideals as follows:

$0\mathbb{Z}_{p^n}, p^{n-1}\mathbb{Z}_{p^n}, p^{n-1}\mathbb{Z}_{p^n}, p^{n-2}\mathbb{Z}_{p^n}, \dots, I_1 = p\mathbb{Z}_{p^n}, I_0 = \mathbb{Z}_{p^n}$. This means that the graph $m(\mathbb{Z}_{p^n})$ is a path P_{n+1}, that is it is a path with $n+1$ vertices.

1. Let $I_r =< p^r >$ and $I_s =< p^s >$ be two ideals of $\mathbb{Z}_{p^n}$. Then exactly one of the following is true. $a)\ r = s$ b) $r > s$ c) $r < s$.

a) If $r = s$, then $|r - s| = 0$ and $I_r = I_s$, consequently, $d(I_r, I_s) = 0 = |r - s|$.

b) If $r > s$, then the chain $I_r \subset I_{r-1} \subset I_{r-2} \subset \ldots \subset I_{s+1} \subset I_s$ is the shortest maximal chain of ideals with the initial ideal I_r and the terminal ideal I_s. So that $d(I_r, I_s) = |r - s|$.

c) Similarly, if $r < s$, then $d(I_r, I_s) = |r - s|$.

The following figure illustrates the distance from $< p^s >$ to $< p^s >$ in the maximal ideal graph $mG(\mathbb{Z}_{p^n})$

2. Since $W\left(m(\mathbb{Z}_{p^n})\right) = W(P_{n+1})$, then by **Theorem 3.12(1),** $W\left(m(\mathbb{Z}_{p^n})\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!3!}$ and

3. By **Theorem 3.12(2)**, $W(m(\mathbb{Z}_{p^n}); x) = W(P_{n+1}; x) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n$.

4. By **Theorem 3.10(1),** $diam\left(m(\mathbb{Z}_{p^n})\right) = degW(P_{n+1}; x) = n.$

5. It is clear that $\varepsilon(< 0 >) = \varepsilon(\mathbb{Z}_{p^n}) = n$, $\varepsilon(< p^{n-1} >) = \varepsilon(< p >) = n - 1$, $\varepsilon(< p^{n-2} >) = \varepsilon(< p^2 >) = n - 2, \dots$. So that for $0 \le i \le n$, $\varepsilon(< p^{n-i} >) = \varepsilon(< p^i >) = n - i$. Now, there are two cases. Case one, if n is an even number, then $\varepsilon\left(< p^{\frac{n}{2}} >\right) \le \varepsilon(< p^t >)$ where $0 \le t \le n$. Case two, if n

is an add number, then $\varepsilon\left(< p^{\frac{n+1}{2}} >\right) \leq \varepsilon(< p^t >)$ where $0 \leq t \leq n$.

Therefore, $rad\left(m(\mathbb{Z}_{p^n})\right) = \begin{cases} \frac{n}{2} & if\ n\ is\ an\ even\ number \\ \frac{n+1}{2} & if\ n\ is\ an\ add\ number \end{cases}$.

Example 3.14. Consider the ring $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$. Then $I_1 =< 0 >= \{0\}$, $I_2 =< 2 > = \{0, 2, 4, 6, 8, 10, 12, 14\}$, $I_3 =< 4 >= \{0, 4, 8, 12\}$ and $I_4 =< 8 >= \{0, 8\}$ are proper ideals of $\mathbb{Z}_{16}$ and $I_1 \subset I_2 \subset I_3 \subset I_4$ is the maximal chain of ideals of $\mathbb{Z}_{16}$.

1. By **Theorem 3.13(3),** the Wiener index of $m(\mathbb{Z}_{16}) = m(\mathbb{Z}_{2^4})$ is $W\left(m(\mathbb{Z}_{16})\right) = \binom{6}{3} = \frac{6!}{3!3!} = 20$
2. By **Theorem 3.13(4)**, the wiener polynomial of $m(\mathbb{Z}_{16})$ is $W(m(\mathbb{Z}_{16}); x) = 4x + 3x^2 + 2x^3 + x^4$.
3. By **Theorem 3.13(5)**, $diam\left(m(\mathbb{Z}_{16})\right) = 4$.
4. By **Theorem 3.13(6)**, $rad\left(m(\mathbb{Z}_{16})\right) = 2$

Example 3.15. Consider the ring $\mathbb{Z}_{128} = \mathbb{Z}_{2^7}$. Then

1. $W\left(m(\mathbb{Z}_{128})\right) = \binom{7+2}{3} = \frac{(7+2)!}{(7-1)!3!}$ =84.
2. $W(m(\mathbb{Z}_{128}); x) = 7x + 6x^2 + 5x^3 + 4x^4 + 3x^5 + 2x^6 + x^7$.
3. $diam\left(m(\mathbb{Z}_{128})\right) = 7$.
4. $rad\left(m(\mathbb{Z}_{128})\right) = 4$.

Definition 3.16 (Sagan , Yeh and Zhang 1996, 960). The Cartesian product of two graphs G_1 and G_2, is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(v_1, v_2): v_1 \in G_1$ and $v_2 \in G_2\}$ and $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2):\ u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2v_2 \in E(G_2)$ and $u_1 = v_1\}$.

Proposition 3.17. Let p and q be any two distinct prime numbers and $n, m \in \mathbb{Z}^+$. Then

1. $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n} = \{(a, b): a \in \mathbb{Z}_{p^m}$ and $b \in \mathbb{Z}_{q^n}\}$ is a ring.
2. $|\mathbb{Z}_{p^m}| = p^m$, $|\mathbb{Z}_{q^n}| = q^n$ and $|\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}| = |\mathbb{Z}_{p^m q^n}| = p^m q^n$
3. The ideals of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$ are of the form $I_1 \times I_2$ where I_1 is an ideal of $\mathbb{Z}_{p^m}$ and I_2 is an ideal of $\mathbb{Z}_{q^n}$.
4. $I_1 \times I_2$ is maximal in $J_1 \times J_2$ if and only if I_1 is maximal in J_1 and $I_2 = J_2$ or I_2 is maximal in J_2 and $I_1 = J_1$.
5. $m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) = m(\mathbb{Z}_{p^m}) \times m(\mathbb{Z}_{q^n}) = m(\mathbb{Z}_{p^m q^n})$.
6. $V\left(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right) = V\left(m(\mathbb{Z}_{p^m})) \times V(m(\mathbb{Z}_{q^n})\right) = V(m(\mathbb{Z}_{p^m q^n}))$
7. $I_1 \times I_2$ is maximal in $J_1 \times J_2$ if and only if $(I_1 \times I_2)(J_1 \times J_2) \in E(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$.

Proof.

1, 2, 3 and 4 are obvious.

5, 6, 7 are direct consequences of **Definition 3.16.**

Note that if $p = q$, then $V\left(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right) \neq V\left(m(\mathbb{Z}_{p^m})) \times V(m(\mathbb{Z}_{q^n})\right)$. For example $V(m(\mathbb{Z}_2 \times \mathbb{Z}_2)) \neq V(m(\mathbb{Z}_2)) \times V(m(\mathbb{Z}_2))$, since $V(m(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \{< 0 > \times < 0 >,\ \ \mathbb{Z}_2 \times < 0 >,\ \ < 0 > \times \mathbb{Z}_2,\ \ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \ \{(0,0), (1,1)\}$ and $V(m(\mathbb{Z}_2)) \times V(m(\mathbb{Z}_2))$=$\{< 0 > \times < 0 >,\ \mathbb{Z}_2 \times < 0 >,\ < 0 > \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2\}$

Definition 3.18 (Sagan , Yeh and Zhang 1996, 961). The ordered Wiener Polynomial defined by $\overline{W}(G; q) = \sum_{(u,v)} x^{d(u,v)}$, where the sum is over all ordered pairs (u, v) of vertices, including those where $u = v$. Thus, $\overline{W}(G; q) = \sum_{(u,v)} x^{d(u,v)} = 2W(G; q) + |V(G)|$.

Theorem 3.19 (Sagan , Yeh and Zhang 1996, 961, Proposition 1.4(2))**.** Suppose that G_1 and G_2 are two connected graphs. Then $\bar{W}(G_1 \times G_2; x) = \bar{W}(G_1; x) \times \bar{W}(G_2; x)$.

Theorem 3.20. Let p and q be any two prime numbers and $n, m \in \mathbb{Z}^+$. Then $W\big(m(\mathbb{Z}_{p^m q^n}); x\big) = 2W\big(m(\mathbb{Z}_{p^m}); x\big)W\big(m(\mathbb{Z}_{q^n}); x\big) + (n+1)W\big(m(\mathbb{Z}_{p^m}); x\big) + (m+1)W\big(m(\mathbb{Z}_{q^n}); x\big)$.

Proof. By **Theorem 3.19**, $\bar{W}\big(\mathbb{Z}_{p^m q^n}; x\big) = \bar{W}\big(\mathbb{Z}_{p^m}; x\big) \times \bar{W}\big(\mathbb{Z}_{q^n}; x\big)$. Then by **Definition 3.18,** $\Big(2W\big(m(\mathbb{Z}_{p^m q^n}); x\big) + |V\left(m(\mathbb{Z}_{p^m q^n})\right)|\Big) = \Big(2W\big(m(\mathbb{Z}_{p^m}); x\big) + |V\left(m(\mathbb{Z}_{p^m})\right)|\Big)\Big(2W\big(m(\mathbb{Z}_{q^n}); x\big) + |V\left(m(\mathbb{Z}_{q^n})\right)|\Big)$. So that $2W\big(m(\mathbb{Z}_{p^m q^n}); x\big)$=$4W\big(m(\mathbb{Z}_{p^m}); x\big)W\big(m(\mathbb{Z}_{q^n}); x\big) + 2\left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W\big(m(\mathbb{Z}_{p^m}); x\big) + 2\left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W\big(m(\mathbb{Z}_{q^n}); x\big)$. Then $W\big(m(\mathbb{Z}_{p^m q^n}); x\big)$=$2W\big(m(\mathbb{Z}_{p^m}); x\big)W\big(m(\mathbb{Z}_{q^n}); x\big) + \left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W\big(m(\mathbb{Z}_{p^m}); x\big) + \left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W\big(m(\mathbb{Z}_{q^n}); x\big) = 2W\big(m(\mathbb{Z}_{p^m}); x\big)W\big(m(\mathbb{Z}_{q^n}); x\big) + (n+1)W\big(m(\mathbb{Z}_{p^m}); x\big) + (m+1)W\big(m(\mathbb{Z}_{q^n}); x\big)$.

Remark 3.21. Consider the ring $\mathbb{Z}_{pq}$where p and q are two prime numbers. Then

1. The wiener polynomial of the maximal ideal graph $m(\mathbb{Z}_{pq})$ is $W(m(\mathbb{Z}_{pq}); x) = 4x + 2x^2$.
2. The wiener index of the maximal ideal graph $m(\mathbb{Z}_{pq})$ is $W(m(\mathbb{Z}_{pq})) = 8$.
3. $diam\left(m(\mathbb{Z}_{pq})\right) = 2$.

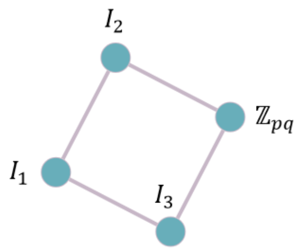
Proof.

1. By **Proposition 3.10**, $W\left(m(\mathbb{Z}_p)\right) = W\left(m(\mathbb{Z}_q)\right) = x$. By **Theorem 3.20**, $W(m(\mathbb{Z}_{pq}); x) = 2W(m(\mathbb{Z}_p); x)W(m(\mathbb{Z}_q); x) + (1+1)W(m(\mathbb{Z}_p); x) + (1+1)W(m(\mathbb{Z}_q); x)$=$4x + 2x^2$.
2. $W(m(\mathbb{Z}_{pq})) = W'(m(\mathbb{Z}_{pq}); 1) = 4 + 4(1) = 8$.
3. By **Theorem 3.10(1),** the result is obtained.

The following diagram illustrates the maximal chains of ideals of $\mathbb{Z}_{pq}$.

$$I_1 \subset \begin{cases} I_2 \subset \mathbb{Z}_{pq} \\ I_3 \subset \mathbb{Z}_{pq} \end{cases}$$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{pq})$



Example 3.22.

1. The wiener polynomial of each of $\mathbb{Z}_6, \mathbb{Z}_{10}, \mathbb{Z}_{14}, \mathbb{Z}_{15}, \mathbb{Z}_{21}$ is $W(m(\mathbb{Z}_{pq}); x) = 4x + 2x^2$.
2. The wiener index of each of $\mathbb{Z}_6, \mathbb{Z}_{10}, \mathbb{Z}_{14}, \mathbb{Z}_{15}, \mathbb{Z}_{21}$ is $W(m(\mathbb{Z}_{pq})) = 8$
3. $diam\left(m(\mathbb{Z}_{pq})\right) = 2$.

Proposition 3.23. Consider the ring $\mathbb{Z}_{p^2q^2}$ where p and q are two prime numbers. Then

1. The wiener polynomial of $\mathbb{Z}_{p^2q^2}$ is $(W(m(\mathbb{Z}_{p^2q^2}); x) = 12x + 14x^2 + 8x^3 + 2x^4$.
2. The wiener index of $\mathbb{Z}_{p^2q^2}$ is $W\left(m(\mathbb{Z}_{p^2q^2})\right) = 68$.

3. $diam\left(m(\mathbb{Z}_{p^2q^2})\right) = 4$.

Proof.

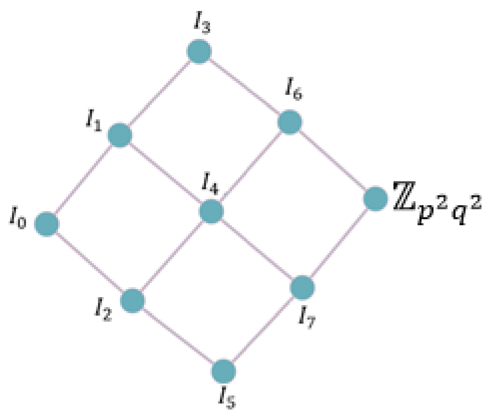
1. By **Theorem 3.13(4)**, $W(m(\mathbb{Z}_{p^2}); x) = W(m(\mathbb{Z}_{q^2}); x) = 2x + x^2$.

   By **Theorem 3.20**, $W(m(\mathbb{Z}_{p^2q^2}); x) = 2W(m(\mathbb{Z}_{p^2}); x)W(m(\mathbb{Z}_{q^2}); x) + (2+1)W(m(\mathbb{Z}_{p^2}); x) + (2+1)W(m(\mathbb{Z}_{q^2}); x) = 2(2x + x^2)^2 + 6(2x + x^2) = 8x^2 + 8x^3 + 2x^4 + 12x + 6x^2 = 12x + 14x^2 + 8x^3 + 2x^4$.
2. By **Theorem 3.10(2),** $W\left(m(\mathbb{Z}_{p^2q^2})\right) = W'(m(\mathbb{Z}_{p^2q^2}); 1) = 12 + 24 + 24 + 8 = 68$
3. By **Theorem 3.10(1),** the result is obtained.

The following diagram illustrates the maximal chain of ideals of the ring $\mathbb{Z}_{p^2q^2}$.

$$I_0 \subset \begin{cases} I_1 \subset \begin{cases} I_3 \subset I_6 \subset \mathbb{Z}_{p^2q^2} \\ I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{p^2q^2} \\ I_7 \subset \mathbb{Z}_{p^2q^2} \end{cases} \end{cases} \\ I_2 \subset \begin{cases} I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{p^2q^2} \\ I_7 \subset \mathbb{Z}_{p^2q^2} \end{cases} \\ I_5 \subset I_7 \subset \mathbb{Z}_{p^2q^2} \end{cases} \end{cases}$$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^2q^2})$



Theorem 3.24. Let $p_1, p_2, p_3, \dots, p_r$ be r distinct prime numbers and $r, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_r \in \mathbb{Z}^+$. Then

1. $\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \times \mathbb{Z}_{{p_2}^{\alpha_2}} \times \dots \times \mathbb{Z}_{{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}} = \bigoplus_{i=1}^{r} \mathbb{Z}_{{p_i}^{\alpha_i}}$.
2. $m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}) = m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right) \times m(\mathbb{Z}_{{p_r}^{\alpha_r}}) = m(\mathbb{Z}_{{p_1}^{\alpha_1}}) \times m(\mathbb{Z}_{{p_2}^{\alpha_2}}) \times \dots \times m(\mathbb{Z}_{{p_r}^{\alpha_r}})$.
3. $V(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}})) = V(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)) \times V(m(\mathbb{Z}_{{p_r}^{\alpha_r}})) = V(m(\mathbb{Z}_{{p_1}^{\alpha_1}})) \times V(m(\mathbb{Z}_{{p_2}^{\alpha_2}})) \times \dots \times V(m(\mathbb{Z}_{{p_r}^{\alpha_r}}))$
4. $W(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}); x) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + (\alpha_r + 1) W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \prod_1^{r-1}(\alpha_i + 1)\, W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$.

Proof.

1. By (Dummit and Foote 2004, 357, Exercises 20(a)) and (Michel n.d., 8, Theorem 2.25), we obtain the result.
2. By **Definition 3.16**, we obtain the result.
3. By **Definition 3.16**, we obtain the result.
4. By **Theorem 3.19**, $\overline{W}(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}); x) = \overline{W}\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) \times \overline{W}(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$. Then by **Definition 3.18**, $\left(2W(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}); x) + |V\left(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}})\right)|\right) = \left(2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + |V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)|\right)\left(2W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + |V\left(m(\mathbb{Z}_{{p_r}^{\alpha_r}})\right)|\right)$.

   So that

   $W(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}); x) = 4W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + 2\left|V\left(m(\mathbb{Z}_{{p_r}^{\alpha_r}})\right)\right| W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + 2\left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right| W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$. Then

$W(m(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_r{}^{\alpha_r}});x)=$

$2W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)+$

$\left|V\left(m(\mathbb{Z}_{p_r{}^{\alpha_r}})\right)\right|W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)+$

$\left|V\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right)\right)\right|W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)=$

$2W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)+(\alpha_r+$

$1)W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)+\prod_1^{r-1}(\alpha_i+1)\,W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)$.

Therefore,

$W(m(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_r{}^{\alpha_r}});x)=$

$2W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)+(\alpha_r+$

$1)W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}p_2{}^{\alpha_2}\ldots p_{(r-1)}{}^{\alpha(r-1)}}\right);x\right)+\prod_1^{r-1}(\alpha_i+1)\,W(m(\mathbb{Z}_{p_r{}^{\alpha_r}});x)$

Proposition 3.25. Consider the ring $\mathbb{Z}_{pqr}$ where p,q and r are three prime numbers. Then

1. The wiener polynomial of $\mathbb{Z}_{pqr}$ is $(W(m(\mathbb{Z}_{pqr});x)=12x+16x^2+4x^3$.
2. The wiener index of $\mathbb{Z}_{pqr}$ is $W\left(m(\mathbb{Z}_{pqr})\right)=48$.
3. $diam\left(m(\mathbb{Z}_{pqr})\right)=3$.

Proof.

1. Since $(W(m(\mathbb{Z}_{pq});x)=4x+2x^2$ and $(W(m(\mathbb{Z}_r);x)=x$, then by **Theorem 3.24**, $W(m(\mathbb{Z}_{pqr});x)=2W(m(\mathbb{Z}_{pq});x)W(m(\mathbb{Z}_r);x)+$ $2W(m(\mathbb{Z}_{pq});x)+4W(m(\mathbb{Z}_r);x)=2x(4x+2x^2)+2(4x+2x^2)+4x$ $=8x^2+4x^3+8x+4x^2+4x=12x+12x^2+4x^3$.
2. $W\left(m(\mathbb{Z}_{pqr})\right)=W'(m(\mathbb{Z}_{pqr});1)=12+24+12=48$.
3. By **Theorem 3.10(1),** the result is obtained.

The following example shows the classical method to find the wiener polynomial of $m(\mathbb{Z}_{pqr})$.

Example 3.26. Consider the proper ideals $I_1 =< 0 >$, $I_2 =< pq >$, $I_3 =< pr >$, $I_4 =< qr >$, $I_5 =< p >$, $I_6 =< q >$ and $I_7 =< r >$ and $I_8 = \mathbb{Z}_{pqr}$ of the ring $R = \mathbb{Z}_{pqr}$where p, q and r are three prime numbers. Then

$$d(I_1, I_2) = d(I_1, I_3) = d(I_1, I_4) = d(I_2, I_5) = d(I_2, I_6) = d(I_3, I_5) = d(I_3, I_7) = d(I_4, I_6) = d(I_4, I_7) = d(I_5, I_8) = d(I_6, I_8) = d(I_7, I_8) = 1.$$

$$d(I_1, I_5) = d(I_1, I_6) = d(I_1, I_7) = d(I_2, I_3) = d(I_2, I_4) = d(I_2, I_8) = d(I_3, I_4) = d(I_3, I_8) = d(I_4, I_8) = d(I_5, I_6) = d(I_5, I_7) = d(I_6, I_7) = 2.$$

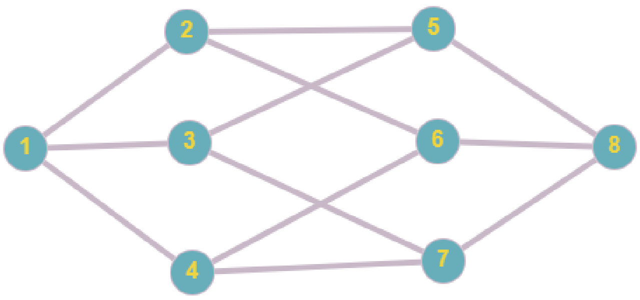
$$d(I_1, I_8) = d(I_2, I_7) = d(I_3, I_6) = d(I_4, I_5) = 3.$$

So that the wiener polynomial of $\mathbb{Z}_{pqr}$ is $W(m(\mathbb{Z}_{pqr}); x) = 12x + 12x^2 + 4x^3$ and the wiener index of $\mathbb{Z}_{pqr}$ is $W(m(\mathbb{Z}_{pqr})) = 48$.

The following diagram illustrates the maximal chain of ideals of $\mathbb{Z}_{pqr}$.

$$I_1 \subseteq \begin{cases} I_2 \subseteq \begin{cases} I_5 \subseteq \mathbb{Z}_{pqr} \\ I_6 \subseteq \mathbb{Z}_{pqr} \end{cases} \\ I_3 \subseteq \begin{cases} I_5 \subseteq \mathbb{Z}_{pqr} \\ I_7 \subseteq \mathbb{Z}_{pqr} \end{cases} \\ I_4 \subseteq \begin{cases} I_7 \subseteq \mathbb{Z}_{pqr} \\ I_6 \subseteq \mathbb{Z}_{pqr} \end{cases} \end{cases}$$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{pqr})$



Proposition 3.27. Consider the ring $\mathbb{Z}_{pqrs}$ where p, q, r and s are four prime numbers. Then

1. The wiener polynomial of $\mathbb{Z}_{pqrs}$ is $(W(m(\mathbb{Z}_{pqrs}); x) = 32x + 48x^2 + 32x^3 + 8x^4$.
2. The wiener index of $\mathbb{Z}_{pqrs}$ is $W\left(m(\mathbb{Z}_{pqrs})\right) = 256$.
3. $diam\left(m(\mathbb{Z}_{p^2q^2})\right) = 4$.

Proof. $W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_r}^{\alpha_r}}\right); x\right) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right)W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right) + (\alpha_r + 1)W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \prod_1^{r-1}(\alpha_i + 1)\, W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right)$

1. Since $(W\left(m(\mathbb{Z}_{pqr}); x\right) = 12x + 12x^2 + 4x^3$ and $(W(m(\mathbb{Z}_s); x) = x$, then by theorem 3.24(4), $W\left(m(\mathbb{Z}_{pqrs}); x\right) = 2W\left(m(\mathbb{Z}_{pqr}); x\right)W(m(\mathbb{Z}_s); x) + 2W\left(m(\mathbb{Z}_{pqr}); x\right) + 8W(m(\mathbb{Z}_s); x) = 2x(12x + 12x^2 + 4x^3) + 2(12x + 12x^2 + 4x^3) + 8x = 32x + 48x^2 + 32x^3 + 8x^4$.
2. $W\left(m(\mathbb{Z}_{pqrs})\right) = W'\left(m(\mathbb{Z}_{pqrs}); 1\right) = 32 + 96 + 96 + 32 = 256$.
3. By **Theorem 3.10(1),** the result is obtained.

The following example shows the classical method to find $W\left(m(\mathbb{Z}_{pqrs})\right)$.

Example 3.28. Consider the ideals $I_1 =< 0 >$, $I_2 =< pqr >$, $I_3 =< pqs >$, $I_4 =< prs >$, $I_5 =< qrs >$, $I_6 =< pq >$, $I_7 =< pr >$, $I_8 =< ps >$, $I_9 =< qr >$, $I_{10} =< qs >$, $I_{11} =< rs >$, $I_{12} =< p >$, $I_{13} =< q >$, $I_{14} =< r >$, $I_{15} =< s >$ and $I_{16} = \mathbb{Z}_{pqrs}$ of the ring $R = \mathbb{Z}_{pqrs}$where p, q, r and s are four prime numbers.

Then we have the following maximal chains:

$d(I_1, I_2) = d(I_1, I_3) = d(I_1, I_4) = d(I_1, I_5) = d(I_2, I_6) = d(I_2, I_7) = d(I_2, I_9) = d(I_3, I_6) = d(I_3, I_8) = d(I_3, I_{10}) = d(I_4, I_7) = d(I_4, I_8) = d(I_4, I_{11}) = d(I_5, I_9) = d(I_5, I_{10}) = d(I_5, I_{11}) = d(I_6, I_{12}) = d(I_6, I_{13}) = d(I_7, I_{12}) = d(I_7, I_{14}) = d(I_8, I_{12}) = d(I_8, I_{15}) = d(I_9, I_{13}) = d(I_9, I_{14}) = d(I_{10}, I_{13}) =$

$d(I_{10}, I_{15}) = d(I_{11}, I_{14}) = d(I_{11}, I_{15}) = d(I_{12}, I_{16}) = d(I_{13}, I_{16}) = d(I_{14}, I_{16}) = d(I_{15}, I_{16}) = 1.$

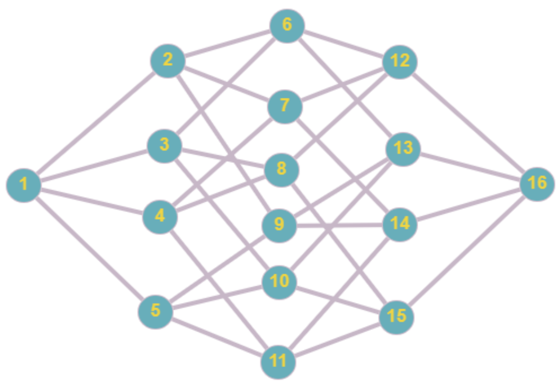
$d(I_1, I_6) = d(I_1, I_7) = d(I_1, I_8) = d(I_1, I_9) = d(I_1, I_{10}) = d(I_1, I_{11}) = d(I_2, I_3) = d(I_2, I_4) = d(I_2, I_5) = d(I_2, I_{12}) = d(I_2, I_{13}) = d(I_2, I_{14}) = d(I_3, I_4) = d(I_3, I_5) = d(I_3, I_{12}) = d(I_3, I_{13}) = d(I_3, I_5) = d(I_4, I_5) = d(I_4, I_{12}) = d(I_4, I_{14}) = d(I_4, I_{15}) = d(I_5, I_{13}) = d(I_5, I_{14}) = d(I_5, I_{15}) = d(I_6, I_7) = d(I_6, I_8) = d(I_6, I_9) = d(I_6, I_{10}) = d(I_6, I_{16}) = d(I_7, I_8) = d(I_7, I_9) = d(I_7, I_{11}) = d(I_7, I_{16}) = d(I_8, I_{10}) = d(I_8, I_{11}) = d(I_8, I_{16}) = d(I_9, I_{10}) = d(I_9, I_{11}) = d(I_9, I_{16}) = d(I_{10}, I_{11}) = d(I_{10}, I_{16}) = d(I_{11}, I_{16}) = d(I_{12}, I_{13}) = d(I_{12}, I_{14}) = d(I_{12}, I_{15}) = d(I_{13}, I_{14}) = d(I_{13}, I_{15}) = d(I_{14}, I_{15}) = 2 .$

$d(I_1, I_{12}) = d(I_1, I_{13}) = d(I_1, I_{14}) = d(I_1, I_{15}) = d(I_2, I_8) = d(I_2, I_{10}) = d(I_2, I_{11}) = d(I_2, I_{16}) = d(I_3, I_7) = d(I_3, I_9) = d(I_3, I_{11}) = d(I_3, I_{16}) = d(I_4, I_6) = d(I_4, I_9) = d(I_4, I_{10}) = d(I_4, I_{16}) = d(I_5, I_6) = d(I_5, I_7) = d(I_5, I_8) = d(I_5, I_{16}) = d(I_6, I_{14}) = d(I_6, I_{15}) = d(I_7, I_{13}) = d(I_7, I_{15}) = d(I_8, I_{13}) = d(I_8, I_{14}) = d(I_9, I_{12}) = d(I_9, I_{15}) = d(I_{10}, I_{12}) = d(I_{10}, I_{14}) = d(I_{11}, I_{12}) = d(I_{11}, I_{13}) = 3.$

$d(I_1, I_{16}) = d(I_2, I_{15}) = d(I_3, I_{14}) = d(I_4, I_{13}) = d(I_5, I_{15}) = d(I_7, I_{10}) = d(I_6, I_{11}) = d(I_8, I_9) = 4.$

So that the wiener polynomial of $\mathbb{Z}_{pqr}$ is $W(m(\mathbb{Z}_{pqrs}); x) = 32x + 48x^2 + 32x^3 + 8x^4$ and the wiener index of $\mathbb{Z}_{pqrs}$ is $W(m(\mathbb{Z}_{pqrs})) = 256.$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{pqrs})$



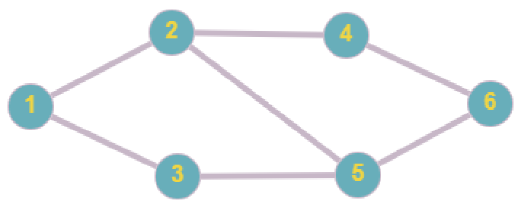
The following diagram illustrates the maximal chain of ideals of the ring $\mathbb{Z}_{pqrs}$

$$I_1 \subseteq \begin{cases} I_2 \subseteq \begin{cases} I_6 \subseteq \begin{cases} I_{12} \subseteq \mathbb{Z}_{pqrs} \\ I_{13} \subseteq \mathbb{Z}_{pqrs} \end{cases} \\ I_7 \subseteq \begin{cases} I_{12} \subseteq \mathbb{Z}_{pqrs} \\ I_{14} \subseteq \mathbb{Z}_{pqrs} \end{cases} \\ I_9 \subseteq \begin{cases} I_{13} \\ I_{14} \end{cases} \subseteq \mathbb{Z}_{pqrs} \end{cases} \\ I_3 \subseteq \begin{cases} I_6 \subseteq \begin{cases} I_{12} \\ I_{13} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_8 \subseteq \begin{cases} I_{12} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_{10} \subseteq \begin{cases} I_{13} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \end{cases} \\ I_4 \subseteq \begin{cases} I_7 \subseteq \begin{cases} I_{12} \\ I_{14} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_8 \subseteq \begin{cases} I_{12} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_{11} \subseteq \begin{cases} I_{14} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \end{cases} \\ I_5 \subseteq \begin{cases} I_9 \subseteq \begin{cases} I_{13} \\ I_{14} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_{10} \subseteq \begin{cases} I_{13} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \\ I_{11} \subseteq \begin{cases} I_{14} \\ I_{15} \end{cases} \subseteq \mathbb{Z}_{pqrs} \end{cases} \end{cases}$$

Examples 3.29.

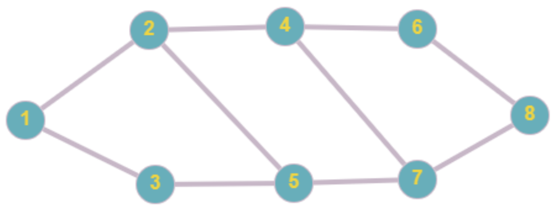
1. The wiener polynomial of the graph $m(\mathbb{Z}_{p^2q})$ is $W(m(\mathbb{Z}_{p^2q});x) = 7x + 6x^2 + 2x^3$.

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^2q})$

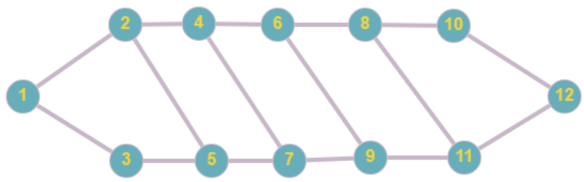


2. The wiener polynomial of the graph $m(\mathbb{Z}_{p^3q})$ is $(m(\mathbb{Z}_{p^3q});x) = 10x + 10x^2 + 6x^3 + 2x^4$.

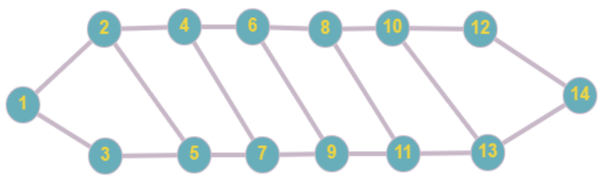
   The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^3q})$



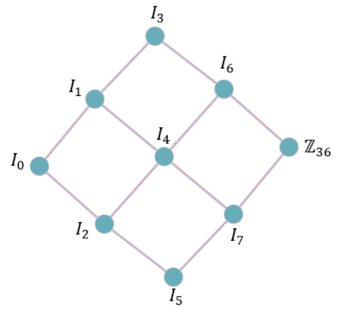
3. The wiener polynomial of the graph $m(\mathbb{Z}_{p^4q})$ is $W(m(\mathbb{Z}_{p^4q});x) = 13x + 14x^2 + 10x^3 + 6x^4 + 2x^5$.
4. The wiener polynomial of the graph $m(\mathbb{Z}_{p^5q})$ is $W(m(\mathbb{Z}_{p^5q});x) = 16x + 18x^2 + 14x^3 + 10x^4 + 6x^5 + 2x^6$. The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^5q})$



5. The wiener polynomial of the graph $m(\mathbb{Z}_{p^6q})$ is $W(m(\mathbb{Z}_{p^6q});x) = 19x + 22x^2 + 18x^3 + 14x^4 + 10x^5 + 6x^6 + 2x^7$. The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^6q})$



6. The wiener polynomial of the graph $m(\mathbb{Z}_{p^2q^2})$ is $W(m(\mathbb{Z}_{p^2q^2});x) = 12x + 14x^2 + 8x^3 + 2x^4$. The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_{p^2q^2})$



7. The wiener polynomial of the graph $m(\mathbb{Z}_{p^3q^2})$ is $W\left(m\left(\mathbb{Z}_{p^3q^2}\right);x\right) = 17x + 22x^2 + 17x^3 + 8x^4 + 2x^5$.
8. The wiener polynomial of the graph $m(\mathbb{Z}_{p^4q^2})$ is $W\left(m\left(\mathbb{Z}_{p^4q^2}\right);x\right) = 22x + 30x^2 + 26x^3 + 17x^4 + 8x^5 + 2x^6$.

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پوختە

لەم پرۆژەیەدا، ئێمە ڕێگایەکی نوێ پێشکەش دەکەین بۆ دۆزینەوەی wiener polynomial **و** wiener index**ی** $m(\mathbb{Z}_n)$ maximal ideal graphs **لە رینگی** $\mathbb{Z}_n$ **کاتێك** n **یەکسانە بە** $p_1^{\alpha_1}\, p_2^{\alpha_2}\, p_3^{\alpha_3} \dots p_k^{\alpha_k}$، p_i **کانیش ژمارەی خۆبەشن جیاوازن لەگەل یەکتر،** α_i **دانەیە لە** $\mathbb{Z}^+$، **و** $1 \leq i \leq k$.

الخلاصة

في هذا المشروع ،نقدم طريقة جديدة لإيجاد متعددة حدود وينر و مؤشر وينر للرسوم البيانية القصوي $m(\mathbb{Z}_n)$ **للحلقات** $\mathbb{Z}_n$ **حيث** n **تساوي** $p_1^{\alpha_1}\, p_2^{\alpha_2}\, p_3^{\alpha_3} \dots p_k^{\alpha_k}$، p_i **هي اعداد اولية متميزة ،**

$\alpha_i \in \mathbb{Z}^+$، $1 \leq i \leq k$.