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Salahaddin University-Erbil

Maximal chain of submodules of a module

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. in Mathematics

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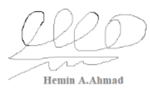
Certification of the Supervisor

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature: 

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Primarily, I would like to thanks my god for helping me to complete this research with success. Then I would like to express special of my supervisor **Mr. Hemin Abdulkarim Ahmad** Whose valuable to guidance has been the once helped me to completing my research.

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Abstract

In this project, we study maximal chain of submodules of the $\mathbb{Z}$-modules $M = \mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_k}^{\alpha_k}}$ where p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k \neq 1$. Then we define the maximal submodule graph $m(M)$ of the module M. Finally we introduce a method to find the wiener polynomial and wiener index of maximal submodule graphs $m(M)$ of modules $M = \mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_k}^{\alpha_k}}$ where p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k$.

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Introduction

Let R be a commutative ring and M be an R-module. A submodule N_1 of M is maximal in a submodule N_2 of M if there is no submodule N_3 of M such that $N_1 \subset N_3 \subset N_2$. A chain of submodules $K_0 \subset K_1 \subset K_2 \subset \cdots$ of an R-module M is called maximal chain of submodules of M if K_{t-1} is a maximal submodule in K_t for each $t \in \mathbb{Z}^+$. If $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_h$ is a finite chain, then K_0 is said to be the initial submodule and K_h is the terminal submodule of the chain. A submodule K_0 of M is called a maximal submodule of length m with respect to the maximal chain of submodules $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset M$. The maximal submodule graph of M, denoted by $m(M)$, is the undirected graph with vertex set, the set of all submodules of M, where two vertices N_1 and N_2 are adjacent if and only if N_1 maximal N_2, or N_2 maximal N_1. In the chapter three we study maximal chain of submodules of $\mathbb{Z}$-modules $\mathrm{M} = \mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \ldots \oplus \mathbb{Z}_{{p_k}^{\alpha_k}}$ where p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k$. Then we find the maximal submodule graph $m(M)$ of the module M. Finally the Wiener index, Wiener polynomial and dimeter of the maximal submodule graphs $m(M)$ are investigated.

Chapter One

Definitions and Back grounds of module theory

Definition 1.1 (Dummit & Foote , 2004, p. 16)**.**

(1) A binary operation $*$ on a set G is a function $* : G \times G \rightarrow G$. For any $a, b \in G$ we shall write $a * b \, for \, *(a, b)$.

(2) A binary operation $*$ on a set G is associative if for all $a, b, c \in G$ we have $a * (b * c) = (a * b) * c$.

(3) If $*$ is a binary operation on a set G we say elements a and b of G commute if $a * b = b * a$. We say $*$ (or G) is commutative if for all a, $b \in G$, $a * b = b * a$.

Definition 1.2 (Dummit & Foote , 2004, p. 46)**.** Let G be a group. The subset H of G is a subgroup of G if H is nonempty and H is closed under products and inverses ($i.e., x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$). If H is a subgroup of G we shall write $H \leq G$.

Definition 1.3 (Dummit & Foote , 2004, p. 62) If A is any subset of the group G define

$$\langle A \rangle = \bigcap_{\substack{A \subset H \\ H \leq G}} H$$

This is called the subgroup of G generated by A.

Definition 1.4 (Dummit & Foote , 2004, p. 223)

(1) A ring R is a set together with two binary operations + and × (called addition and multiplication) satisfying the following axioms:

(I) $(R, +)$ is an abelian group,

(ii) x is associative : $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$;

(iii) the distributive laws hold in R : for all $a, b, c \in R$ $(a + b) \times c = (a \times c) + (b \times c) and\ a \times (b + c) = (a \times b) + (a \times c)$

(2) The ring R is commutative if multiplication is commutative.

(3) The ring R is said to have an identity (or contain a 1) if there is an element $1 \in R$ with $1 \times a = a \times 1 = a$ for all $a \in R$.

Definition 1.5 (Dummit & Foote , 2004, p. 228) A subring of the ring R is a subgroup of R that is closed under multiplication

Definition 1.6 (Dummit & Foote , 2004, p. 337) Let R be a ring (not necessarily commutative nor with 1). A left R-module or a left module over R is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \rightarrow M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

(a) $(r + s)m = rm + sm$, for all $r, s \in R,\ m \in M,$

(b)$(rs)m = r(sm)$, for all $r, s \in R, m \in M$, and

(c) $r(m + n) = rm + rn$, for all $r \in R, m, n \in M$. If the ring R has a 1 we impose the additional axiom:

(d) $1m = m$, for all $m \in M$.

Examples 1.7 (Dummit & Foote , 2004)

1. If R is a field then an R-module is the same as an R-vector space.

2. If $M = R$ and scalar multiplication is given by multiplication in R then

3. $M = \mathbb{Z}$ itself becomes an R-module.

Definition 1.8 (Dummit & Foote , 2004, p. 337) Let R be a ring and let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of ring element $i.e., rn \in N,$ for all $r \in R, n \in N$.

Definition 1.9 (Ahmad & Hummadi, 2023)**.** A submodule N of an R-module M is said to be a maximal submodule of M if $M \neq N$ and there is no proper submodule of M strictly containing N.

Definition 1.10 (Dummit & Foote , 2004, p. 751) . An R- module $\boldsymbol{M}$ is said to be Artinian or to satisfy the descending chain condition on submodule (or D. C. C. on module) if there is no infinite decreasing chain of submodules in $\boldsymbol{M}$, i.e., whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a decreasing chain of submodules of $\boldsymbol{M}$, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Proposition 1.11. The following are equivalent:

(1) $\boldsymbol{M}$ is an Artinian submodule.

(2) Every nonempty set of submodule of R contains a minimal element under inclusion.

Definition 1.12. (Dummit & Foote , 2004, p. 458) An R- module $\boldsymbol{M}$ is said to be Noetherian or to satisfy the ascending chain condition on submodule (or A. C. C. on module) if there is no infinite decreasing chain of submodules in $\boldsymbol{M}$, i.e., whenever $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ is an increasing chain of submodules of $\boldsymbol{M}$, then there is a positive integer m such that $M_k = M_m$ for all $k \geq m$.

Chapter Two

Definitions and back grounds of graph theory

Definition 2.1 (Naduvath, 2017, p. 3)A graph G can be considered as an ordered triple (V, E, ψ), where .

(i) $V = \{ v_1, v_2, v_3, ...\}$ is called the vertex set of G and the elements of V are called the vertices (or points or nodes);

(ii) $E = \{e_1, e_2, e_3,...\}$ is the called the edge set of G and the elements of E are called edges (or lines or arcs); and

(iii) ψ is called the adjacency relation, defined by $\psi : E \to V \times V$, which defines the association between each edge with the vertex pairs of G.

Definition 2.2 (Naduvath, 2017, p. 3) The order of a graph G, denoted by $\nu(G)$, is the number of its vertices and the size of G, denoted by $\varepsilon(G)$, is the number of its edge

Definition 2.3 (Naduvath, 2017, p. 4)A graph with a finite number of vertices as well as a finite number of edges is called a finite graph. Otherwise, it is an infinite graph

Definition 2.4 (Naduvath, 2017, p. 4)An edge of a graph that joins a node to itself is called loop or a self-loop. That is, a loop is an edge uv, where $u = v$.

Definition 2.5 (Naduvath, 2017, p. 5) The edges connecting the same pair of vertices are called multiple edges or parallel edges.

Definition 2.6 (Naduvath, 2017, p. 5) A graph G which does not have loops or parallel edges is called a simple graph. A graph which is not simple is generally called a multigraph

Definition 2.7 (Naduvath, 2017, p. 5) number of edges incident on a vertex v, with self-loops counted twice, is called the degree of the vertex v and is denoted by deg(v) or deg(v) or simply $d(v)$.

Definition 2.8 (Naduvath, 2017, p. 5) A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are those with zero degree.

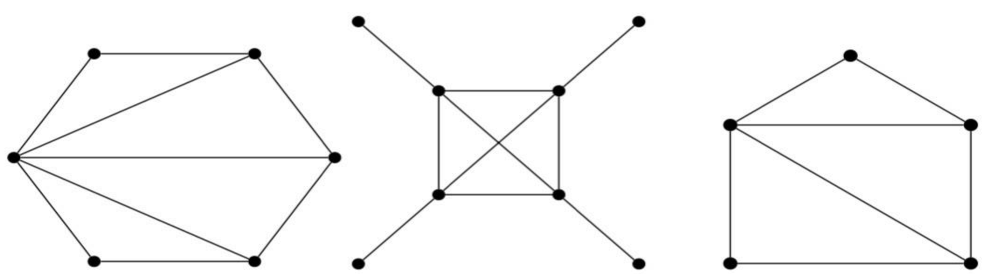
Definition 2.9 (Naduvath, 2017, p. 5) A vertex, which is neither a pendent vertex nor an isolated vertex, is called an internal vertex or an intermediate vertex.

Definition 2.10 (Naduvath, 2017, p. 5) The maximum degree of a graph G, denoted by $\Delta(G)$, is defined to be $\Delta(G) = max\{d(v) : v \in V(G)\}$. Similarly, the minimum degree of a graph G, denoted by $\delta(G)$, is defined to be $\delta(G) = min\{d(v) : v \in V(G)\}$. Note that for any vertex v in G, we have $\delta(G) \leq d(v) \leq \Delta(G)$.

Definition 2.11 (Naduvath, 2017, p. 7) The neighborhood (or open neighbourhood) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v. That is, $N(v) = \{x \in V : vx \in E\}$. The closed neighbourhood of a vertex v, denoted by $N[v]$, is simply the set $N(v) \cup \{v\}$.

Definition 2.12 (Naduvath, 2017, p. 8) A graph $H(V_1, E_1)$ is said to be a subgraph of a graph $G(V, E)$ if $V_1 \subseteq V \; and \; E_1 \subseteq E$.

Definition 2.13 (Naduvath, 2017, p. 8) A graph $H(V_1,E_1)$ is said to be a spanning subgraph of a graph $G(V, E) \; if \; V_1 = V \; and \; E_1 \subseteq E$.



Definition 2.14 (Naduvath, 2017, p. 8)**.** Suppose that V' be a subset of the vertex set V of a graph G. Then, the subgraph of G whose vertex set is $V'$and whose edge set is the set of edges of G that have both end vertices in V' is denoted by $G[V]$ or $\langle V \rangle$ called an induced subgraph of G

Definition 2.15 (Naduvath, 2017, p. 8)**.** Suppose that E' be a subset of the edge set V of a graph G. Then, the subgraph of G whose edge set is E' and whose vertex set is the set of end vertices of the edges in E' is denoted by $G[E]$ or $\langle E \rangle$ called an edge-induced subgraph of G.

Definition 2.16 (Naduvath, 2017, p. 8)**.** A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph…

Definition 2.17 (Naduvath, 2017, p. 11)**.** An isomorphism of two graphs G and H is a bijective function $f : V(G) \rightarrow V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. This bijection is commonly described as edge-preserving bijection. If an isomorphism exists between two graphs, then the graphs are called isomorphic graphs and denoted $as\ G \simeq H\ or\ G \cong H.$

(Naduvath, 2017)

Remark 2.18. Every two graphs G and H are said to be isomorphic if

(i) $|V(G)| = |V(H)|$,

(ii) $|E(G)| = |E(H)|$,

(iii) $v_i v_j \in E(G) \Longrightarrow f(v_i)\, f(v_i) \in E(H)$.

Definition 2.19 (Naduvath, Sudev, 2017, p. 23)**.** A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a closed walk.

Definition 2.20 (Naduvath, 2017, p. 23)**.** A trail is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A tour is a trail that begins and ends on the same vertex.

Definition 2.21 (Naduvath, 2017, p. 23)**.** A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A cycle or a circuit is a path that begins and ends on the same vertex.

Definition 2.22 (Naduvath, 2017)**.** The length of a walk or circuit or path or cycle is the number of edges in it.

Definition 2.23 (Naduvath, 2017, p. 24)**.** The distance between two vertices u and v in a graph G, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the length (number of edges) of a shortest path (also called a graph geodesic) connecting them. This distance is also known as the geodesic distance.

Definition 2.24 (Naduvath, 2017, p. 24)**.** The eccentricity of a vertex v, denoted by $\varepsilon(v)$, is the greatest geodesic distance between v and any other vertex. It can be thought of as how far a vertex is from the vertex most distant from it in the graph.

Definition 2.25 (Naduvath, 2017, p. 24)**.** The radius r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = \min_{v \in v_{(G)}} \varepsilon(v)$.

Definition 2.26 (Naduvath, 2017, p. 24)**.** The diameter of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = \max_{v \in v_{(G)}} \varepsilon(v)$.

Definition 2.27 (Naduvath, 2017, p. 24)**.** A center of a graph G is a vertex of G whose eccentricity equal to the radius of G.

Definition 2.28 (Naduvath, 2017, p. 25)**.** Two vertices u and v are said to be connected if there exists a path between them. If there is a path between two vertices u and v, then u is said to be reachable from v and vice versa. A graph G is said to be connected if there exist paths between any two vertices in G.

Definition 2.29 (Naduvath, 2017, p. 26)**.** A connected component or simply, a component of a graph G is a maximal connected subgraph of G.

Definition 2.30 (Sagan , Yeh, & Zhang, 1996)**.** Let $d(u, v)$ denote the distance between vertices u and v in a graph G. The Wiener index of G is defined as $W(G) = \sum_{\{u,v\}} d(u, v)$ where the sum is over all unordered pairs $\{u, v\}$ of distinct vertices in G. If x is a parameter, then the Wiener polynomial of G is $W(G; x) = \sum_{\{u,v\}} x^{d(u,v)}$ where the sum is taken over the same set of pairs.

Definition 2.31 (Sagan , Yeh, & Zhang, 1996, p. 961). The ordered Wiener Polynomial defined by $\overline{W}(G; q) = \sum_{(u,v)} x^{d(u,v)}$, where the sum is over all ordered pairs (u, v) of vertices, including those where $u = v$. Thus, $\overline{W}(G; q) = \sum_{(u,v)} x^{d(u,v)} = 2W(G; q) + |V(G)|$.

Theorem 2.32 (Sagan , Yeh, & Zhang, 1996, pp. 961, Proposition 1.4(2))**.** Suppose that G_1 and G_2 are two connected graphs. Then $\overline{W}(G_1 \times G_2; x) = \overline{W}(G_1; x) \times \overline{W}(G_2; x)$.

Chapter Three

In this chapter, we study maximal chain of submodules of $\mathbb{Z}$-modules $M = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_k^{\alpha_k}}$ where p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$, and $1 \leq i \leq k \neq 1$. Then we find the maximal submodules graph $m(M)$ of the module M. Finally the Wiener index, Wiener polynomial, dimeter and radical of the maximal submodule graphs $m(M)$ are investigated.

Definition 3.1. A submodule N_1 of an R-module M is maximal in a submodule N_2 of M if there is no submodule N_3 of M such that $N_1 \subset N_3 \subset N_2$.

Example 3.2. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{36}$. Then

1. The submodules of $\mathbb{Z}_{36}$ are the form $n\mathbb{Z}_{36}$ where $n \in \{0, 1, 2, 3, 4, 6, 12, 18\}$
2. For each prime number p, if $n = pm$, then $n\mathbb{Z}_{36}$ is maximal in $m\mathbb{Z}_{36}$.

Definition 3.3 (Ahmad & Hummadi, 2023)**.** A chain of submodules $K_0 \subset K_1 \subset K_2 \subset \cdots$ of an R-module M is called maximal chain of submodules of M if K_{t-1} is a maximal submodule in K_t for each $t \in \mathbb{Z}^+$. If $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_h$ is a finite chain, then K_0 is said to be the initial submodule and K_h is the terminal submodule of the chain. A submodule K_0 of M is called a maximal submodule of length m with respect to the maximal chain of submodules $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset M$. The length of K_0 is said to be ∞, if there is no such finite maximal chain of submodules with initial submodule K_0.

Definition 3.4. Let M be an R-module. The maximal submodule graph of M, denoted by $m(M)$, is the undirected graph with vertex set, the set of all submodules of M, where two vertices N_1 and N_2 are adjacent if and only if N_1 maximal N_2, or N_2 maximal N_1.

Remark 3.5. Let M be an R-module. If $|V(m(M))| > 2$, then the $m(M)$ graph is not complete.

Proof. Let M be an R-module with at least three submodules $I =< 0 >, J$ and K. Without loss of generality if I is a maximal in both J and K, then neither J maximal in K nor K maximal in J. So that two vertices J and K are not adjacent.

Theorem 3.6. Let M be an R-module. If M is an Artinian and Noetherian module, then the graph $m(M)$ is connected. But the converse is not true.

The following proposition is easy to prove

Proposition 3.7. Consider an R-module $\boldsymbol{M}$. Then

1. If $\boldsymbol{M} =< 0 >$, then the wiener polynomial of $\boldsymbol{M}$ is 1, that is $W(m(\boldsymbol{M}); x) = 1$, $rad(G) = diam(G) = 0$.
2. If $\boldsymbol{M}$ is a simple module, then $W(m(\boldsymbol{M}); x) = x$, $rad(G) = diam(G) = 1$.

Theorem 3.8. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ where p is a prime number and $n \in \mathbb{Z}^+$. Let $I_i =< p^i >$ for $0 \leq i \leq n$. Then

1. For any two submodules I_r, I_s of $\mathbb{Z}_{p^n}$, $d(I_r, I_s) = |r - s|$.
2. $W\left(m(\mathbb{Z}_{p^n})\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!}$
3. $W\left(m(\mathbb{Z}_{p^n}); x\right) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n$

4. $diam\left(m(\mathbb{Z}_{p^n})\right) = n.$

5. $rad\left(m(\mathbb{Z}_{p^n})\right) = \frac{n}{2}$

Proof. It is clear that the submodules of $\mathbb{Z}_{p^n}$ are of the form $I_i =< p^i >=$ for $0 \leq i \leq n$. That is there are $n + 1$ submodules as follows:

$0\mathbb{Z}_{p^n,}\ p^{n-1}\mathbb{Z}_{p^n}, p^{n-1}\mathbb{Z}_{p^n}, p^{n-2}\mathbb{Z}_{p^n}, \ldots, I_1 = p\mathbb{Z}_{p^n}, I_0 = \mathbb{Z}_{p^n}$. This means that the graph $m(\mathbb{Z}_{p^n})$ is a path P_{n+1}that is it is a path with $n + 1$ vertices.

1. Let $I_r =< p^r >$ and $I_s =< p^s >$ be two submodules of $\mathbb{Z}_{p^n}$. Then exactly one of the following is true. $a)\ r = s$ b) $r > s$ c) $r < s$.

a) If $r = s$ then $|r - s| = 0$ and $I_r = I_s$, consequently, $d(I_r, I_s) = 0 = |r - s|$.

b) If $r > s$, then the chain $I_r \subset I_{r-1} \subset I_{r-2} \subset \ldots \subset I_{s+1} \subset I_s$ is the shortest maximal chain of submodules with the initial submodule I_r and the terminal submodule I_s. So that $d(I_r, I_s) = |r - s|$.

c) Similarly, if $r < s$, then $d(I_r, I_s) = |r - s|$.

The following figure illustrates the maximal submodule graph $mG(\mathbb{Z}_{p^n})$

2. Since $W\left(m(\mathbb{Z}_{p^n})\right) = W(P_{n+1})$, then by (Sagan , Yeh, & Zhang, 1996, pp. 960, theorem 1.3(5)), $W\left(m(\mathbb{Z}_{p^n})\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!3!}$ and

3. By (Sagan , Yeh, & Zhang, 1996, pp. 960, theorem 1.2(5)), $W\left(m(\mathbb{Z}_{p^n}); x\right) = W(P_{n+1}; x) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n = \frac{((n+1)-[n])x}{1-x}$.

4. By (Sagan , Yeh, & Zhang, 1996, pp. 960, theorem 1.1(1)), $diam\left(m(\mathbb{Z}_{p^n})\right) = degW(P_{n+1}; x) = n.$

5. It is clear.

Example 3.9. Consider the ring $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$. Then

1.The proper submodules of $\mathbb{Z}_{16}$ are as follows:
$I_1 =< 0 >= \{0\}$, $I_2 =< 2 >= \{0,2,4,6,8,10,12,14\}$,
$I_3 =< 4 >= \{0,4,8,12\}$ and $I_4 =< 8 >= \{0,8\}$.
2.The following diagram illustrates the maximal chains of submodules of $\mathbb{Z}_{16}$.

$$I_1 \subset I_2 \subset I_3 \subset I_4$$

3.The following figure illustrates the maximal submodules of graph $m(\mathbb{Z}_{16})$

$$< 0 > — < 2^3 > — < 2^2 > — < 2 > — < \mathbb{Z}_{2^4} >$$

4.The Wiener index of $m(\mathbb{Z}_{16})$ is $W\big(m(\mathbb{Z}_{16})\big) = \binom{6}{3} = \frac{6!}{3!3!} = 20$
5.The wiener polynomial for $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is $w(x) = 4x + 3x^2 + 2x^3 + x^4$.
6.$diam\big(m(\mathbb{Z}_{16})\big) = 4$.
7.$rad\big(m(\mathbb{Z}_{16})\big) = 2$

Definition 3.10 (Sagan , Yeh, & Zhang, 1996, p. 960). The Cartesian product of two graphs G_1 and G_2, is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(v_1, v_2): v_1 \in G_1$ and $v_2 \in G_2\}$ and $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2)$: $u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2v_2 \in E(G_2)$ and $u_1 = v_1\}$.

Theorem 3.11. Consider the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_q$ where p and q are two prime numbers. Then

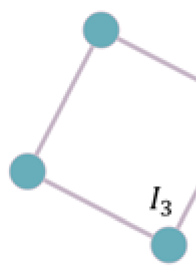
1. $rad\big(m(\mathbb{Z}_p \oplus \mathbb{Z}_q)\big) = 2$ and $diam\big(m(\mathbb{Z}_p \oplus \mathbb{Z}_q)\big) = 2$.
2. $W\big(m(\mathbb{Z}_p \oplus \mathbb{Z}_q)\big) = 8$ and $W\big(m(\mathbb{Z}_p \oplus \mathbb{Z}_q); x\big) = 4x + 2x^2$.

Proof. It is well known that $I_1 =< 0 >= \{0\}$, $I_2 =< p >= \{0, p, 2p, \ldots, (q-1)p\}$ and $I_3 =< q >= \{0, q, 2q, \ldots, (p-1)q\}$ are proper submodules of. Since $\mathbb{Z}_p \oplus \mathbb{Z}_q$ e $d(I_1, I_2) = d(I_1, I_3) = d(I_2,)\mathbb{Z}_p \oplus \mathbb{Z}_q = d(I_3,) = 1$, $d(I_2, I_3) = d(I_1, \mathbb{Z}_p \oplus \mathbb{Z}_q) = 2$, then $rad(m()) = diam(m(\mathbb{Z}_p \oplus \mathbb{Z}_q)) = 2$, $W(m()) = 8$ and $W(m(\mathbb{Z}_p \oplus \mathbb{Z}_q); x) = 4x + 2x^2$.

The following diagram illustrates the maximal chains of submodules of. $\mathbb{Z}_p \oplus \mathbb{Z}_q$.

$$I_1 \subset \begin{cases} I_2 \subset \mathbb{Z}_p \oplus \mathbb{Z}_q \\ I_3 \subset \mathbb{Z}_p \oplus \mathbb{Z}_q \end{cases}$$

The following figure illustrates the maximal submodule graph $m(\mathbb{Z}_p \oplus \mathbb{Z}_q)$



Theorem 3.12. Let p and q be any two prime numbers and $n, m \in \mathbb{Z}^+$. Then $W(m(\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{q^n}); x) = 2W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) + (n+1)W(m(\mathbb{Z}_{p^m}); x) + (m+1)W(m(\mathbb{Z}_{q^n}); x)$.

Proof. By **Theorem 2.32,** $\bar{W}(\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{q^n}; x) = \bar{W}(\mathbb{Z}_{p^m}; x) \oplus \bar{W}(\mathbb{Z}_{q^n}; x)$. Then by **Definition 2.31,** $\left(2W(m(\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{q^n}); x) + |V\left(m(\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{q^n})\right)|\right) = \left(2W(m(\mathbb{Z}_{p^m}); x) + |V\left(m(\mathbb{Z}_{p^m})\right)|\right)\left(2W(m(\mathbb{Z}_{q^n}); x) + |V\left(m(\mathbb{Z}_{q^n})\right)|\right)$. So that

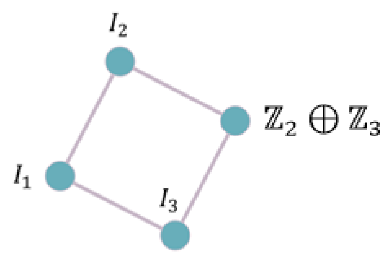
$2W(m(\mathbb{Z}_{p^m} \ \oplus \ \mathbb{Z}_{q^n}); x)$=$4W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) +$
$2\left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W(m(\mathbb{Z}_{p^m}); x) + 2\left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W(m(\mathbb{Z}_{q^n}); x)$. Then
$W(m(\mathbb{Z}_{p^m} \ \oplus \mathbb{Z}_{q^n}); x)$=$2W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) +$
$\left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W(m(\mathbb{Z}_{p^m}); x) + \left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W(m(\mathbb{Z}_{q^n}); x) =$
$2W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) + (n+1)W(m(\mathbb{Z}_{p^m}); x) + (m+1)W(m(\mathbb{Z}_{q^n}); x)$.

Example 3.13. Consider the $\mathbb{Z}$-module $\mathbb{Z}_2 \ \oplus \ \mathbb{Z}_3$. $I_1 =< 0 >= \{0\}$, $I_2 =< 2 >= \{0,2,4\}$ and $I_3 =< 3 >= \{0,3\}$ are the proper submodules of $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $diam(m(\mathbb{Z}_2 \ \oplus \ \mathbb{Z}_3)) = diam\, \mathbb{Z}_p \ \oplus \ \mathbb{Z}_q = 2$, $W(m(\mathbb{Z}_2 \ \oplus \ \mathbb{Z}_3)) = 8$ and $W(m(\mathbb{Z}_2 \ \oplus \ \mathbb{Z}_3); x) = 4x + 2x^2$.

1. The following diagram illustrates the maximal chains of submodules of $\boldsymbol{M}$.

$$I_1 \subset \begin{cases} I_2 \subset \ \mathbb{Z}_2 \oplus \mathbb{Z}_3 \\ I_3 \subset \ \mathbb{Z}_2 \oplus \mathbb{Z}_3 \end{cases}$$

2. The following figure illustrates the maximal submodule graph $mG(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$



Theorem 3.14. Let $p_1,\ p_2,\ p_3, \dots, p_r$ be r distinct prime numbers and $r, \alpha_1,\ \alpha_2, \alpha_3, \dots, \alpha_r \in \mathbb{Z}^+$. Then

1. $\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \times \mathbb{Z}_{{p_2}^{\alpha_2}} \times \dots \times \mathbb{Z}_{{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}} = \bigoplus_{i=1}^{r} \mathbb{Z}_{{p_i}^{\alpha_i}}$.
2. $m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}}) = m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha(r-1)}}\right) \times m(\mathbb{Z}_{{p_r}^{\alpha_r}}) = m(\mathbb{Z}_{{p_1}^{\alpha_1}}) \times m(\mathbb{Z}_{{p_2}^{\alpha_2}}) \times \dots \times m(\mathbb{Z}_{{p_r}^{\alpha_r}})$.

3. $V(m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}})) = V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right) \times V(m(\mathbb{Z}_{{p_r}^{\alpha_r}})) = V(m(\mathbb{Z}_{{p_1}^{\alpha_1}})) \times V(m(\mathbb{Z}_{{p_2}^{\alpha_2}})) \times \dots \times V(m(\mathbb{Z}_{{p_r}^{\alpha_r}}))$
4. $W(m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}}); x) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + (\alpha_r + 1) W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \prod_1^{r-1}(\alpha_i + 1)\, W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x).$

Proof.

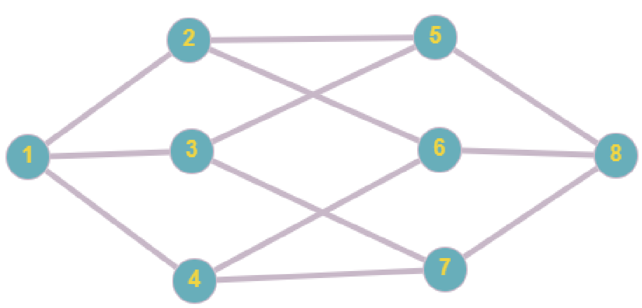
1. By (Dummit and Foote 2004, 357, Exercises 20(a)) and (Michel n.d., 8, Theorem 2.25), we obtain the result.
2. By **Definition 3.10**, we obtain the result.
3. By **Definition 3.10**, we obtain the result.
4. By **Theorem 2.32**, $\bar{W}(m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}}); x) = \bar{W}\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) \times \bar{W}(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$. Then by **Definition 3.10**, $\left(2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right|\right) = \left(2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right|\right)\left(2W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + \left|V\left(m(\mathbb{Z}_{{p_r}^{\alpha_r}})\right)\right|\right)$. So that $W(m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}}); x) = 4W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + 2\left|V\left(m(\mathbb{Z}_{{p_r}^{\alpha_r}})\right)\right| W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + 2\left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right| W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$. Then $W(m(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_r}^{\alpha_r}}); x) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + \left|V\left(m(\mathbb{Z}_{{p_r}^{\alpha_r}})\right)\right| W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right| W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x) + (\alpha_r + 1) W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}} \oplus \mathbb{Z}_{{p_2}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \prod_1^{r-1}(\alpha_i + 1)\, W(m(\mathbb{Z}_{{p_r}^{\alpha_r}}); x)$.

Therefore, $W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}} \oplus \mathbb{Z}_{p_2{}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_r{}^{\alpha_r}}\right); x\right)$=$2W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}} \oplus \mathbb{Z}_{p_2{}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_{(r-1)}{}^{\alpha(r-1)}}\right); x\right) W\left(m\left(\mathbb{Z}_{p_r{}^{\alpha_r}}\right); x\right) + (\alpha_r + 1) W\left(m\left(\mathbb{Z}_{p_1{}^{\alpha_1}} \oplus \mathbb{Z}_{p_2{}^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_{(r-1)}{}^{\alpha(r-1)}}\right); x\right) + \prod_1^{r-1}(\alpha_i + 1)\, W\left(m\left(\mathbb{Z}_{p_r{}^{\alpha_r}}\right); x\right)$

Examples. 3.15. Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ as an $\mathbb{Z}$-module. The proper submodules of M are $N_1 =< 0 >\oplus< 0 >\oplus< 0 >$, $N_2 = \mathbb{Z}_2 \oplus< 0 >\oplus< 0 >$, $N_3 =< 0 >\oplus \mathbb{Z}_3 \oplus< 0 >$, $N_4 =< 0 >\oplus< 0 >\oplus \mathbb{Z}_5$, $N_5 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus< 0 >$, $N_6 = \mathbb{Z}_2 \oplus< 0 >\oplus \mathbb{Z}_5$ and $N_7 =< 0 >\oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$. By **Example 3.13,** $W(m(\mathbb{Z}_2 \oplus \mathbb{Z}_3); x) = 4x + 2x^2$ and $W(m(\mathbb{Z}_5); x) = x$. So that by above theorem, the wiener polynomial of the graph $m(M)$ is $W(m(M); x) = 2x(4x + 2x^2) + 2(4x + 2x^2) + 4x = 12x + 12x^2 + 4x^3$ and the wiener index of $m(M)$ is $W(m(M)) = 48$. The following diagram illustrates the maximal chains of submodules of $\mathbb{Z}$ module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

$$N_1 \subset \begin{cases} N_2 \subset \begin{cases} N_5 \subset M \\ N_6 \subset M \end{cases} \\ N_3 \subset \begin{cases} N_5 \subset M \\ N_7 \subset M \end{cases} \\ N_4 \subset \begin{cases} N_6 \subset M \\ N_7 \subset M \end{cases} \end{cases}$$

The following figure illustrates the maximal ideal graph $mG(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5)$



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پوختە

لەم پڕۆژەیەدا، ئێمە ڕێگایەکی نوێ پێشکەش دەکەین بۆ دۆزینەوەی wiener polynomial و wiener index ی maximal ideal graphs $m(M)$ بو مۆدیولی $M = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_r^{\alpha_r}}$ لە کاتێك ، p_i کانیش ژمارەی خۆبەشی جیاوازن لەگەڵ یەکتر، α_i دانەیە لە $\mathbb{Z}^+$، و $1 \leq i \leq k$.

الخلاصة

في هذا المشروع ،نقدم طريقة جديدة لإيجاد متعددة حدود وينر و مؤشر وينر للرسوم البيانية القصوي $m(\mathbb{Z}_n)$ للمقاسات $M = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \dots \oplus \mathbb{Z}_{p_r^{\alpha_r}}$ حيث p_i هي اعداد اولية متميزة ،

$1 \leq i \leq k$، $\alpha_i \in \mathbb{Z}^+$.