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Salahaddin University-Erbil

Maximal chain of subgroups of a group

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. In Mathematics

By:

Roshnar jangy ibrahem

Supervised by:

hemin Abdulkarim Ahmad

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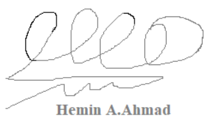
Certification of the Supervisor

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature: 

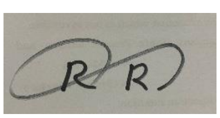
Supervisor: **Mr. Hemin Abdulkarim Ahmad**

Scientific grade: Lecturer

Date: 8 / 4 / 2023

In view of the available recommendations, I forward this report for debate by the examining committee.

Signature:



Name: **Dr. Rashad Rashid Haji**

Scientific grade: Assistant Professor

Chairman of the Mathematics Department

Date: / 4 / 2023

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Abstract

In this work we study maximal chain of subgroups of abelian groups G of order less than 26. Then we find the maximal graph $m(G)$ of those groups. Finally we investigated the Wiener index, the Wiener polynomial, the dimeter and the radical of some of the maximal subgroup graphs $m(G)$ where $|G| < 26$.

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Introduction

Let be a group. A subgroup N_1 of M is maximal in a subgroup N_2 of M if there is no subgroup N_3 of M such that $N_1 \subset N_3 \subset N_2$. A chain of subgroups $K_0 \subset K_1 \subset K_2 \subset \cdots$ of a group G is called maximal chain of subgroups of G if K_{t-1} is a maximal subgroup in K_t for each $t \in \mathbb{Z}^+$. If $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_h$ is a finite chain, then K_0 is said to be the initial subgroup and K_h is the terminal subgroup of the chain. A subgroup K_0 of G is called a maximal subgroup of length m with respect to the maximal chain of subgroups $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset M$. The maximal subgroup graph of G, denoted by $m(M)$, is the undirected graph with vertex set, the set of all subgroups of M, where two vertices N_1 and N_2 are adjacent if and only if N_1 maximal N_2, or N_2 maximal N_1. In the chapter three we study maximal chain of subgroups of groups less than 26. Then we find the maximal subgroup graph $m(G)$ of groups G. Finally we find each of the Wiener index, Wiener polynomial and dimeter of the maximal subgroup graphs $m(G)$.

Chapter One

Definitions and Back grounds of group theory

Definition 1.1 (Dummit & Foote, 2004, p. 15).A group is an ordered pair

$(G,*)$ where G is a set and $*$ is a binary operation on G satisfying the following axioms:

(i) $(a * b) * c = a * (b * c)$, for all $a, b, c \epsilon G$, i.e., * is associative,

(ii) there exists an element e in G, called an identity of G, such that for all

$a \epsilon G$ we have $a * e = e * a = a$,

(iii) for each $a \in G$ there is an element $a^{-1} of\ G$, called an inverse of a,

such that $a * a^{-1} = a^{-1} * a = e$.

Definition 1.2 (Dummit & Foote, 2004, p. 46). Let G be a group. The subset H of G is a subgroup of G if H is nonempty and H is closed under products and inverses ($i.e., x\ \ , y \epsilon H$ implies $x^{-1} \epsilon H\ and\ x\ y \epsilon H$). If H is a subgroup of G we shall write $H \leq G$.

Example.1.3

1. Consider the group $\mathbb{Z}_{36} = \{0, 1, 2, \dots, 35\}$. The proper subgroups of $\mathbb{Z}_{36}$ are $H_0 =< 0 >= \{0\}$, $H_1 =< 18 >= \{0, 18\}$, $H_2 =< 12 >= \{0, 12, 24\}$, $H_3 =< 9 >= \{0, 9, 18, 27\}$, $H_4 =< 6 >= \{0, 6, 12, 18, 24, 30\}$, $H_5 =< 4 >= \{0, 4, 8, 12, 16, 20, 24, 28, 32\}$ and $H_1 =< 2 >= \{0, 2, 4, \dots, 34\}$
2. Consider the symmetric group $S_3 = \{(),(1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. The proper subgroups of S_3 are $L_0 = \{e\}$, $L_1 = \{e, (1\ 2)\}$, $L_2 = \{e, (1\ 3)\}$, $L_3 = \{e, (2\ 3)\}$, $L_4 = \{e, (12\ 3), (132)\}\}$.

The following table illustrates the multiplication table of the symmetry group S_3.

∘	()	(1 2)	(2 3)	(1 3)	(1 2 3)	(1 3 2)
()	()	(1 2)	(2 3)	(1 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	()	(1 2 3)	(1 3 2)	(2 3)	(1 3)
(2 3)	(2 3)	(1 3 2)	()	(1 2 3)	(1 3)	(1 2)
(1 3)	(1 3)	(1 3 2)	(1 3 2)	()	(1 2)	(2 3)
(1 2 3)	(1 2 3)	(1 3)	(1 2)	(2 3)	(1 2 3)	()
(1 3 2)	(1 3 2)	(2 3)	(1 3)	(1 2)	()	(1 2 3)

Definition 1.4 (Dummit & Foote, 2004). For G a group and $x \in G$ define the order of x to be the smallest positive integer n such that $x^n = 1$, and denote this integer by $|x|$ order n. If no positive power of x is the identity. The order of a finite group is the number of its elements. If a group is not finite, one says that its order is infinite.

Definition 1.5 (Dummit & Foote, 2004, p. 37). The map φ: G → H is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if

(1) φ is a homomorphism $(i.e., \varphi\,(xy) = \varphi\,(x)\,\varphi\,(y))$, and

(2) φ is a bijection.

Definition 1.6 (Dummit & Foote, p. 65). A subgroup M of a group G is called a maximal subgroup if $M \neq G$ and the only subgroups of G which contain M are M and G.

Chapter Two

Definitions and Back grounds of Graph Theory

Definition 2.1 (Naduvath, 2017, p. 3)**.** A graph G can be considered as an ordered triple (V, E, ψ), where

(i) $V = \{ v_1, v_2, v_3, ...\}$ is called the vertex set of G and the elements of V are called the vertices (or points or nodes);

(ii) $E = \{e_1, e_2, e_3, ...\}$ is the called the edge set of G and the elements of E are called edges (or lines or arcs); and

(iii) ψ is called the adjacency relation, defined by $\psi : E \to V \times V$, which defines the association between each edge with the vertex pairs of G.

Definition 2.2 (Naduvath, 2017, p. 4)**.** The order of a graph G, denoted by $\nu(G)$, is the number of its vertices and the size of G, denoted by $\varepsilon(G)$, is the number of its edge

Definition 2.3 (Naduvath, 2017, p. 4)**.** A graph with a finite number of vertices as well as a finite number of edges is called a finite graph. Otherwise, it is an infinite graph

Definition 2.4 (Naduvath, 2017, p. 4)**.** An edge of a graph that joins a node to itself is called loop or a self-loop. That is, a loop is an edge uv, where $u = v$.

Definition 2.5 (Naduvath, 2017, p. 5)**.** The edges connecting the same pair of vertices are called multiple edges or parallel edges.

Definition 2.6 (Naduvath, 2017, p. 5)**.** A graph G which does not have loops or parallel edges is called a simple graph. A graph which is not simple is generally called a multigraph

Definition 2.7 (Naduvath, 2017, p. 5)**.** number of edges incident on a vertex v, with self-loops counted twice, is called the degree of the vertex v and is denoted by deg(v) or deg(v) or simply $d(v)$.

Definition 2.8 (Naduvath, 2017, p. 5)**.** A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are those with zero degree.

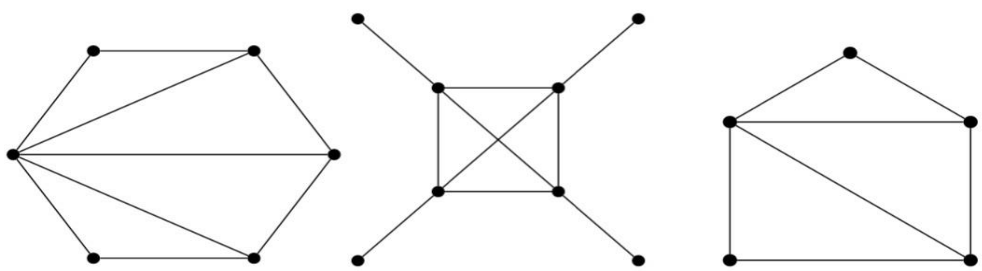
Definition 2.9 (Naduvath, 2017, p. 5)**.** A vertex, which is neither a pendent vertex nor an isolated vertex, is called an internal vertex or an intermediate vertex.

Definition 2.10 (Naduvath, 2017, p. 5)**.** The maximum degree of a graph G, denoted by $\Delta(G)$, is defined to be $\Delta(G) = max\{d(v) : v \in V(G)\}$. Similarly, the minimum degree of a graph G, denoted by $\delta(G)$, is defined to be $\delta(G) = min\{d(v) : v \in V(G)\}$. Note that for any vertex v in G, we have $\delta(G) \leq d(v) \leq \Delta(G)$.

Definition 2.11 (Naduvath, 2017, p. 7)**.** The neighborhood (or open neighbourhood) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v. That is, $N(v) = \{x \in V : vx \in E\}$. The closed neighbourhood of a vertex v, denoted by $N[v]$, is simply the set $N(v) \cup \{v\}$.

Definition 2.12 (Naduvath, 2017, p. 8)**.** A graph $H(V_1, E_1)$ is said to be a subgraph of a graph $G(V, E)$ if $V_1 \subseteq V \text{ and } E_1 \subseteq E$.

Definition 2.13 (Naduvath, 2017, p. 8)**.**A graph H(V_1,E_1) is said to be a spanning subgraph of a graph $G(V, E)\ if\ V_1 = V\ and\ E_1 \subseteq E$.



Definition 2.14 (Naduvath, 2017, p. 8)**.** Suppose that V' be a subset of the vertex set V of a graph G. Then, the subgraph of G whose vertex set is $V'$and whose edge set is the set of edges of G that have both end vertices in V' is denoted by $G[V]$ or $\langle V \rangle$ called an induced subgraph of G

Definition 2.15 (Naduvath, 2017, p. 8). Suppose that E' be a subset of the edge set V of a graph G. Then, the subgraph of G whose edge set is E' and whose vertex set is the set of end vertices of the edges in E' is denoted by $G[E]$ or $\langle E \rangle$ called an edge-induced subgraph of G.

Definition 2.16 (Naduvath, 2017, p. 8). A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph

Definition 2.17 (Naduvath, 2017, p. 11)**.** An isomorphism of two graphs G and H is a bijective function $f : V(G) \rightarrow V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. This bijection is commonly described as edge-preserving bijection. If an isomorphism exists between two graphs, then the graphs are called isomorphic graphs and denoted $as\, G \simeq H\, or\, G \cong H$.

Definition 2.18 (Naduvath, 2017, p. 23)**.**A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a closed walk.

Definition 2.19 (Naduvath, 2017, p. 23)**.** A trail is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A tour is a trail that begins and ends on the same vertex.

Definition 2.20 (Naduvath, 2017, p. 23)**.** A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A cycle or a circuit is a path that begins and ends on the same vertex.

Definition 2.21 (Naduvath, 2017, p. 23)**.** The length of a walk or circuit or path or cycle is the number of edges in it.

Definition 2.22 (Naduvath, 2017, p. 24)**.** The distance between two vertices u and v in a graph G, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the length (number of edges) of a shortest path (also called a graph geodesic) connecting them. This distance is also known as the geodesic distance.

Definition 2.23 (Naduvath, 2017, p. 24)**.**The eccentricity of a vertex v, denoted by ε(v), is the greatest geodesic distance between v and any other vertex. It can be thought of as how far a vertex is from the vertex most distant from it in the graph.

Definition 2.24 (Naduvath, 2017, p. 24)**.**The radius r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = \min_{v \in V(G)} \varepsilon(v)$.

Definition 2.25 (Naduvath, 2017, p. 24)**.** The diameter of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = \max_{v \in V(G)} \varepsilon(v)$.

Definition 2.26 (Naduvath, 2017, p. 24).A center of a graph G is a vertex of G whose eccentricity equal to the radius of G.

Definition 2.27 (Naduvath, 2017, p. 25)**.** Two vertices u and v are said to be connected if there exists a path between them. If there is a path between two vertices u and v, then u is said to be reachable from v and vice versa. A graph G is said to be connected if there exist paths between any two vertices in G.

Definition 2.28 (Naduvath, 2017, p. 26)**.** A connected component or simply, a component of a graph G is a maximal connected subgraph of G.

Definition 2.29 (Sagan , et al., 1996, p. 27)**.** Let $d(u, v)$ denote the distance between vertices u and v in a graph G. The Wiener index of G is defined as $W(G) = \sum_{\{u,v\}} d(u, v)$ where the sum is over all unordered pairs $\{u, v\}$ of distinct vertices in G. If x is a parameter, then the Wiener polynomial of G is $W(G; x) = \sum_{\{u,v\}} x^{d(u,v)}$ where the sum is taken over the same set of pairs.

Chapter three

In this chapter, we study maximal chain of subgroups of abelian groups G of order less than 26. Then we find the maximal graph $m(G)$ of those groups. Finally we investigated the Wiener index, the Wiener polynomial, the dimeter and the radical of some of the maximal subgroup graphs $m(G)$ where $|G| < 26$.

Definition 3.1 (Ahmad & Hummadi, 2023, p. 2)**.** A subgroup H_1 of a group G is maximal in a subgroup H_2 of G if there is no subgroup H_3 of G such that $H_1 \subset H_3 \subset H_2$.

Example 3.2 Consider the group of integers $\mathbb{Z}$. Then

1. The subgroups of $\mathbb{Z}$ are the form $n\mathbb{Z}$ where $n \in \mathbb{Z}^+ \cup \{\mathbf{0}\}$**.**
2. The nonzero maximal subgroups of $\mathbb{Z}$ are the form $n\mathbb{Z}$ where n is a prime number.
3. For each prime number p, if $n = pm$, then $n\mathbb{Z}$ is maximal in $m\mathbb{Z}$.
4. In the group of integers $\mathbb{Z}$, the zero subgroup is not maximal in any another subgroup.

Definition 3.3 (Ahmad & Hummadi, 2023, p. 2)**.** A chain of proper subgroups $I_0 \subset I_1 \subset I_2 \subset \cdots$ of a group G is called maximal chain of subgroups of R if I_{t-1} is maximal in I_t for each $t \in \mathbb{Z}^+$. If $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ is a finite chain, then I_0 is said to be the initial subgroup and I_h is the terminal subgroup of the chain. A subgroup K_0 of M is called a maximal subgroup of length m with respect to the maximal chain of subgroups $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset M$. The length of K_0 is said to be ∞, if there is no such finite maximal chain of subgroups with initial subgroup K_0.

Definition 3.4. Let G be a group. The maximal subgroup graph of G, denoted by $m(G)$, is the undirected graph with vertex set, the set of all subgroups of G, where two vertices I and J are adjacent if and only if I maximal in J, or J maximal in I.

Remark 3.5. Let G be a group and $m(G)$ is the maximal subgroup graph of G. Then

1. The length of the maximal chain $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ of G is h and the length of the path $I_0\ e_1\ I_1\ \ e_2\ I_2\ e_3\ ...\ \ e_h\ I_h$ of $m(G)$ is h.
2. $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_h$ is a shortest maximal chain of subgroups of G with the initial subgroup I_0 and terminal subgroup I_h if and only if $I_0\ e_1\ I_1\ \ e_2\ I_2\ e_3\ ...\ \ e_h\ I_h$ is a shortest path of $m(G)$ with the initial vertex I_0 and terminal vertex I_h where $e_i = (I_{i-1},\ I_i)$.

Remark 3.6. Let G be a group. If $|V(m(G))| > 2$, then the $m(G)$ graph is not complete.

Proof. Suppose G has at least three Let G be a groups $I =< 0 >, J$ and K. Without loss of generality if I is a maximal in both J and K, then neither J maximal in K nor K maximal in J. So that two vertices J and K are not adjacent.

Definition 3.7 (Dummit & Foote, 2004, p. 751)**.** A Group $\boldsymbol{G}$ is said to be Artinian or to satisfy the descending chain condition on subgroups (or D. C. C. on subgroups) if there is no infinite decreasing chain of subgroups in $\boldsymbol{G}$, i.e., whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a decreasing chain of subgroups of $\boldsymbol{G}$, then there is a positive integer m such that $I_m = I_k$ for all $k > m$.

Definition 3.8 (Dummit & Foote, 2004, p. 458)A Group $\boldsymbol{G}$ is said to be Noetherian or to satisfy the ascending chain condition on subgroups (or A.C.C. on subgroups) if there are no infinite increasing chains of subgroups, i.e., whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an increasing chain of subgroups of $\boldsymbol{G}$, then there is a positive integer m such that for all $k \geq m$, $I_m = I_k$.

Theorem 3.9 (Ahmad & Hummadi, 2023, p. 8)**.** If a group G is Artinian and Noetherian, then the maximal graph $mG(G)$ is connected.

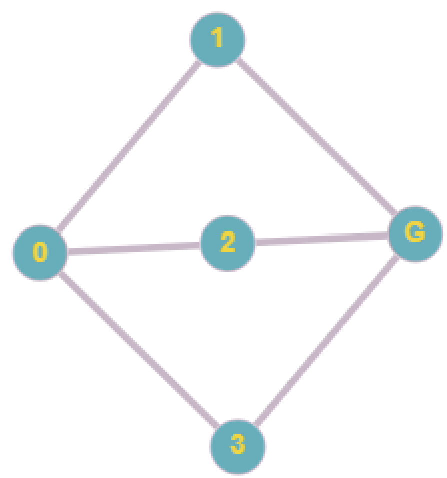
Example 3.10. Consider the group $= \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),\ (0,1), (1,0), (1,1)\}$. G has the following proper subgroups: $I_0 =< (0,0) >=< 0 >\times< 0 >= \{(0,0)\}$,

$I_1 =< (1,0) >= \mathbb{Z}_2 \times < 0 >= \{(0,0),\ (1,0)\}$, $I_2 =< (0,1) >=< 0 > \times \mathbb{Z}_2 = \{(0,0), (0,1)\}$ and $I_3 =< (1,1) >= \{(0,0), (1,1)\}$.

The following diagram illustrates the maximal chain of subgroups of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$I_0 \subset \begin{cases} I_1 \subset I_4 \\ I_2 \subset I_4 \\ I_3 \subset I_4 \end{cases}$$

The following figure illustrates the maximal subgroup graph $mG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ where I_i denoted by i for each $0 \leq i < 4$.



Definition 3.11 (Sagan , et al., 1996, p. 1)**.** Let $d(u,v)$ denote the distance between vertices u and v in a graph G. The Wiener index of G is defined as $W(G) = \sum_{\{u,v\}} d(u,v)$ where the sum is over all unordered pairs $\{u,v\}$ of distinct vertices in G. If x is a parameter, then the Wiener polynomial of G is $W(G;x) = \sum_{\{u,v\}} x^{d(u,v)}$ where the sum is taken over the same set of pairs.

Theorem 3.12. Let G be a graph and $W(G)$, $W(G;x)$ be the Wiener index and Wiener polynomial of G respectively. Then

1. $deg(W(G;\ q))$ equals the diameter of G.
2. $W(G) = f'(1)$

Proof.

1. By (Sagan , et al., 1996, pp. 960 , Theorem 1.1), the result is obtained.
2. By (Sagan , et al., 1996, pp. 960, theorem 1.1(5)), the result is obtained.

The following proposition is easy to prove

Proposition 3.13. If $\boldsymbol{G}$ is a group and $G \simeq m(\mathbb{Z}_p)$ where p is a prime number, then

1. $W(m(\boldsymbol{G})) = 1$ and $W(m(\boldsymbol{G}); x) = x$.
2. $rad(m(\boldsymbol{G})) = diam(m(\boldsymbol{G})) = 1$.

Proof. It is clear that $V(m(\mathbb{Z}_p)) = \{< 0 >, \mathbb{Z}_p\}$. Then $d(< 0 >, \mathbb{Z}_p) = 1$. Therefore, that $W(m(\boldsymbol{G})) = 1$, $W(m(\boldsymbol{G}); x) = x$ and $rad(m(\boldsymbol{G})) = diam(m(\boldsymbol{G})) = 1$.

Corollary 3.14.

1. $W(m(\mathbb{Z}_2)) = W(m(\mathbb{Z}_3)) = W(m(\mathbb{Z}_5)) = W(m(\mathbb{Z}_7)) = W(m(\mathbb{Z}_{11})) = W(m(\mathbb{Z}_{13})) = W(m(\mathbb{Z}_{17})) = W(m(\mathbb{Z}_{19})) = W(m(\mathbb{Z}_{23})) = 1$.
2. $W(m(\mathbb{Z}_2); x) = W(m(\mathbb{Z}_3); x) = W(m(\mathbb{Z}_5); x) = W(m(\mathbb{Z}_7); x) = W(m(\mathbb{Z}_{11}); x) = W(m(\mathbb{Z}_{13}); x) = W(m(\mathbb{Z}_{17}); x) = W(m(\mathbb{Z}_{19}); x) = W(m(\mathbb{Z}_{23}); x) = x$.

Theorem 3.15. Let P_n be a path with n vertices for some $n \in \mathbb{Z}^+$. Then

1. $W(P_n) = \binom{n+1}{3} = \frac{(n+1)!}{(n-2)!3!}$;
2. $W(P_n;x) = (n-1)x + (n-2)x^2 + (n-3)x^3 + \cdots + 2x^{n-2} + x^{n-1}$.

Proof.

1. By (Sagan , et al., 1996, p. Theorem 1.3(5)), the result is obtained.
2. By (Sagan , et al., 1996, p. Theorem 1.2(5)), the result is obtained.

Theorem 3.16. Consider the group $\mathbb{Z}_{p^n}$ where p is a prime number and $n \in \mathbb{Z}^+$. Let $I_i =< p^i >$ for $0 \leq i \leq n$. Then

1. For any two subgroups I_r, I_s of $\mathbb{Z}_{p^n}$, $d(I_r, I_s) = |r-s|$.
2. $W\left(m(\mathbb{Z}_{p^n})\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!}$
3. $W\left(m(\mathbb{Z}_{p^n}); x\right) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n$
4. $diam\left(m(\mathbb{Z}_{p^n})\right) = n$.
5. $rad\left(m(\mathbb{Z}_{p^n})\right) = \begin{cases} \frac{n}{2} & if\ n\ is\ an\ even\ number \\ \frac{n+1}{2} & if\ n\ is\ an\ add\ number \end{cases}$

Proof. It is clear that the subgroups of $\mathbb{Z}_{p^n}$ are of the form $I_i =< p^i >=$ for $0 \leq i \leq n$. That is there are $n+1$ subgroups as follows:

$0\mathbb{Z}_{p^n}, p^{n-1}\mathbb{Z}_{p^n}, p^{n-1}\mathbb{Z}_{p^n}, p^{n-2}\mathbb{Z}_{p^n}, \ldots, I_1 = p\mathbb{Z}_{p^n}, I_0 = \mathbb{Z}_{p^n}$. This means that the graph $m(\mathbb{Z}_{p^n})$ is a path P_{n+1}, that is it is a path with $n+1$ vertices.

1. Let $I_r =< p^r >$ and $I_s =< p^s >$ be two subgroups of $\mathbb{Z}_{p^n}$. Then exactly one of the following is true. $a)$ $r = s$ b) $r > s$ c) $r < s$.

a) If $r = s$, then $|r-s| = 0$ and $I_r = I_s$, consequently, $d(I_r, I_s) = 0 = |r-s|$.

b) If $r > s$, then the chain $I_r \subset I_{r-1} \subset I_{r-2} \subset \ldots \subset I_{s+1} \subset I_s$ is the shortest maximal chain of subgroups with the initial subgroup I_r and the terminal subgroup I_s. So that $d(I_r, I_s) = |r-s|$.

c) Similarly, if $r < s$, then $d(I_r, I_s) = |r-s|$.

The following figure illustrates the distance from $< p^s >$ to $< p^s >$ in the maximal subgroup graph $mG(\mathbb{Z}_{p^n})$

2. Since $W\left(m\left(\mathbb{Z}_{p^n}\right)\right) = W(P_{n+1})$, then by **Theorem 3.15(1),** $W\left(m\left(\mathbb{Z}_{p^n}\right)\right) = \binom{n+2}{3} = \frac{(n+2)!}{(n-1)!3!}$ and
3. By **Theorem 3.15(2)**, $W\left(m\left(\mathbb{Z}_{p^n}\right); x\right) = W(P_{n+1}; x) = nx + (n-1)x^2 + (n-2)x^3 + \cdots + x^n$.
4. By **Theorem 3.12(1),** $diam\left(m\left(\mathbb{Z}_{p^n}\right)\right) = degW(P_{n+1}; x) = n.$
5. It is clear that $\varepsilon(< 0 >) = \varepsilon\left(\mathbb{Z}_{p^n}\right) = n$, $\varepsilon(< p^{n-1} >) = \varepsilon(< p >) = n-1$, $\varepsilon(< p^{n-2} >) = \varepsilon(< p^2 >) = n-2, \dots$. So that for $0 \le i \le n$, $\varepsilon\left(< p^{n-i} >\right) = \varepsilon\left(< p^i >\right) = n - i$. Now, there are two cases. Case one, if n is an even number, then $\varepsilon\left(< p^{\frac{n}{2}} >\right) \le \varepsilon(< p^t >)$ where $0 \le t \le n$. Case two, if n is an add number, then $\varepsilon\left(< p^{\frac{n+1}{2}} >\right) \le \varepsilon(< p^t >)$ where $0 \le t \le n$. Therefore, $rad\left(m\left(\mathbb{Z}_{p^n}\right)\right) = \begin{cases} \frac{n}{2} & if\ n\ is\ an\ even\ number \\ \frac{n+1}{2} & if\ n\ is\ an\ add\ number \end{cases}$.

Corollary 3.17.

1. Consider the maximal graphs $m(\mathbb{Z}_4), m(\mathbb{Z}_8), m(\mathbb{Z}_9), m(\mathbb{Z}_{16})$ and $m(\mathbb{Z}_{25})$. Then

a) $W\left(m(\mathbb{Z}_4)\right) = W\left(m(\mathbb{Z}_9)\right) = W\left(m(\mathbb{Z}_{25})\right) = \binom{2+2}{3} = \frac{4!}{1!3!} = 4.$

b) $W(m(\mathbb{Z}_4); x) = W(m(\mathbb{Z}_9); x) = W(m(\mathbb{Z}_{25}); x) = 2x + x^2.$

c) $diam\left(m(\mathbb{Z}_4)\right) = diam\left(m(\mathbb{Z}_9)\right) = diam\left(m(\mathbb{Z}_{25})\right) = 2.$

d) $rad\left(m(\mathbb{Z}_4)\right) = diam\left(m(\mathbb{Z}_9)\right) = diam\left(m(\mathbb{Z}_{25})\right) = 1.$

2. Consider the maximal graph $m(\mathbb{Z}_8)$. Then

a) $W\big(m(\mathbb{Z}_8)\big) = \binom{2+3}{3} = \frac{5!}{2!3!} = 10.$

b) $W(m(\mathbb{Z}_8); x) = 3x + 2x^2 + x^3.$

c) $diam\big(m(\mathbb{Z}_8)\big) = 3.$

d) $rad\big(m(\mathbb{Z}_8)\big) = 2.$

3. Consider the maximal graph $m(\mathbb{Z}_{16})$. Then

a) $W\big(m(\mathbb{Z}_{16})\big) = \binom{2+4}{3} = \frac{6!}{3!3!} = 20.$

b) $W(m(\mathbb{Z}_{16}); x) = 4x + 3x^2 + 2x^3 + x^4.$

c) $diam\big(m(\mathbb{Z}_{16})\big) = 4.$

d) $rad\big(m(\mathbb{Z}_{16})\big) = 2.$

Definition 3.18 (Sagan , et al., 1996, p. 960). The Cartesian product of two graphs G_1 and G_2, is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(v_1, v_2): v_1 \in G_1$ and $v_2 \in G_2\}$ and $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2)$: $u_1 v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2 v_2 \in E(G_2)$ and $u_1 = v_1\}$.

Proposition 3.19. Let p and q be any two distinct prime numbers and $n, m \in \mathbb{Z}^+$. Then

1. $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n} = \{(a, b): a \in \mathbb{Z}_{p^m}$ and $b \in \mathbb{Z}_{q^n}\}$ is a group.
2. $\left|\mathbb{Z}_{p^m}\right| = p^m$, $\left|\mathbb{Z}_{q^n}\right| = q^n$ and $\left|\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}\right| = |\mathbb{Z}_{p^m q^n}| = p^m q^n$
3. The subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$ are of the form $I_1 \times I_2$ where I_1 is a subgroup of $\mathbb{Z}_{p^m}$ and I_2 is a subgroup of $\mathbb{Z}_{q^n}$.
4. $I_1 \times I_2$ is maximal in $J_1 \times J_2$ if and only if I_1 is maximal in J_1 and $I_2 = J_2$ or I_2 is maximal in J_2 and $I_1 = J_1$.
5. $m\big(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}\big) = m\big(\mathbb{Z}_{p^m}\big) \times m\big(\mathbb{Z}_{q^n}\big) = m\big(\mathbb{Z}_{p^m q^n}\big).$
6. $V\Big(m\big(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}\big)\Big) = V\Big(m\big(\mathbb{Z}_{p^m}\big)\big) \times V\big(m(\mathbb{Z}_{q^n}\big)\Big) = V\big(m\big(\mathbb{Z}_{p^m q^n}\big)\big)$
7. $I_1 \times I_2$ is maximal in $J_1 \times J_2$ if and only if $(I_1 \times I_2)(J_1 \times J_2) \in E\big(m\big(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}\big)\big)$.

Proof.
1, 2, 3 and 4 are obvious.
5, 6, 7 are direct consequences of **Definition 3.18.**

Note that if $p = q$, then $V\left(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right) \neq V\left(m(\mathbb{Z}_{p^m})) \times V(m(\mathbb{Z}_{q^n})\right)$. For example, $V(m(\mathbb{Z}_2 \times \mathbb{Z}_2)) \neq V(m(\mathbb{Z}_2)) \times V(m(\mathbb{Z}_2))$, since $V(m(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \{< 0 > \times < 0 >,\ \mathbb{Z}_2 \times < 0 >,\ < 0 > \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \{(0,0),(1,1)\}$ and $V(m(\mathbb{Z}_2)) \times V(m(\mathbb{Z}_2))$=$\{< 0 > \times < 0 >,\ \mathbb{Z}_2 \times < 0 >,\ < 0 > \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2\}$

Definition 3.20 (Sagan , et al., 1996, p. 961). The ordered Wiener Polynomial defined by $\overline{W}(G;q) = \sum_{(u,v)} x^{d(u,v)}$, where the sum is over all ordered pairs (u,v) of vertices, including those where $u = v$. Thus, $\overline{W}(G;q) = \sum_{(u,v)} x^{d(u,v)} = 2W(G;q) + |V(G)|$.

Theorem 3.21 (Sagan , et al., 1996, pp. 961, Proposition 1.4(2))**.** Suppose that G_1 and G_2 are two connected graphs. Then $\overline{W}(\ G_1 \times G_2; x) = \overline{W}(G_1; x) \times \overline{W}(G_2; x)$.

Theorem 3.22. Let p and q be any two prime numbers and $n, m \in \mathbb{Z}^+$. Then $W(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}); x) = 2W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) + (n + 1)W(m(\mathbb{Z}_{p^m}); x) + (m + 1)W(m(\mathbb{Z}_{q^n}); x)$.

Proof. By **Theorem 3.21**, $\overline{W}(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}; x) = \overline{W}(\mathbb{Z}_{p^m}; x) \times \overline{W}(\mathbb{Z}_{q^n}; x)$. Then by **Definition 3.20,** $\left(2W(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}); x) + |V\left(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right)|\right) = \left(2W(m(\mathbb{Z}_{p^m}); x) + |V\left(m(\mathbb{Z}_{p^m})\right)|\right)\left(2W(m(\mathbb{Z}_{q^n}); x) + |V\left(m(\mathbb{Z}_{q^n})\right)|\right)$. So that $2W(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}); x)$=$4W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) + 2\left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W(m(\mathbb{Z}_{p^m}); x) + 2\left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W(m(\mathbb{Z}_{q^n}); x)$. Then $W(m(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}); x)$=$2W(m(\mathbb{Z}_{p^m}); x)W(m(\mathbb{Z}_{q^n}); x) + \left|V\left(m(\mathbb{Z}_{q^n})\right)\right| W(m(\mathbb{Z}_{p^m}); x) + \left|V\left(m(\mathbb{Z}_{p^m})\right)\right| W(m(\mathbb{Z}_{q^n}); x) =$

$2W\left(m\left(\mathbb{Z}_{p^m}\right);x\right)W\left(m\left(\mathbb{Z}_{q^n}\right);x\right)+(n+1)W\left(m\left(\mathbb{Z}_{p^m}\right);x\right)+(m+1)W\left(m\left(\mathbb{Z}_{q^n}\right);x\right)$.

Corollary 3.23. Consider the group $\mathbb{Z}_p\times\mathbb{Z}_q$ where p and q are two prime numbers. Then

1. The wiener polynomial of the maximal subgroup graph $m(\mathbb{Z}_p\times\mathbb{Z}_q)$ is $W(m(\mathbb{Z}_p\times\mathbb{Z}_q);x)=4x+2x^2$.
2. The wiener index of the maximal subgroup graph $m(\mathbb{Z}_p\times\mathbb{Z}_q)$ is $W(m(\mathbb{Z}_p\times\mathbb{Z}_q))=8$.

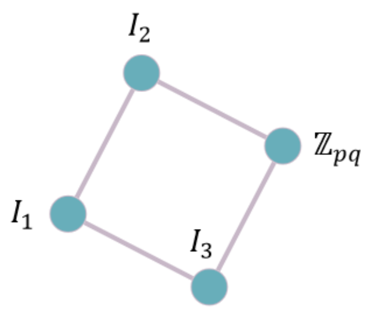
Proof.

1. By **Proposition 3.13**, $W\left(m\left(\mathbb{Z}_p\right)\right)=W\left(m\left(\mathbb{Z}_q\right)\right)=x$. By **Theorem 3.22**, $W\left(m\left(\mathbb{Z}_p\times\mathbb{Z}_q\right);x\right)=2W\left(m\left(\mathbb{Z}_p\right);x\right)W\left(m\left(\mathbb{Z}_q\right);x\right)+(1+1)W\left(m\left(\mathbb{Z}_p\right);x\right)+(1+1)W\left(m\left(\mathbb{Z}_q\right);x\right)=4x+2x^2$.
2. $W(m(\mathbb{Z}_p\times\mathbb{Z}_q))=W'(m(W(m(\mathbb{Z}_p\times\mathbb{Z}_q);1)=4+4(1)=8$.

The following diagram illustrates the maximal chains of subgroups of $\mathbb{Z}_p\times\mathbb{Z}_q$.

$$I_1\subset\begin{cases}I_2\subset\ \mathbb{Z}_p\times\mathbb{Z}_q\\ I_3\subset\ \mathbb{Z}_p\times\mathbb{Z}_q\end{cases}$$

The following figure illustrates the maximal subgroup graph $mG(\mathbb{Z}_p\times\mathbb{Z}_q)$



Example 3.24. Consider the maximal graphs $m(\mathbb{Z}_2\times\mathbb{Z}_3)$, $m(\mathbb{Z}_2\times\mathbb{Z}_5)$, $m(\mathbb{Z}_2\times\mathbb{Z}_7)$, $m(\mathbb{Z}_2\times\mathbb{Z}_{11})$, $m(\mathbb{Z}_3\times\mathbb{Z}_5)$ and $m(\mathbb{Z}_3\times\mathbb{Z}_7)$. If G is one of the above group, then

a) $W\left(m(G)\right) = 8$.

b) $W(m(G); x) = 4x + 2x^2$.

c) $diam\left(m(G)\right) = 2$.

d) $rad\left(m(G)\right) = 2$.

Theorem 3.25. Let $p_1,\ p_2,\ p_3, \dots, p_r$ be r distinct prime numbers and $r, \alpha_1,\ \alpha_2,\ \alpha_3, \dots, \alpha_r \in \mathbb{Z}^+$. Then

1. $\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \times \mathbb{Z}_{{p_2}^{\alpha_2}} \times \dots \times \mathbb{Z}_{{p_r}^{\alpha_r}} = \mathbb{Z}_{{p_1}^{\alpha_1}} \bigoplus \mathbb{Z}_{{p_2}^{\alpha_2}} \bigoplus \dots \bigoplus \mathbb{Z}_{{p_r}^{\alpha_r}} = \bigoplus_{i=1}^{r} \mathbb{Z}_{{p_i}^{\alpha_i}}$.
2. $m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right) = m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right) \times m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right) = m(\mathbb{Z}_{{p_1}^{\alpha_1}}) \times m(\mathbb{Z}_{{p_2}^{\alpha_2}}) \times \dots \times m(\mathbb{Z}_{{p_r}^{\alpha_r}})$.
3. $V(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right)) = V(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)) \times V(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right)) = V(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}}\right)) \times V(m\left(\mathbb{Z}_{{p_2}^{\alpha_2}}\right)) \times \dots \times V(m(\mathbb{Z}_{{p_r}^{\alpha_r}}))$
4. $W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right); x\right) = 2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right) + (\alpha_r + 1) W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + \prod_{1}^{r-1}(\alpha_i + 1)\, W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right)$

Proof.

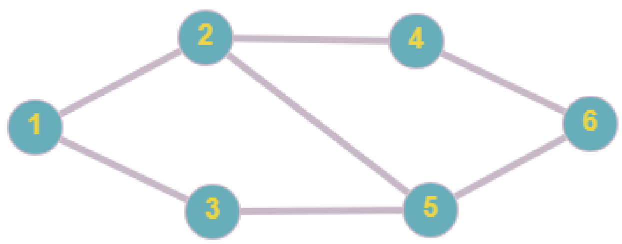
1. By (Dummit & Foote, 2004, pp. 357, Exercises 20(a)), we obtain the result.
2. By **Definition 3.18**, we obtain the result.
3. By **Definition 3.18**, we obtain the result.
4. By **Theorem 3.21**, $\bar{W}\left(m(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}); x\right) = \bar{W}\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) \times \bar{W}\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right)$. Then by **Definition 3.20**, $\left(2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right); x\right) + |V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right)\right)|\right) = \left(2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) + |V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)|\right)\left(2W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right) + |V\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right)\right)|\right)$. So that $2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_r}^{\alpha_r}}\right); x\right) = 4W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right); x\right) + 2\left|V\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right)\right)\right| W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_{(r-1)}}^{\alpha_{(r-1)}}}\right); x\right) +$

$2\left|V\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_{(r-1)}}^{\alpha_{(r-1)}}}\right)\right)\right| W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right);x\right)$. Then $W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_r}^{\alpha_r}}\right);x\right)$ $2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_{(r-1)}}^{\alpha_{(r-1)}}}\right);x\right) W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right);x\right) + (\alpha_r + 1)W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_{(r-1)}}^{\alpha_{(r-1)}}}\right);x\right) + \prod_1^{r-1}(\alpha_i+1)\, W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right);x\right)$.

Therefore,

$$W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_r}^{\alpha_r}}\right);x\right)=2W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_{(r-1)}}^{\alpha_{(r-1)}}}\right);x\right) W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right);x\right) + (\alpha_r + 1)W\left(m\left(\mathbb{Z}_{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_{(r-1)}}^{\alpha_{(r-1)}}}\right);x\right) + \prod_1^{r-1}(\alpha_i+1)\, W\left(m\left(\mathbb{Z}_{{p_r}^{\alpha_r}}\right);x\right)$$

Corollary 3.26. The wiener polynomial of the graph $m(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $W(m(\mathbb{Z}_{p^2} \times \mathbb{Z}_q);x) = 7x + 6x^2 + 2x^3$ and the wiener index of the graph $m(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $7 + 12 + 6 = 25$. The following figure illustrates the maximal subgroup graph $mG(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ where I_i denoted by i for each $1 \leq i < 7$



Example 3.27. Consider the maximal graphs $m(\mathbb{Z}_4 \times \mathbb{Z}_3)$, $m(\mathbb{Z}_4 \times \mathbb{Z}_5)$, $m(\mathbb{Z}_2 \times \mathbb{Z}_7)$, $m(\mathbb{Z}_2 \times \mathbb{Z}_{11})$, $m(\mathbb{Z}_3 \times \mathbb{Z}_5)$ and $m(\mathbb{Z}_3 \times \mathbb{Z}_7)$. If G is one of the above group, then

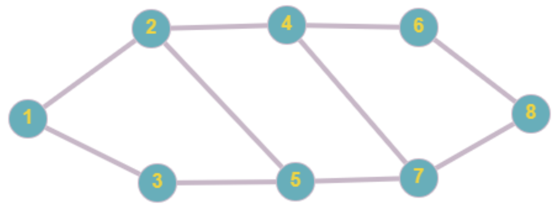
a) $W(m(G)) = 8$.

b) $W(m(G);x) = 4x + 2x^2$.

c) $diam(m(G)) = 2$.

d) $rad(m(G)) = 2$.

Example 3.28. The wiener polynomial of the graph $m(\mathbb{Z}_8 \times \mathbb{Z}_3)$ is $(m(\mathbb{Z}_8 \times \mathbb{Z}_3); x) = 10x + 10x^2 + 6x^3 + 2x^4$ and the wiener index of the graph $m(\mathbb{Z}_{p^3} \times \mathbb{Z}_q)$ is 10+20+18+8=56 The following figure illustrates the maximal subgroup graph $mG(\mathbb{Z}_8 \times \mathbb{Z}_3)$ where I_i denoted by i for each $1 \leq i < 9$.



e) $W(m(G)) = 56$.

f) $W(m(G); x) = 10x + 10x^2 + 6x^3 + 2x^4$.

g) $diam(m(G)) = 4$.

h) $rad(m(G))$ =4.

References

Ahmad, H. A. & Hummadi, P. A., 2023. Maximal Chain Of Ideals And N-Maximal Ideal. *Commun. Korean Math. Soc.*.

Dummit, D. S. & Foote, R. M., 2004. *Abstract Algebra.* United States of America: Wiley Hoboken.

M, F. A. & I, G. M., 1969. *Introduction to Commutative Algebra.* London: Addison wesley company.

Michel, O., n.d. An Introduction To The Zariski Topology.

Naduvath, S., 2017. *Lecture Notes On Graph Theory.* Thrissur-India: Vidya Academy of Science & Technology.

Naduvath, S., 2017. *Lecture Notes on Graph Theory.* Kerala, India: Sudev Naduvath.

Sagan , E. B., Yeh, Y.-N. & Zhang, P., 1996. The Wiener Polynomial Of A Graph. *International Journal Of Quantum Chemistry,* 60(5), pp. 959--969.

پوختە

لەم پرۆژەیەدا، ئێمە هەریەك لە wiener polynomial و graphs wiener index و diameter **بۆ** $m(G)$ maximal subgroup graph **دەدۆزینەوە لە کاتێك دا گروپی** G **ئۆردەری بچوکترە لە** 26 **ە**.

الخلاصة

في هذا المشروع ، نجد متعددة حدود وينر و مؤشر وينر للرسوم البيانية القصوي $m(G)$ **للزمرات** G **حيث** $|G| < 26$.