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Salahaddin University-Erbil

The

The Sombor coindex of maximal ideals gaphs

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. in Mathematics

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Certification of the Supervisor

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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I would like to thank my family, friend and library staff whose support has helped me to conceive this research

Abstract

In this project, we study the Sombor coindex of various specialized graphs, including Path graphs, Circle graphs, Wheel graphs, Ladder graphs, Grid graphs, and Crystal Lattice graphs. Subsequently, our focus shifts to the maximal chain of ideals of rings $\mathbb{Z}_n$ where $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_k}^{\alpha_k}$, p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$,and $1 \leq i \leq k$. Finally, we find a technique to find the Sombor coindex of some maximal ideal graphs $m(\mathbb{Z}_n)$ of rings $\mathbb{Z}_n$.

Table of Contents

Introduction

Let R be a ring. An ideal I_1 of R is maximal in an ideal I_2 of R if there is no ideal I_3 of R such that $I_1 \subset I_3 \subset I_2$ (Ahmad and Hummadi 2023). A chain of proper ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ of R is called maximal chain of ideals of R if I_{t-1} is maximal in I_t for each $t \in \mathbb{Z}^+$. The maximal ideal graph of R , denoted by $m(R)$, is the undirected graph with vertex set, the set of all ideals of R , where two vertices $I$and J are adjacent if and only if I maximal in J, or J maximal in I (Ahmad & Hummadi, 2023). Let $G = (V, E)$ be a finite simple graph. The Sombor index $SO(G)$ of G is defined as $\sum_{u,v \in E(G)} \sqrt{d_u^2 + d_v^2}$ and the Sombor coindex $\overline{SO(G)}$ of G is defined as $\sum_{u,v \notin E(G)} \sqrt{d_u^2 + d_v^2}$ where d_u is the degree of vertex u in G (Du, et al., 2023) and (Ghanbari & Alikhani, Sat, 20 Feb 2021). In this work, we study both indexes for some certain graphs. In the chapter three we focus on finding the Sombor coindex of maximal ideal graphs $m(\mathbb{Z}_n)$ where $= {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_k}^{\alpha_k}$, p_i's are distinct primes, $\alpha_i \in \mathbb{Z}^+$ and $1 \le i \le k$.

Chapter One

Definitions and Backgrounds of ring theory

Definition 1.1 (ATIYAH & MACDONALD, 1969)**.** A ring R is a set with two binary operations (addition and multiplication) such that

1) R is an abelian group with respect to addition (so that R has a zero element, denoted by 0, and every $x \in R$ has an (additive) inverse, $-x$).
2) Multiplication is associative ($(xy)z = x(yz)$) and distributive over addition

$(x(y + z) = xy + xz$, $(y + z)x = yx + zx)$.

We shall consider only rings which are commutative:

3) $xy = yx$ for all x, $y \in R$, and have an identity element (denoted by 1):
4) $\exists 1 \in R$ such that $x1 = 1x = x$ for all $x \in R$.

Example 1.2 (DUMMIT & FOOTE, 2004)**.**

1. The ring of integers $\mathbb{Z}$, under the usual operations of addition and multiplication is a commutative ring with identity (the integer 1).
2. The quotient group $\mathbb{Z}/n\,\mathbb{Z}$ is a commutative ring with identity (the element 1) under the operations of addition and multiplication of residue classes.

Definition 1.3 (DUMMIT & FOOTE, 2004)**.** A subring of the ring R is a subgroup of R that is closed under multiplication.

Definition 1.4 (DUMMIT & FOOTE, 2004, p. 242)**.** Let R be a ring, let I be a subset of R and let $r \in R$.

1) $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$.

2) A subset I of R is a left ideal of R if

a. I is a subring of R, and

b. I is closed under left multiplication by elements from R, i.e., $rI \subseteq I$ for all $r \subseteq R$.

Similarly I is a right ideal if (a) holds and in place of (b) one has

c. I is closed under right multiplication by elements from R, i.e., $Ir \subseteq I$ for all $r \in R$.

3) A subset I that is both a left ideal and a right ideal is called an ideal (or, for added emphasis, a two-sided ideal) of R.

Example 1.5. Consider the ring of all rational numbers $\mathbb{Q}$. Then $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but it is not an ideal of $\mathbb{Q}$.

Definition 1.6 (DUMMIT & FOOTE, 2004, p. 255)**.** Assume R is commutative. An ideal P is called a prime ideal if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Definition 1.7 (DUMMIT & FOOTE, 2004, p. 253)**.** An ideal M in an arbitrary ring R is called a maximal ideal if $M \neq R$ and the only ideals containing M are M and R.

Definition 1.8 (Ahmad & Hummadi, 2023)**.** Let R be a ring. An ideal I_1 of R is maximal in an ideal I_2 of R if there is no ideal I_3 of R such that $I_1 \subset I_3 \subset I_2$.

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Definition 1.9 (Ahmad & Hummadi, 2023). A chain of proper ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ of R is called maximal chain of ideals of R if I_{t-1} is maximal in I_t for each $t \in \mathbb{Z}^+$.

Example 1.10. Consider the ring $\mathbb{Z}_{36} = \{0, 1, 2, \ldots, 35\}$. The ring $\mathbb{Z}_{36}$ has the following proper ideals: $I_0 =< 0 >$, $I_1 =< 18 >= \{0, 18\}$, $I_2 =< 12 >$ = {0, 12, 24}, $I_3 =< 9 >$={0, 9, 18, 27}, $I_4 =< 6 >$ ={0, 6, 12, 18, 24, 30}, $I_5 =< 4 >$ ={0, 4, 8, 12, 16, 20, 24, 28, 32}, $I_6 =< 3 >$={0, 3, 6, 12, 15,18, 21, 24, 27, 30, 33}, $I_7 =< 2 >$={0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34}.

The following diagram illustrates the maximal chain of ideals of the ring $\mathbb{Z}_{36}$.

$$I_0 \subset \begin{cases} I_1 \subset \begin{cases} I_3 \subset I_6 \subset \mathbb{Z}_{36} \\ I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{36} \\ I_7 \subset \mathbb{Z}_{36} \end{cases} \end{cases} \\ I_2 \subset \begin{cases} I_4 \subset \begin{cases} I_6 \subset \mathbb{Z}_{36} \\ I_7 \subset \mathbb{Z}_{36} \end{cases} \\ I_5 \subset I_7 \subset \mathbb{Z}_{36} \end{cases} \end{cases}$$

Chapter Two

Definitions and Backgrounds of Graph Theory

Definition 2.1 (Gross, et al., 2014, p. 2)**.** A graph G = (V, E) consists of two sets $V$and E.

1) The elements of V are called vertices (or nodes).
2) The elements of E are called edges.
3) Each edge has a set of one or two vertices associated to it, which are called its endpoints. An edge is said to join its endpoints.

Definition 2.2 (NADUVATH, 2017, p. 23)**.** A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges. If the starting and end vertices of a walk are the same, then such a trail is called a closed walk.

Definition 2.3 (NADUVATH, 2017, p. 23)**.** A trail is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time. A tour is a trail that begins and ends on the same vertex.

Definition 2.4 (NADUVATH, 2017, p. 23)**.** A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A cycle or a circuit is a path that begins and ends on the same vertex.

Definition 2.5 (NADUVATH, 2017, p. 23)**.** The length of a walk or circuit or path or cycle is the number of edges in it.

Definition 2.6. The number of edges incident on a vertex u, is called the degree of the vertex v and is denoted by $deg_G(u)$ or $deg(u)$ or simply $d(u)$ or d_u.

Definition 2.7 (Ghanbari & Alikhani, Sat, 20 Feb 2021). Let $G = (V, E)$ be a finite simple graph. The Sombor index $SO(G)$ of G is defined as $\sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$ and the Sombor coindex $\overline{SO}(G)$ of G is defined as $\sum_{uv \notin E(G)} \sqrt{d_u^2 + d_v^2}$ where d_u is the degree of vertex u in G.

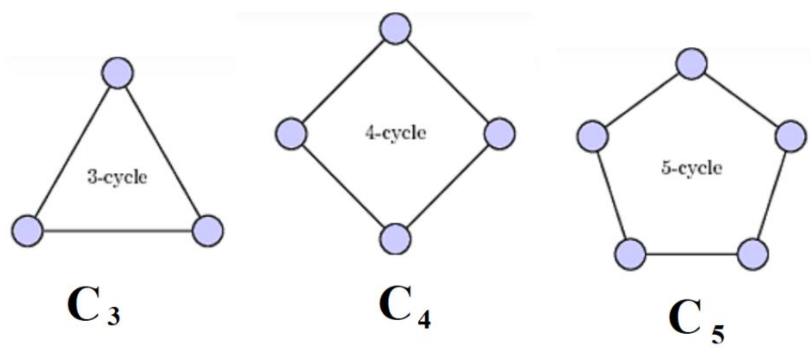
Note that for each $uv \in E(G)$ or $uv \notin E(G)$, $SO(uv) = \sqrt{d_u^2 + d_v^2}$. So that $SO(G) = \sum_{uv \in E(G)} SO(uv) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$

Notation. Let $G = (V, E)$ be a finite simple graph, where V is the set of vertices and E is the set of edges of graph G. The complement of $E(G)$, denoted by $\overline{E(G)}$, is the set of all possible edges that do not belong to $E(G)$.

Therefore, $\overline{SO}(G)$ of G is defined as $\sum_{uv \notin E(G)} \sqrt{d_u^2 + d_v^2} = \sum_{uv \in \overline{E(G)}} \sqrt{d_u^2 + d_v^2}$ where d_u is the degree of vertex u in G.

Definition 2.8 (Bondy & Murty, 1976)**.** A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3, if the graph is simple) connected in a closed chain. The cycle graph with n vertices is called C_n. The number of vertices in C_n equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.

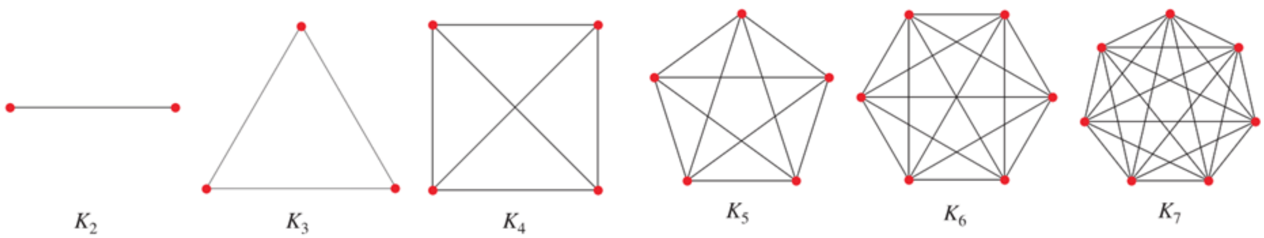
Example 2.9. Consider the following three cycle graphs:



1. $V(C_3)$ =$\{v_1, v_2, v_3\}$, $E(C_3)$= $\{v_1v_2, v_2v_3,\ v_3v_1\}$and $\overline{E(C_3)} = \emptyset$. Then $SO(C_3) = 6\sqrt{2}$ and $\overline{SO}(C_3) = 0$.
2. $V(C_4)$ =$\{v_1, v_2, v_3, v_4\}$, $E(C_4)$= $\{v_1v_2, v_2v_3,\ v_3v_4,\ v_4v_1\}$ and $\overline{E(C_4)} = \{v_1v_3,\ v_2v_4\}$. Then $SO(C_4) = 8\sqrt{2}$ and $\overline{SO}(C_4) = 4\sqrt{2}$.
3. $V(C_5)$ =$\{v_1, v_2, v_3, v_4, v_5\}$, $E(C_5)$= $\{v_1v_2, v_2v_3,\ v_3v_4,\ v_4v_5,\ v_5v_1\}$ and $\overline{E(C_4)} = \{v_1v_3, v_1v_4\ \ v_2v_4, v_2v_5,\ v_3v_5\}$. Then $SO(C_5) = \overline{SO}(C_5) = 10\sqrt{2}$

Definition 2.10. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. The complete graph on n vertices is denoted by K_n.

Example 2.11. Consider the following complete graphs:

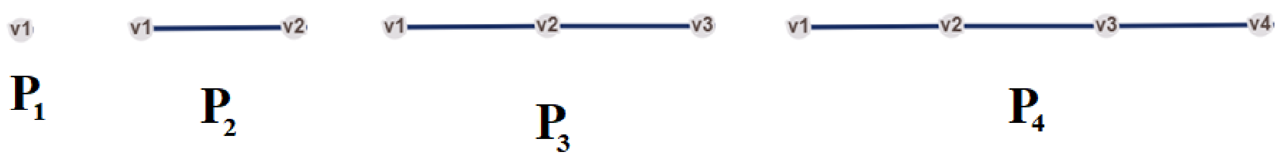


1. $SO(K_2) = \sqrt{2}$ and $\overline{SO}(K_2) = 0$.
2. $SO(K_3) = 6\sqrt{2}$ and $\overline{SO}(K_3) = 0$.
3. $SO(K_4) = 18\sqrt{2}$ and $\overline{SO}(K_4) = 0$.
4. $SO(K_n) = \frac{n(n-1)}{2}\sqrt{(n-1)^2 + (n-1)^2} = \frac{n(n-1)^2\sqrt{2}}{2}$ and $\overline{SO}(K_n) = 0$.

Remark 2.12. Let $G = (V, E)$ be a finite simple graph, where V is the set of vertices and E is the set of edges of graph G. Then the complement of $E(G)$, denoted by $\overline{E(G)}$, is the set of all possible edges that do not belong to $E(G)$. Therefore, $\overline{SO}(G)$ of G is defined as $\sum_{uv \notin E(G)} \sqrt{d_u^2 + d_v^2} = \sum_{uv \in \overline{E(G)}} \sqrt{d_u^2 + d_v^2}$ where d_u is the degree of vertex u in G.

Definition 2. 13. A path graph (or linear graph) is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges are $v_i v_{i+1}$ where $i = 1, 2, \dots, n-1$. Equivalently, a path with at least two vertices is connected and has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2. The path graph with n vertices is called P_n. The number of edges in P_n equals $n-1$.

Example 2.14. Consider the following four path graphs:



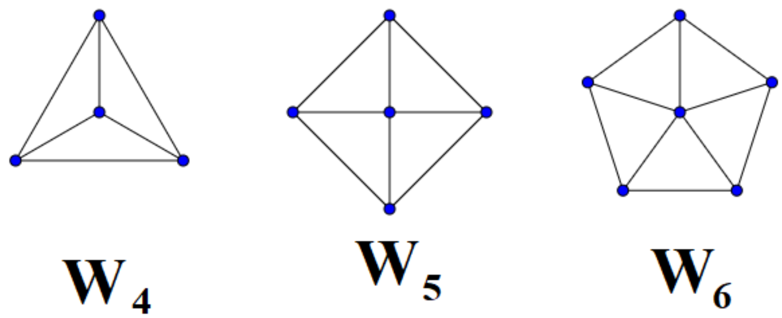
1. $V(P_1) = \{v_1\}$, $E(P_1) = \emptyset$ and $\overline{E(P_1)} = \emptyset$;
2. $V(P_2) = \{v_1, v_2\}$, $E(P_2) = \{v_1 v_2\}$ and $\overline{E(P_2)} = \emptyset$;
3. $V(P_3) = \{v_1, v_2, v_3\}$, $E(P_3) = \{v_1 v_2, v_2 v_3\}$ and $\overline{E(P_3)} = \{v_1 v_3\}$;
4. $V(P_4) = \{v_1, v_2, v_3, v_4\}$, $E(P_4) = \{v_1 v_2, v_2 v_3, v_3 v_4\}$ and $\overline{E(P_4)} = \{v_1 v_3, v_1 v_4, v_2 v_4\}$;

Then we obtain the following results:

1. $SO(P_1) = \overline{SO}(P_1) = 0$;
2. $SO(P_2) = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\overline{SO}(P_2) = 0$;
3. $SO(P_3) = 2\sqrt{1^2 + 2^2} = 2\sqrt{5}$ and $\overline{SO}(P_3) = \sqrt{1^2 + 1^2} = \sqrt{2}$;
4. $SO(P_4) = 2\sqrt{1^2 + 2^2} + \sqrt{2^2 + 2^2} = 2\sqrt{5} + 2\sqrt{2}$ and $\overline{SO}(P_4) = \sqrt{1^2 + 1^2} + 2\sqrt{1^2 + 2^2} = \sqrt{2} + 2\sqrt{5}$

Definition 2.15. A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. A wheel graph with n vertices can also be defined as the 1-skeleton of an (n – 1)-gonal pyramid. The wheel graph with n vertices is called W_n which is formed by connecting a single vertex to all vertices of a cycle of length n-1.

Example 2.16. Consider the following three wheel graphs:

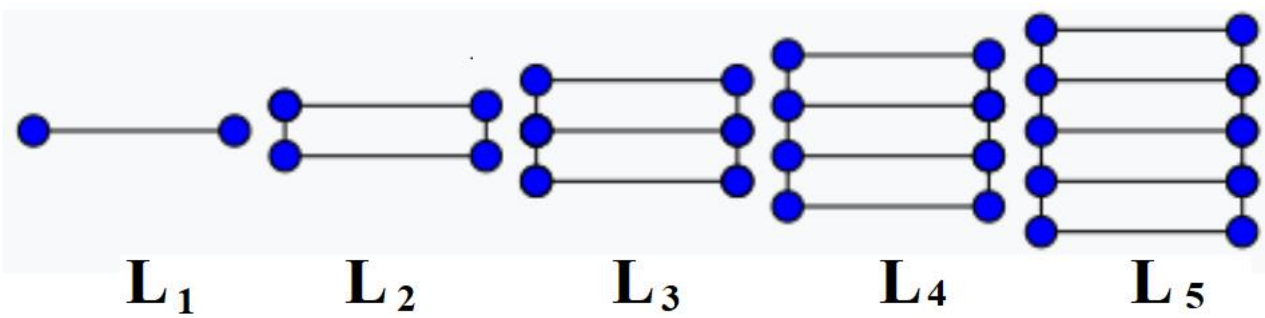


$V(W_4)$ =$\{v_0, v_1, v_2, v_3\}$, $E(W_4)$= $\{v_1v_2, v_2v_3, v_3v_1, v_1v_0, v_2v_0, v_3v_0\}$, $\overline{E(W_4)} = \emptyset$, $V(W_5)$ =$\{v_0, v_1, v_2, v_3, v_4\}$, $E(W_5)$= $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_0, v_2v_0, v_3v_0, v_4v_0\}$, $\overline{E(W_5)} = \{v_1v_3, v_2v_4\}$, $V(W_6) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and $E(W_6) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_0, v_2v_0, v_3v_0, v_4v_0, v_5v_0\}$, $\overline{E(W_6)} = \{v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5\}$. So that in W_6, $d_{v_i} = 3$ for $1 \le i \le 5$ and $d_{v_0} = 5$. Then $So(v_iv_j) = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ and $So(v_iv_0) = \sqrt{3^2 + 5^2} = \sqrt{34}$ for $1 \le i, j \le 5$. Therefore, $SO(W_6) = 5(3\sqrt{2}) + 5(\sqrt{34}) = 15\sqrt{2} + 5\sqrt{34}$ and $\overline{SO}(W_6) = 5(3\sqrt{2}) = 15\sqrt{2}$. Similarly $SO(W_5) = 4(3\sqrt{2}) + 4(5) = 12\sqrt{2} + 20$, $\overline{SO}(W_5) = 2(3\sqrt{2}) = 6\sqrt{2}$, $SO(W_4) = 6(3\sqrt{2}) + 4(5) = 18\sqrt{2}$ and $\overline{SO}(W_4) = \emptyset$.

Definition 2.17 (Sagan , et al., 1996, p. 960). The Cartesian product of two graphs G_1 and G_2, is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(v_1, v_2): v_1 \in G_1$ and $v_2 \in G_2\}$ and $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2)$: $u_1 v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2 v_2 \in E(G_2)$ and $u_1 = v_1\}$.

Definition 2.18. The ladder graph L_n is undirected graph with $2n$ vertices and $3n - 2$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_n = P_n \times P_2$.

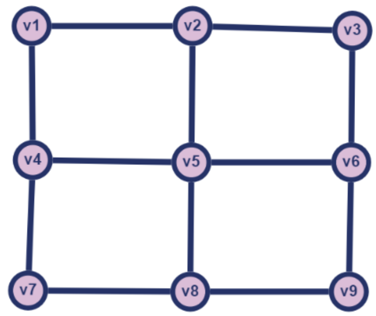
Example 2.19. Consider the following ladder graphs:



Then $SO(L_1) = \sqrt{2}$, $\overline{SO}(L_1) = \emptyset$ and $SO(L_2) = \overline{SO}(L_2) = 4(2\sqrt{2}) = 8\sqrt{2}$.

Definition 2.20. The grid graph $G_{m,n}$ is undirected graph can be obtained as the Cartesian product of two path graphs $P_m \times P_n$, that is $G_{m,n} = P_m \times P_n$.

Example 2.21. Consider the following grid graphs:



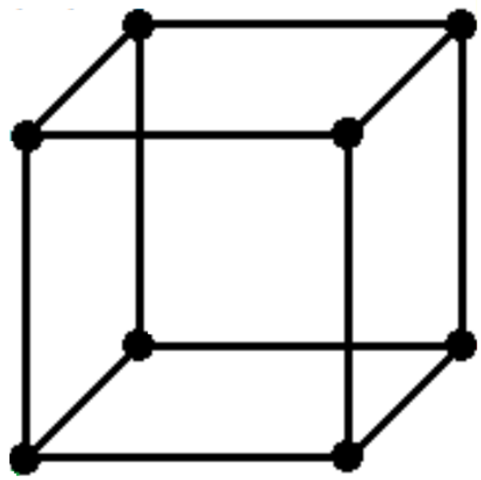
$\mathbf{G_{3,3}}$

Then $E(G_{3,3}) = A_{2,3} \cup A_{3,4}$ where $\overline{E(G_{3,3})} = B_{2,2} \cup B_{2,3} \cup B_{2,4} \cup B_{3,3}$ where both $A_{i,j}$ and $B_{i,j}$ contains edges of degree i and j. $A_{2,3}$ contains 8 edges, $A_{3,4}$ contains 4 edges, $B_{2,2}$ contains 2 edges, $B_{2,3}$ contains 16 edges, $B_{2,4}$ contains 4 edges and $B_{3,3}$ contains 6 edges. See Remark 3.2.

$SO(G_{3,3}) = 8\sqrt{2^2+3^2} + 4\sqrt{3^2+4^2} = 8\sqrt{13} + 20$ and $\overline{SO(G_{3,3})} = 2\sqrt{2^2+2^2} + 16\sqrt{2^2+3^2} + 4\sqrt{2^2+4^2} + 6\sqrt{3^2+3^2} = 4\sqrt{2} + 16\sqrt{13} + 4\sqrt{20} + 18\sqrt{2} = 22\sqrt{2} + 16\sqrt{13} + 4\sqrt{20}$.

Definition 2.22. The Crystal Lattice graph $C_{L,m,n}$ is undirected graph can be obtained as the Cartesian product of three path graphs $P_L \times P_m \times P_n$, that is $C_{L,m,n} = (P_L \times P_m) \times P_n$.

Example 2.23. Consider the Crystal Lattice graph $C_{2,2,2}$:



$\mathbf{C}_{2,2,2}$

Then $SO(C_{2,2,2}) = \overline{SO(C_{2,2,2})} = 12\sqrt{3^2 + 3^2} = 36\sqrt{2}$ and $\overline{SO(C_{2,2,2})} = 16\sqrt{3^2 + 3^2} = 48\sqrt{2}$.

Chapter Three

Definition 3.1 (Ahmad, 2023)**.** Let R be a commutative ring with identity. The maximal ideal graph of R, denoted by $m(R)$, is the undirected graph with vertex set, the set of all ideals of R, where two vertices I and J are adjacent if and only if I maximal in J, or J maximal in I.

Remark 3.2. Let G be a graph, $V(G) = \{v_i; 1 \leq i \leq t\} = \bigcup_{i=1}^{l} A_{\alpha_i}^{x_i}$, $E(G) = \bigcup_{i=1}^{m} B_{\beta_i,\gamma_i}^{y_i}$ and $\overline{E(G)} = \bigcup_{i=1}^{n} C_{\delta_i,\varepsilon_i}^{z_i}$ where $A_{\alpha_i}^{x_i}$ is a set of vertices which contains x_i vertices of degree α_i, $B_{\beta_i,\gamma_i}^{y_i}$ is a set of edges which contains y_i edges with endpoints of degree β_i and γ_i, $C_{\delta_i,\varepsilon_i}^{z_i}$ is a set of edges contains z_i edges with endpoints of degree δ_i and ε_i. Then

1. $E(G) \cap \overline{E(G)} = \emptyset$;
2. $E(G) \cup \overline{E(G)} = \{v_i v_j; 1 \leq i < j \leq t\}$;
3. $\left|E(G) \cup \overline{E(G)}\right| = \left|K_{|G|}\right| = \frac{|G|(|G|-1)}{2} = \frac{t(t-1)}{2}$
4. $E(G) \cup \overline{E(G)} = (\bigcup_{i=1}^{l} A_{\alpha_i,\alpha_i}^{\frac{x_i(x_i-1)}{2}}) \cup \left(\bigcup_{i=2}^{m}\left(\bigcup_{j=1}^{i-1} A_{\alpha_i,\alpha_j}^{\left|A_{\alpha_i}^{x_i}\right| . \left|A_{\alpha_j}^{x_j}\right|}\right)\right)$ and $\overline{E(G)} = \bigcup_{i=1}^{n} C_{\delta_i,\varepsilon_i}^{z_i} = (\bigcup_{i=1}^{l} A_{\alpha_i,\alpha_i}^{\frac{x_i(x_i-1)}{2}}) \cup \left(\bigcup_{i=2}^{m}\left(\bigcup_{j=1}^{i-1} A_{\alpha_i,\alpha_j}^{\left|A_{\alpha_i}^{x_i}\right| . \left|A_{\alpha_j}^{x_j}\right|}\right)\right) - E(G)$;

So that $SO(G) = \sum_{i=1}^{m} y_i \sqrt{{\beta_i}^2 + {\gamma_i}^2}$ and $\overline{SO(G)} = \sum_{i=1}^{n} z_i \sqrt{{\delta_i}^2 + {\varepsilon_i}^2}$.

Furthermore, and $\overline{SO(G)} = (\bigcup_{i=1}^{l} A_{\alpha_i,\alpha_i}^{\frac{x_i(x_i-1)}{2}}) \cup \left(\bigcup_{i=2}^{m} \left(\bigcup_{j=1}^{i-1} A_{\alpha_i,\alpha_j}^{|A_{\alpha_i}^{x_i}| \cdot |A_{\alpha_j}^{x_j}|} \right) \right) - \sum_{i=1}^{m} y_i \sqrt{{\beta_i}^2 + {\gamma_i}^2}$

Remark 3.3. The maximal ideal graph $m(\mathbb{Z}_{p^n})$ is the path graph P_{n+1} where p prime number and n is a positive integer. See the following graph.

$m(\mathbb{Z}_{p^n})$

Theorem 3.4. Consider the path $P_n = \{v_1, v_2, \dots, v_n\}$ where is a positive integer. Then

1. $E(P_1) = \overline{E(P_1)} = \emptyset$, then $SO(P_1) = \overline{SO(P_1)} = 0$;
2. $E(P_2) = A_{1,1}^1$ and $\overline{E(P_2)} = \emptyset$, $SO(P_1) = \sqrt{2}$ and $\overline{SO(P_1)} = 0$;
3. $E(P_3) = A_{2,1}^2$ and $\overline{E(P_3)} = A_{1,1}^1$, $SO(P_3) = 2\sqrt{5}$ and $\overline{SO(P_3)} = \sqrt{2}$;
4. For $n \geq 4$, $E(P_n) = A_{2,1}^2 \cup A_{2,2}^{n-3}$, and $\overline{E(P_n)} = A_{1,1}^1 \cup A_{2,2}^{\frac{n^2-7n+12}{2}} \cup A_{1,2}^{2n-6}$;
5. $SO(P_n) = 2\sqrt{5} + 2(n-3)\sqrt{2}$ and $\overline{SO(P_n)} = (n^2 - 7n + 13)\sqrt{2} + (2n-6)\sqrt{5}$;

Proof 1, 2, 3. They are obvious.

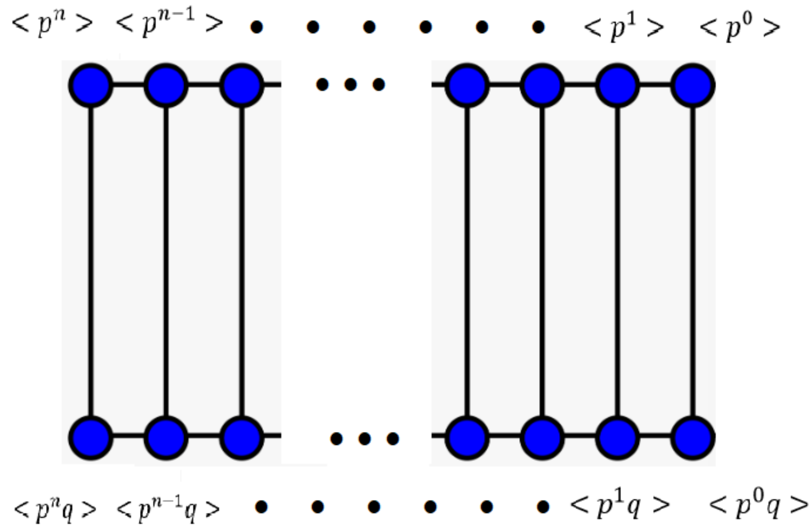
Proof 4, 5. It is clear that $E(P_n)$ contains two edges with endpoints of degree 1 and 2, contains n-3 edges with endpoints of degree 2. Therefore, $E(P_n) = A_{1,2}^2 \cup A_{2,2}^{n-3}$ and $SO(P_n) = 2\sqrt{5} + 2(n-3)\sqrt{2}$. On the other hand, $E(G) \cup \overline{E(G)} =$

$(\cup_{i=1}^{l} A_{\alpha_i,\alpha_i}^{\frac{x_i(x_i-1)}{2}}) \cup \left(\cup_{i=2}^{m}\left(\cup_{j=1}^{i-1} A_{\alpha_i,\alpha_j}^{\left|A_{\alpha_i}^{x_i}\right|.\left|A_{\alpha_j}^{x_j}\right|}\right)\right) = A_{1,1}^{1} \cup A_{2,2}^{\frac{(n-2)(n-3)}{2}} \cup A_{1,2}^{2(n-2)}$. So that $\overline{E(G)} = A_{1,1}^{1} \cup A_{2,2}^{\frac{(n-2)(n-3)}{2}} \cup A_{1,2}^{2(n-2)} - A_{1,2}^{2} \cup A_{2,2}^{n-3} = A_{1,1}^{1} \cup A_{2,2}^{\frac{(n-2)(n-3)}{2}-(n-3)} \cup A_{1,2}^{2(n-2)-2}$ $= A_{1,1}^{1} \cup A_{2,2}^{\frac{(n-2)(n-3)-2n+6}{2}} \cup A_{1,2}^{2\text{n}-6} = A_{1,1}^{1} \cup A_{2,2}^{\frac{n^2-5n+6-2n+6}{2}} \cup A_{1,2}^{2\text{n}-6} =$ $A_{1,1}^{1} \cup A_{2,2}^{\frac{n^2-7n+12}{2}} \cup A_{1,2}^{2\text{n}-6}$. So that $\overline{E(P_n)}$ contains one edges with endpoints of degree 1, $2n-6$ edges with endpoints of degree 1 and 2, and $\frac{n^2-7n+12}{2}$ edges with endpoints of degree 2. Therefore, $\overline{SO(P_n)} = \sqrt{2} + (2n-6)\sqrt{5} + 2(\frac{n^2-7n+12}{2})\sqrt{2} = (n^2-7n+13)\sqrt{2} + (2n-6)\sqrt{5}$;

Corollary 3.5. Consider the ring $\mathbb{Z}_{p^n}$ where p is a prime number and n is a positive integer. Then for $n > 3$, $\overline{SO(m(\mathbb{Z}_{p^n}))} = (n^2-5n+7)\sqrt{2} + (2n-4)\sqrt{5}$;

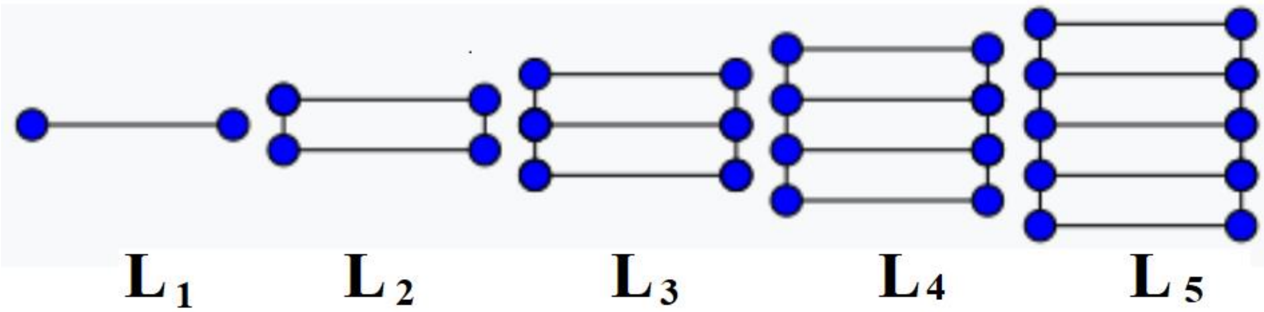
Proof. It is obvious.

Remark 3.6. Consider the ring $\mathbb{Z}_{p^m q}$ where p and q are two prime numbers and m is a positive integer. The maximal ideal graph $m(\mathbb{Z}_{p^m q})$ is the ladder graph $L_n = P_n \times P_2$ where $n = m+1$. See the following graph



Theorem 3.7. Consider a ladder graph $P_n \times P_2$ where n is a positive integer. Then

1. If $n = 1$, then $E(G) = A_{1,1}^1$ and $\overline{E(G)} = \emptyset$ and $So(P_n \times P_2) = \sqrt{2}$ and $\overline{So(P_n \times P_2)} = 0$;
2. If $n = 2$, then $E(G) = A_{2,2}^4$ and $\overline{E(G)} = A_{2,2}^2$ and $So(P_2 \times P_2) = 8\sqrt{2}$ and $\overline{So(P_2 \times P_2)} = 4\sqrt{2}$;
3. If $n = 3$, then $E(G) = A_{2,2}^2 \cup A_{2,3}^4 \cup A_{3,3}^1$ and $\overline{E(G)} = A_{2,2}^4 \cup A_{2,3}^4 \cup A_{3,3}^0$ and $So(P_3 \times P_2) = 2\sqrt{2^2+2^2} + 4\sqrt{2^2+3^2} + \sqrt{3^2+3^2} = 7\sqrt{2} + 4\sqrt{13}$ and $\overline{So(P_3 \times P_2)} = 4\sqrt{2^2+2^2} + 4\sqrt{2^2+3^2} = 7\sqrt{2} + 8\sqrt{13}$;
4. If $n > 3$, $E(G) = A_{2,2}^2 \cup A_{2,3}^4 \cup A_{3,3}^{3n-8}$ and $\overline{E(G)} = A_{2,2}^4 \cup A_{2,3}^{12n-36} \cup A_{3,3}^{9n^2-54n+80}$ and $So(P_n \times P_2) = (9n-20)\sqrt{2} + 4\sqrt{13}$ and $\overline{So(P_n \times P_2)} = (27n^2 - 162n + 248)\sqrt{2} + (12n-36)\sqrt{13}$;



Proof 1. If $n = 1$, then $E(G) = A_{1,1}^{1}$ and $\overline{E(G)} = \emptyset$ and $So(P_n \times P_2) = \sqrt{2}$ and $\overline{So(P_n \times P_2)} = 0$;

Proof 2. If $n = 2$, then $E(G) = A_{2,2}^{4}$ and $\overline{E(G)} = A_{2,2}^{2}$ and $So(P_2 \times P_2) = 8\sqrt{2}$ and $\overline{So(P_2 \times P_2)} = 4\sqrt{2}$;

Proof 3. If $n = 3$, then $E(G) = A_{2,2}^{2} \cup A_{2,3}^{4} \cup A_{3,3}^{1}$ and $\overline{E(G)} = A_{2,2}^{4} \cup A_{2,3}^{4} \cup A_{3,3}^{0}$ and $So(P_3 \times P_2) = 2\sqrt{2^2+2^2} + 4\sqrt{2^2+3^2} + \sqrt{3^2+3^2} = 7\sqrt{2} + 4\sqrt{13}$ and $\overline{So(P_3 \times P_2)} = 4\sqrt{2^2+2^2} + 4\sqrt{2^2+3^2} = 7\sqrt{2} + 8\sqrt{13}$;

Proof 4. It is clear that the ladder graph $P_n \times P_2$, consisting of rungs and rails. It has $3n - 2$ edges, including n rungs and $2n - 2$ rails.

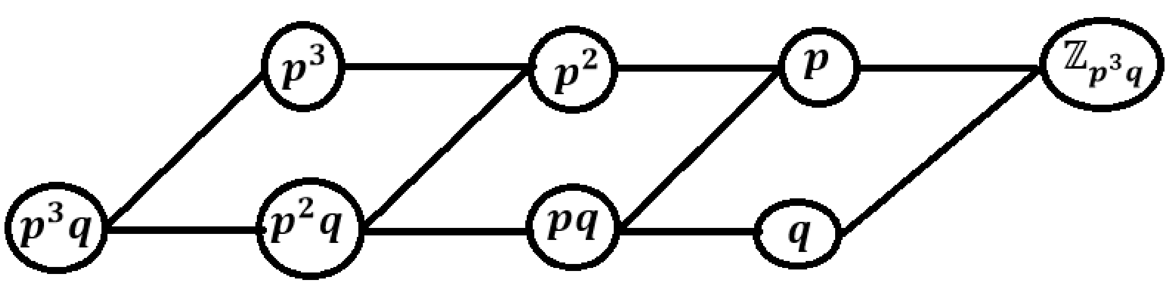
1. The first and final rungs each have endpoints of degree 2, while the others have endpoints of degree 3.
2. The two side rails of the graph represent two paths. Each path includes two edges (first and final) with endpoints of degree 2 and 3, while the remaining edges have endpoints of degree 3.

Therefor, $E(G) = A_{2,2}^{2} \cup A_{2,3}^{4} \cup A_{3,3}^{3n-8}$ and $\overline{E(G)} = A_{2,2}^{6} \cup A_{2,3}^{4(3n-8)} \cup A_{3,3}^{(3n-8)(3n-9)} - A_{2,2}^{2} \cup A_{2,3}^{4} \cup A_{3,3}^{3n-8} = A_{2,2}^{4} \cup A_{2,3}^{12n-36} \cup A_{3,3}^{(3n-8)(3n-9)-(3n-8)} = A_{2,2}^{4} \cup A_{2,3}^{12n-36} \cup A_{3,3}^{9n^2-54n+80}$ and $So(P_n \times P_2) = 2\sqrt{2^2+2^2} + 4\sqrt{2^2+3^2} + (3n-8)\sqrt{3^2+3^2} = (9n-20)\sqrt{2} + 4\sqrt{13}$ and $\overline{So(P_n \times P_2)} = 4\sqrt{2^2+2^2} + (12n-36)\sqrt{2^2+3^2} + (9n^2-54n+80)\sqrt{3^2+3^2} = \big(8 + 3(9n^2-54n+80)\big)\sqrt{2} + (12n-36)\sqrt{13} = (27n^2-162n+248)\sqrt{2} + (12n-36)\sqrt{13}$;

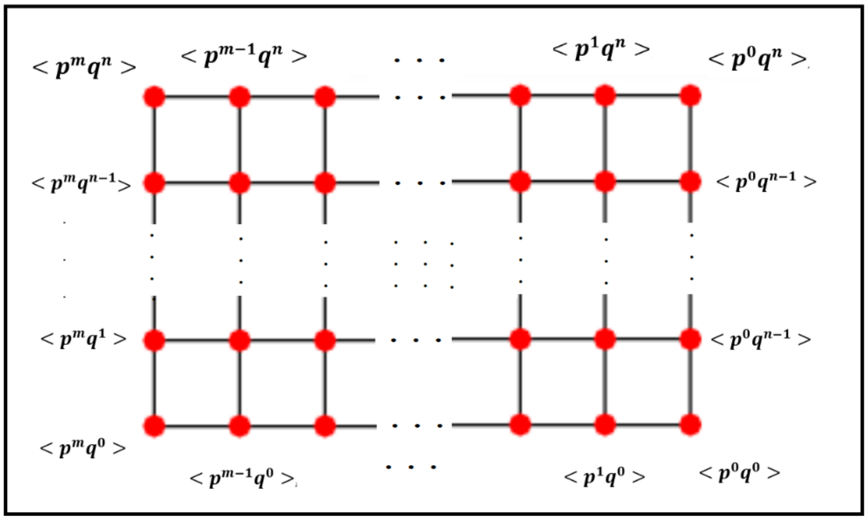
Note that the Sombor index of the ladder graph was determined incorrectly in (Ghanbari & Alikhani, Sat, 20 Feb 2021).

Corollary 3.8. Consider the ring $\mathbb{Z}_{p^nq}$ where p and q are two prime number and n is a positive integer greater than 2. Then $SO\big(m(\mathbb{Z}_{p^nq})\big) = 4\sqrt{13} + (9(n+1)-20)\sqrt{2} = 4\sqrt{13} + (9n-11)\sqrt{2}$ and $\overline{So\big(m(\mathbb{Z}_{p^nq})\big)} = (27n^2-108n+275)\sqrt{2} + (12n-24)\sqrt{13}$;

Example 3.9. Consider the maximal graph $m\big(\mathbb{Z}_{p^3q}\big)$. Then $So\left(m\big(\mathbb{Z}_{p^3q}\big)\right) = 4\sqrt{13} + 16\sqrt{2}$ and $\overline{So\left(m\big(\mathbb{Z}_{p^3q}\big)\right)} = 275\sqrt{2} + 24\sqrt{13}$.



Remark 3.10. Consider the ring $\mathbb{Z}_{p^m q^n}$ where p and q are two prime numbers and m and n are two positive integers. The maximal ideal graph $m(\mathbb{Z}_{p^m q^n})$ is the grid graph $P_{m+1} \times P_{n+1}$. See the following graph.



$\boldsymbol{m}(\mathbb{Z}_{p^m q^n})$

Remark 3.11. Consider two paths, P_m and P_n where $m, n > 2$. Then

1. $V(P_m \times P_n) = \{(u_i, v_j): 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(P_m \times P_n) = \{(u_1, u_2)(v_1, v_2):\ u_1 v_1 \in E(P_m)$ and $u_2 = v_2$ or $u_2 v_2 \in E(P_n)$ and $u_1 = v_1\}$.
2. $|V(P_m \times P_n)| = mn$ and $|E(P_m \times P_n)| = (m-1)n + m(n-1) = 2mn - (m+n)$
3. $d_{(u_i, v_j)} = d_{u_i} + d_{v_j}$, that is
   a. $d_{(v_1, v_1)} = d_{(v_1, v_n)} = d_{(v_m, v_1)} = d_{(v_m, v_n)} = 2$;
   b. $d_{(v_i, v_1)} = d_{(v_1, v_j)} = 3$ where $1 < i < m$ and $1 < j < n$;
   c. $d_{(v_i, v_j)} = 4$ where $1 < i < m$ and $1 < j < n$;

Theorem 3.12. Consider two paths, P_m and P_n where m and n are two positive integers. Then

1. $SO(P_1 \times P_1) = \overline{SO(P_1 \times P_1)} = \overline{SO(P_1 \times P_2)} = 0$; $SO(P_1 \times P_2) = SO(P_2 \times P_1) = \sqrt{2}$;
2. $SO(P_2 \times P_2) = 8\sqrt{2}$ and $\overline{SO(P_2 \times P_2)} = 4\sqrt{2}$;
3. For $n > 2$, $SO(P_2 \times P_n) = SO(P_n \times P_2) = 4\sqrt{13} + (9n - 20)\sqrt{2}$ and $\overline{So(P_n \times P_2)} = (27n^2 - 162n + 248)\sqrt{2} + (12n - 36)\sqrt{13}$;
4. For $m > 2$, $SO(P_3 \times P_m) = SO(P_m \times P_3) = 10(m-1) + 8\sqrt{13} + (10m - 30)\sqrt{2}$ and $\overline{SO(P_3 \times P_m)} = (8m^2 - 35m + 63)\sqrt{2} + 8(m-2)\sqrt{13} + 4(m - 2)\sqrt{20} + 5(2m^2 - 8m + 6)$
5. $SO(P_m \times P_n) = 10(m + n - 4) + 8\sqrt{13} + (8mn - 14m - 14n - 12)\sqrt{2}$ and $\overline{SO(P_m \times P_n)} = (14m^2 + 14n^2 + 34mn - 41m - 41n + 2m^2n^2 - 8m^2n - 8mn^2 + 54)\sqrt{2} + (8m + 8n - 24)\sqrt{13} + (4mn - 8m - 8n + 16)\sqrt{20} + 5(2m^2n - 4m^2 - 12mn + 14m + 2mn^2 - 4n^2 + 14n - 8)$

Proof 1, 2, 3. They are obvious.

Proof 4. From the graph $P_m \times P_3$ we obtain the following facts:

$\mathrm{V}(P_m \times P_3) = A_2^4 \cup A_3^{2\mathrm{m}-2} \cup A_4^{\mathrm{m}-2}$ and $\mathrm{E}(P_m \times P_3) \cup \overline{\mathrm{E}(P_m \times P_3)} = A_{2,2}^6 \cup A_{3,3}^{\frac{(2\mathrm{m}-2)(2\mathrm{m}-3)}{2}} \cup A_{4,4}^{\frac{(\mathrm{m}-2)(\mathrm{m}-3)}{2}} \cup A_{2,3}^{4(2m-2)} \cup A_{2,4}^{4(m-2)} \cup A_{3,4}^{(2m-2)(m-2)}$. We have to find $E(P_m \times P_3)$ and $\overline{E(P_m \times P_3)}$ as follows:

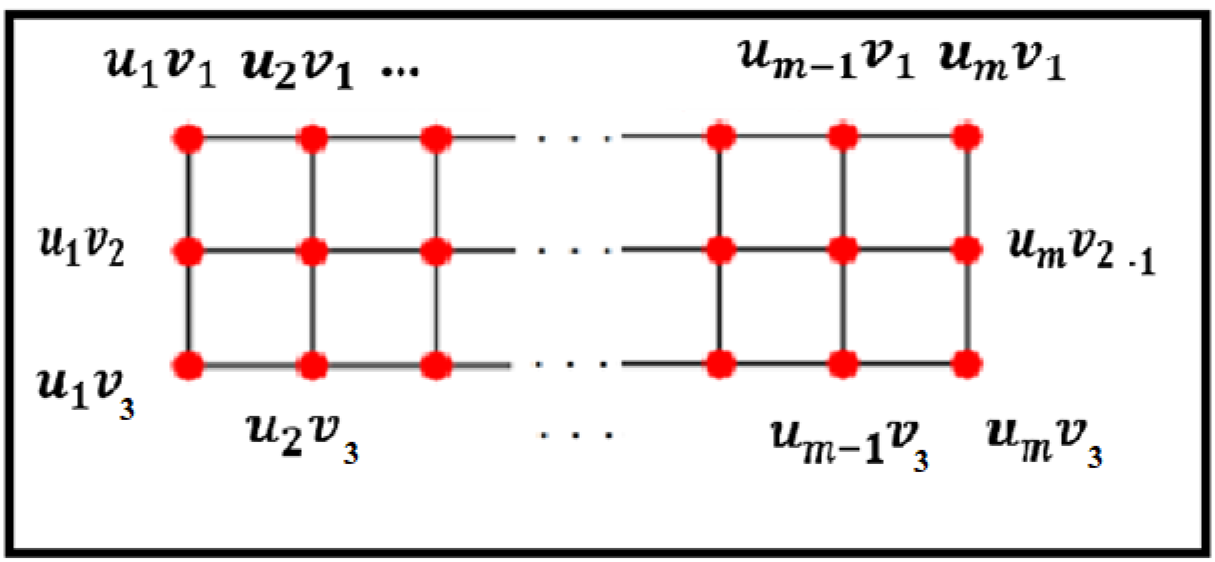
1. There are eight edges with endpoints of degree 2 and 3.
2. There are 2(m-3) edges with endpoints of degree 3.
3. There are 2(m-1) edges with endpoints of degree 3 and 4.
4. There are $m - 3$ edges with endpoints of degree 4.

So that $E(P_m \times P_3) = A_{2,3}^8 \cup A_{3,3}^{2\mathrm{m}-6} \cup A_{3,4}^{2m-2} \cup A_{4,4}^{m-3}$ and

$\overline{E(P_m \times P_3)} = A_{2,2}^{6-0} \cup A_{3,3}^{\frac{(2m-2)(2m-3)}{2}-(2m-6)} \cup A_{4,4}^{\frac{(m-2)(m-3)}{2}-(m-3)} \cup A_{2,3}^{4(2m-2)-8} \cup A_{2,4}^{4(m-2)-0} \cup A_{3,4}^{(2m-2)(m-2)-(2m-2)} = A_{2,2}^{6} \cup A_{3,3}^{(2m^2-7m+9)} \cup A_{4,4}^{\frac{m^2}{2}-\frac{7m}{2}+6} \cup A_{2,3}^{8m-16} \cup A_{2,4}^{4(m-2)} \cup A_{3,4}^{2m^2-8m+6}$.

Therefore, $SO(P_m \times P_3) = 8\sqrt{2^2+3^2} + 2(m-3)\sqrt{3^2+3^2} + 2(m-1)\sqrt{3^2+4^2} + (m-3)\sqrt{4^2+4^2} = 8\sqrt{13} + 6(m-3)\sqrt{2} + 10(m-1) + 4(m-3)\sqrt{2} = 10(m-1) + 8\sqrt{13} + (10m-30)\sqrt{2}$.

Also $\overline{SO(P_m \times P_3)} = 6\sqrt{2^2+2^2} + (2m^2-7m+9)\sqrt{3^2+3^2} + \left(\frac{m^2}{2} - \frac{7m}{2} + 6\right)\sqrt{4^2+4^2} + 8(m-2)\sqrt{2^2+3^2} + 4(m-2)\sqrt{2^2+4^2} + (2m^2-8m+6)\sqrt{3^2+4^2} = (12 + 6m^2 - 21m + 27 + 2m^2 - 14m + 24)\sqrt{2} + 8(m-2)\sqrt{13} + 4(m-2)\sqrt{20} + 5(2m^2-8m+6) = (8m^2-35m+63)\sqrt{2} + 8(m-2)\sqrt{13} + 4(m-2)\sqrt{20} + 5(2m^2-8m+6)$



$P_m \times P_3$

Proof 5. From the graph $P_m \times P_n$ we obtain the following facts:

$V(P_m \times P_n) = A_2^4 \cup A_3^{2m+2n-4} \cup A_4^{(m-2)(n-2)}$ and $E(P_m \times P_n) \cup \overline{E(P_m \times P_n)} = A_{2,2}^6 \cup A_{3,3}^{\frac{(2m+2n-4)(2m+2n-5)}{2}} \cup A_{4,4}^{\frac{(mn-2m-2n+4)(mn-2m-2n+3)}{2}} \cup A_{2,3}^{4(2m+2n-4)} \cup A_{2,4}^{4(mn-2m-2n+4)} \cup A_{3,4}^{(2m+2n-4)(mn-2m-2n+4)} = A_{2,2}^6 \cup A_{3,3}^{(m+n-2)(2m+2n-5)} \cup A_{4,4}^{\frac{(mn-2m-2n+4)(mn-2m-2n+3)}{2}} \cup A_{2,3}^{4(2m+2n-4)} \cup A_{2,4}^{4(mn-2m-2n+4)} \cup A_{3,4}^{(2m+2n-4)(mn-2m-2n+4)}$. We have to find $E(P_m \times P_n)$ and $\overline{E(P_m \times P_n)}$ as follows:

1. There are eight edges with endpoints of degree 2 and 3.
2. There are 2(n-3)+ 2(m-3) edges with endpoints of degree 3.
3. There are 2(n-2)+ 2(m-2) edges with endpoints of degree 3 and 4.
4. The other edges with endpoints of degree 4 which are $2mn - (m+n) - \Big(8 + \big(2(n-3) + 2(m-3)\big) + \big(2(n-2) + 2(m-2)\big)\Big) = 2mn - 5m - 5n + 12$.

So that $E(P_m \times P_n) = A_{2,2}^0 \cup A_{3,3}^{2(n-3)+2(m-3)} \cup A_{4,4}^{(2mn-5m-5n+12)} \cup A_{2,3}^8 \cup A_{2,4}^0 \cup A_{3,4}^{2(n-2)+2(m-2)}$. Therefore, $SO(P_m \times P_n) = 8\sqrt{2^2+3^2} + (2m+2n-12)\sqrt{3^2+3^2} + (2m+2n-8)\sqrt{3^2+4^2} + (2mn-5m-5n+12)\sqrt{4^2+4^2} = 8\sqrt{13} + (6m+6n-36)\sqrt{2} + (10m+10n-40) + (8mn-20m-20n+48)\sqrt{2} = 8\sqrt{13} + (8mn-20m-20n+48+6m+6n-36)\sqrt{2} + (10m+10n-40) = 8\sqrt{13} + (8mn-14m-14n+12)\sqrt{2} + (10m+10n-40)$.

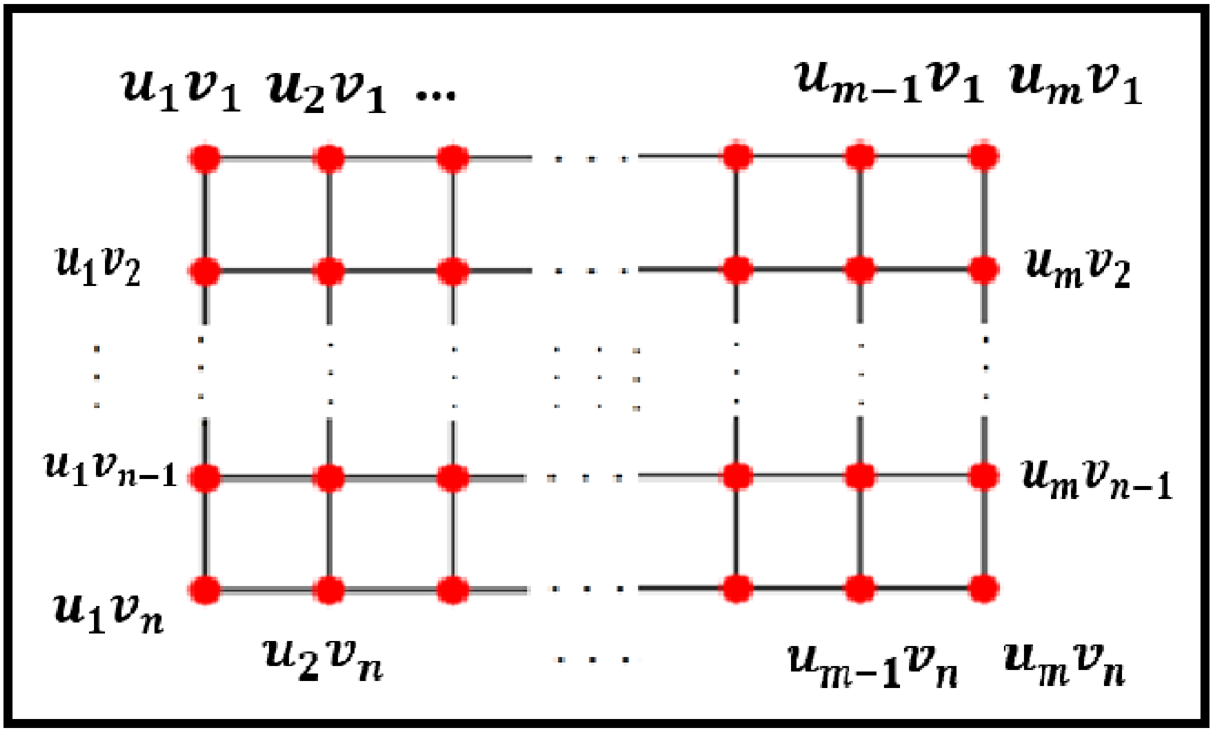
$\overline{E(P_m \times P_n)} = A_{2,2}^{6-0} \cup A_{3,3}^{(m+n-2)(2m+2n-5)-(2m+2n-12)} \cup A_{4,4}^{\frac{(mn-2m-2n+4)(mn-2m-2n+3)}{2}-(2mn-5m-5n+12)} \cup A_{2,3}^{4(2m+2n-4)-8} \cup A_{2,4}^{4(mn-2m-2n+4)-0} \cup A_{3,4}^{(2m+2n-4)(mn-2m-2n+4)-(2m+2n-8)} =$

$$A_{2,2}^{6} \cup A_{3,3}^{2m^2+2n^2+4mn-11m-11n+22} \cup A_{4,4}^{\frac{m^2n^2-4m^2n+4m^2-4mn^2+11mn-4m+4n^2-4n-12}{2}}$$
$$\cup A_{2,3}^{8m+8n-24} \cup A_{2,4}^{4mn-8m-8n+16}$$
$$\cup A_{3,4}^{2m^2n-4m^2-12mn+14m+2mn^2-4n^2+14n-8}$$

$$A_{2,2}^{6} \cup A_{3,3}^{2m^2+2n^2+4mn-11m-11n+22} \cup A_{4,4}^{\frac{m^2n^2-4m^2n+4m^2-4mn^2+11mn-4m+4n^2-4n-12}{2}}$$
$$\cup A_{2,3}^{8m+8n-24} \cup A_{2,4}^{4mn-8m-8n+16}$$
$$\cup A_{3,4}^{2m^2n-4m^2-12mn+14m+2mn^2-4n^2+14n-8}$$

Therefore, $\overline{SO(P_m \times P_n) =}$ $6\sqrt{2^2+2^2} + (2m^2+2n^2+4mn-11m-11n+22)\sqrt{3^2+3^2} + \left(\frac{m^2n^2-4m^2n+4m^2-4mn^2+11mn-4m+4n^2-4n-12}{2}\right)\sqrt{4^2+4^2} + (8m+8n-24)\sqrt{2^2+3^2} + (4mn-8m-8n+16)\sqrt{2^2+4^2} + (2m^2n-4m^2-12mn+14m+2mn^2-4n^2+14n-8)\sqrt{3^2+4^2}$

$= (14m^2+14n^2+34mn-41m-41n+2m^2n^2-8m^2n-8mn^2+54)\sqrt{2} + (8m+8n-24)\sqrt{13} + (4mn-8m-8n+16)\sqrt{20} + 5(2m^2n-4m^2-12mn+14m+2mn^2-4n^2+14n-8)$;


u_1v_1 u_2v_1 ...
$u_{m-1}v_1$ u_mv_1
u_1v_2
u_mv_2
u_1v_{n-1}
u_mv_{n-1}
u_1v_n
u_2v_n
. . .
$u_{m-1}v_n$ u_mv_n

$P_m \times P_n$

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پوخته

لەم پرۆژەیەدا ئێمە لێکۆلینەوە لە سەر سۆمبۆر کۆئینديکس هەندێك جۆری گراف دەکەین وەکو گرافی هێلی، گرافی بازنەیی ، گرافی ویل ، گرافی پەیژەیی ، گرافی تۆریی و گرافی تۆریی بلوریی، پاشان سەرنج دەخەینە سەر زۆرترین زنجیرەی ئایدیالەکانی ئەلقەیی $\mathbb{Z}_n$ کاتێك n یەکسانە $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ کاتێك p_i جیاوازن وە $\alpha_i \in \mathbb{Z}^+$ و $1 \le i \le k$. لە کۆتایدا ، تەکنیکێک پێشکەش دەکەین بۆ دۆزینەوەی سۆمبۆر کۆئینديکس هەندێک گرافی زۆرترین زنجیرەی ئایدیاڵ $m(\mathbb{Z}_n)$ بۆ ئەلقەکانی $\mathbb{Z}_n$.

الخلاصة

في هذا المشروع نقوم بدراسة سومبور کۆئینديکس لبعض الرسام البيانية الخاصة مثل الرسام البيانية المسارية، الرسام البيانية الدائرية، الرسام البيانية العجمة ، الرسام البيانية السلمية ، الرسام البيانية الشبكية ، الرسام البيانية الشبكية البلورية ثم نركز على السلسلة القصوى للحلقات $\mathbb{Z}_n$ حيث $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ ، p_i العدد الأولية مختلفة $\alpha_i \in \mathbb{Z}^+$ و $1 \le i \le k$. وأخيرا ، ونقدم تقنية لأيجاد سومبور کۆئینديکس لبعض الرسام البيانية المثالية القصوى $m(\mathbb{Z}_n)$ للحلقات $\mathbb{Z}_n$.