Chapter one

Historically, the first finite geometry to be considered was a three-dimensional geometry, each plane of which contained seven points and seven lines. The modernity of finite geometries is emphasized by the fact that Fano explored this first finite geometry in 1892. It was not until 1906 that finite projective geometries were studied by Veblen and Bussey. Since that time, a great many finite geometries have been or are being studied.

An *axiomatic system* consists of some undefined terms (primitive terms) and a list of statements, called *axioms* or *postulates*, concerning the undefined terms. One obtains a mathematical theory by proving new statements, called *theorems*, using only the axioms (postulates), logic system, and previous theorems. Definitions are made in the process in order to be more concise.

Axiomatic System (Postulate System)

1. Undefined terms/primitive terms
2. Defined terms
3. Axioms/postulates - accepted unproved statements
4. Theorems - proved statements

Most early Greeks made a distinction between axioms and postulates. Evidence exists that Euclid made the distinction that an axiom (common notion) is an assumption common to all sciences and that a postulate is an assumption peculiar to the particular science being studied. Now in modern times no distinction is made between the two; an axiom or postulate is an assumed statement.

Usually an axiomatic system does not stand alone, but other systems are also assumed to hold. For example, we will assume:

1. The real number system,
2. Some set theory,
3. The English language.

We will not develop any of these but use what we need from them.

Language rules 1: Point and line are undefined technical terms.

Language rules 2: Line is a set of points.

Language rules 3:

1. If a point p is an element of the line α, then we say the following:

a-α pass through P.

b-α contains P.

c-P is on α.

d-P lies on α.

2. Each of the following statements are logically equivalent:

P: If P and Q are any two points then there exist at least one line on both P and Q.

Q: Each two distinct points are on at least one line.

R: There is at least one line on any two distinct points.

P: If α is a line then there exist exactly three points on it.

Q: There are exactly three points on every line.

R: Each two distinct points are on exactly one line.

Language rules 4: -If $p \in \alpha \cap \beta$ then we say the following:

a-α meet β in P. b- α intersect β at P.

Question: What is different between the following two statements?

P: If A and B are any two points then there exist exactly one line α on both A and B.

Q: If α is a line then there exist exactly two points A and B on it.

Answer: According to statement P there are two points to find(or construct) a line but According to the statement Q there is a line to find two points.

Axiom set I

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least three points on it.

3- If α is a line then there exists at least a point P not on it.

4-There exists at least one line.

Note that we can write axiom set I as follows:

1-a- There is at least one line on any two distinct points.
1-a- There is at most one line on any two distinct points.
2- There are at least three points on every line.
3- Not all points are on the same line.
4- There exists at least one line.

Theorem 1. There is at least three distinct points.
Proof. By **A4,** there exists a line α. By **A2,** it has at least three distinct points, say P, Q and R.

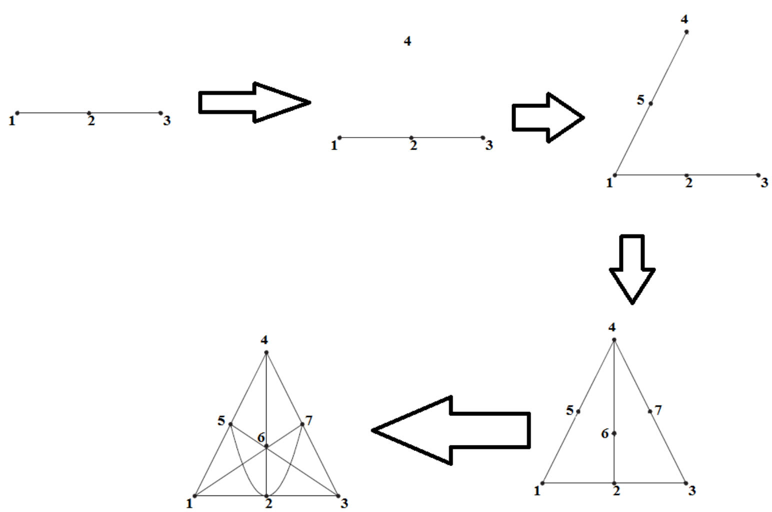
Theorem. There is at least four distinct points. **H.W**

Theorem 2. There exist at least three distinct lines.
Proof. By A4, there exists a line α. By A2, α contains at least three distinct points P, Q and R. By A3, there exists a point, say S, not on α. By A1 there exist three lines through each pairs PS, QS and RS. Suppose that any two of them were equal say RS = QS then there would be two lines containing Q, R namely PQR, QSR which is contradiction by A1b.

Theorem. There is at least four distinct lines. **H.W**

Theorem 3. There exist at least seven points and seven lines.

Proof. by **A4,** there exists a line. By **A2,** there exist at least three distinct points on it, say 1, 2 and 3. Let the line denoted by {1, 2, 3}. By **A3**, there exists a point say 4 not on {1, 2, 3}. By **A1a,** there exists a line containing 1, 4 and, by **A1b** cannot be contains 2,3. By **A2,** there exists a point 5 on this line. Let the line denoted by {1, 4, 5}, furthermore, the points 1, 2, 3, 4 and 5 are distinct. By similar arguing there exist lines {2, 4, 6} and {3, 4, 7}. Thus there exists at least seven points 1, 2, 3, 4, 5, 6 and 7. By **A1a,** the following lines are possible 16- , 17-, 25-, 27-, 35-, 36- , 57- , 67-. Consider the line 16- , by **A2** and **A1b** it must contains 7. Hence we have a new line {1, 6, 7}, by **A1b** now rules out 17- , and 67-. Consider the line 25-, by **A2** and **A1b** it must contains 7. Hence we have a new line {2, 5, 7}, by **A1b**, rules out 27- and 57-. Consider the line 35- , by **A2** and **A1b** it must contains 6. Hence we have a new line 356, **by A1b**, rules out 36- and 56-. Hence we have generated distinct lines {1, 2, 3}, {1, 4, 5}, {2, 4, 6}, {3, 4, 7}, {1, 6, 7}, {2, 5, 7} and {3, 5, 6}.



Theorem 4. Two lines have at most one point in common. **H.W**

Theorem 5. If P be any point, then there exists a line β not through P. **H.W**

Theorem 6. There exist at least three lines through any point.

Proof. Let P be any point. By theorem 5, there exists a line β not contains P. By **A2,** there exist at least three points R, S, T on β. By **A1a** there exist lines PR, PS, PT. (each lines PR, PS, PT are distinct) hence there exist at least three lines through a point P.

Definition. Two lines α and β are called parallel if have no point in common ($\alpha \cap \beta = \emptyset$).

Axiom set II

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least three points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-a- If α is a line and P be any point not on α, then there exists at least one line β through P and parallel to α.

5-b- If α is a line and P be any point not on α, then there exists at most one line β through P and parallel to α.

Theorem 7. There exist at least **nine points** and **twelve lines**.

Proof. By **A4,** there exist a line and by **A2,** there exist at least three distinct points on it say {1, 2, 3}. Let the line denoted by {1, 2, 3}. By **A3,** there exists a point 4 not on the line {1, 2, 3}. By **A5,** there exists a line through point 4 parallel to the line {1, 2, 3} and by **A2,** there exist points 5,6 on this line and this line denoted by {4, 5, 6}. By **A1a** there exist a line through 1,4 and by **A1b** it cannot through the points 2,3,5,6 and by **A2,** there exist another point say 7 on this line denoted by {1, 4, 7}. By **A5,** there exists a line containing 7 parallel to {1, 2, 3} and by **A2**, **A5a** there exist points 8,9 on this line and this line denoted by {7, 8, 9} hence there exist at least nine points 1, 2, 3, 4, 5, 6, 7, 8, 9. Now by **A1a** and **A2** the following are possible:

{1, 5, 8}, {1, 5, 9}, {1, 6, 8}, {1, 6, 9},

{2, 4, 8}, {2, 4, 9}, {2, 5, 7}, {2, 5, 8}, {2, 5, 9}, {2, 6, 7}, {2, 6, 8}, {2, 6, 9},

{3, 4, 8}, {3, 4, 9}, {3, 5, 7}, {3, 5, 8}, {3, 5, 9}, {3, 6, 7}, {3, 6, 8}, {3, 6, 9}

We choose {1, 5, 9} by **A1b** rules out {1, 5, 8}, {1, 6, 9}, {2, 5, 9}, {3, 5, 9} and we must have {1, 6, 8} by **A1b** rules out {2, 6, 8}, {3, 6, 8}.

We choose {2, 5, 8} by **A1b** rules out {3, 5, 8} and we must have {2, 4, 9} by **A1b** rules out {2, 4, 8}, {3, 4, 9} we must have {2, 6, 7} by **A1b** rules out {2, 6, 9}, {3, 6, 7}.

We choose {3, 6, 9} by **A1b** rules out {3, 5, 9} and we must have {3, 4, 8} and {3, 5,7}. Hence we have generated distinct lines {1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {1, 4, 7}, {1, 5, 9}, {1, 6, 8}, {2, 5, 8}, {2, 6, 7}, {2, 4, 9}, {3, 6, 9}, {3, 5, 7} and{3, 4, 8}.

Exercise. Consider the axiom set II, then deduce the following theorems:

Theorem 8. There exist at least two lines parallel to a given line.

Theorem 9. If α and β are parallel and γ intersects α at a point p, then and γ intersects β.

Theorem 10. If α and β are parallel lines and β and $\gamma$are parallel lines, then α and $\gamma$are parallel lines.

Theorem 11. There exist at least four lines through any point.

Young System

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least three points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-a- If α is a line and P be any point not on α, then there exists at least one line β through P and parallel to α.

5-b- If α is a line and P be any point not on α, then there exists at most one line β through P and parallel to α.

6-If α is a line then there exist at most three points on it.

Theorem 1Y. There exist exactly nine points.

Proof. By theorem 7, there exist at least nine points 1, 2, 3, 4, 5, 6, 7, 8, 9 and at least twelve lines {1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {1, 4, 7}, {1, 5, 9}, {1, 6, 8}, {2, 5, 8}, {2, 6, 7}, {2, 4, 9},{3, 6, 9}, {3, 5, 7} and{3, 4, 8}. If there is another point say X. By A1a, there exists a line X1. By theorem 9, line X1 intersects with {4, 5, 6}, {7, 8, 9}. Therefore, line X1 contains at least four points contradict with A6.

Theorem 2Y. There exist exactly twelve lines.

Proof. By theorem 1Y and by theorem 7, there exist exactly nine points 1, 2, 3, 4, 5, 6, 7, 8, 9 and at least twelve lines {1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {1, 4, 7}, {1, 5, 9}, {1, 6, 8}, {2, 5, 8}, {2, 6, 7}, {3, 6, 9}, {2, 4, 9}, {3, 5, 7} and{3, 4, 8}. If there is another line say α, by A2 and A6, α has three points of 1, 2, 3, 4, 5, 6, 7, 8, 9 but any two of these points are related in the first twelve lines contradict with A1b.

Theorem 3Y. there exist exactly four lines through any point. H.W

Axiom set III

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least three points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

Remark. One can prove the theorems 1, 2, 3, 4, 5 and 6 by axiom set III.

Exercise. In set II, prove the following:

1. If one line α contains exactly n points, then any line parallel to α contains exactly n points.
2. If one line α contains exactly n points, then there exist exactly n-1 lines parallel to α.
3. If one line α contains exactly n points, then every lines contains exactly n points.
4. If one line α contains exactly n points, then every point has exactly n + 1 lines through it.
5. If one line α contains exactly n points, then there exist exactly n^2 points.
6. If one line α contains exactly n points, then there exist exactly n(n+1) lines.

Proof 1. Suppose α is a line with exactly n points $P_1, P_2, \ldots P_n$ and β iparallel to α . Since by Theorem 8 there exists still another line say γ, parallel to α(and also β by theorem 10) and by Axiom 2 it has a point Q on it, which by the definition of parallel cannot be on either β or α. Now by Axiom la there exist lines $QP_1, QP_2, \ldots QP_n$ which by Axiom lb are distinct. By Theorem 9 these lines intersect β and by Theorem 4 it must be in n distinct points. Hence there exist at least n distinct points on β. Suppose there exists another point on β, say P_{n+1}. Then there must be a line connecting Q with P_{n+1}, and by Theorem 9 and Theorem 4 it must intersect α in some point other than $P_1, P_2, \ldots P_n$ contradicting the assumption that α has exactly n points. Hence there exist at most n distinct points on β.

Exercise. In set I and III, prove the following:

1. If one line α contains exactly n points, then every lines contains exactly n points.
2. If one line α contains exactly n points, then every point has exactly n lines through it.
3. If one line α contains exactly n points, then there exist exactly n^2 –n+1 points.
4. If one line α contains exactly n points, then there exist exactly n^2 –n+1 lines.

Gino Fano (1871–1952) is credited with being the first person to explore finite geometries beginning in 1892. He worked primarily in projective and algebraic geometry. He was born in 1871 in Mantua, Italy. He initially studied in Turin. Later, he moved to Göttingen and worked with Felix Klein. Fano served as a professor of mathematics in Turin until he was forced to leave during World War II. He also taught in Switzerland and the United States. He died in Verona, Italy in 1952.

Fano system

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least three points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

6-If α is a line then there exist at most three points on it.

Theorem 1F. There exist exactly seven points.

Proof. By Theorem3 there exist at least seven points 1, 2, 3, 4, 5, 6 and 7 and seven lines {1, 2, 3}, {1, 4, 5}, {2, 4, 6}, {3, 4, 7}, {1, 6, 7}, {2, 5, 7} and {3, 5, 6}. If there is another points say X. By A1a, there exists a line 1X. By A5, the line 1X intersects with {2, 4, 6}, by A1b this point is not 2, 4, 6. Therefore, {2, 4, 6} contains at least four points contradict with A6.

Theorem 2F. There exist exactly seven lines.

Proof. By Theorem 1F and Theorem3,There exist exactly seven points 1, 2, 3, 4, 5, 6 and 7 and at least seven lines {1, 2, 3}, {1, 4, 5}, {2, 4, 6}, {3, 4, 7}, {1, 6, 7}, {2, 5, 7}, {3, 5, 6}. If there is another lines say α by A2 and A6, α has three points of 1, 2, 3, 4, 5, 6, 7 but any two of this points are related in the first seven lines contradict with A1b.

Theorem 3F. There exist exactly three lines through any point.

Proof. Let P be any point. By Theorem 5, there exist a line α does not contains P. By A2 and A6, there exist exactly three points R, S, T on α. By A1a, there exist lines PR, PS and PT. Hence there exist at least three lines through a point P. If there is another line say β through P, by A5, β intersects α, by A1b, β cannot through the points R, S, T.Therefore, α must have four points contradict with A6.

Axiom set I(n)

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least n points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

Theorem 1. There exist at least $n^2 - n + 1$ points.

Theorem 2. There exist at least n lines through any point.

Theorem 3. There exist at least $n^2 - n + 1$ lines.

Axiom set II(n)

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least n points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-a- If α is a line and p be any point not on α, then there exists at least one line β through p and parallel to α.

5-b- If α is a line and p be any point not on α, then there exists at most one line β through p and parallel to α.

Theorem 1. There exist at least n^2 points.

Theorem 2. There exist at least $n + 1$ lines through any point.

Theorem 3. There exist at least $n(n + 1)$ lines.

Axiom set III(n)

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least n points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

Theorem 1. There exist at least $n^2 - n + 1$ points.

Theorem 2. There exist at least n lines through any point.

Theorem 3. There exist at least $n^2 - n + 1$ lines.

Young(n) System

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least n points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-a- If α is a line and p be any point not on α, then there exists at least one line β through p and parallel to α.

5-b- If α is a line and p be any point not on α, then there exists at most one line β through p and parallel to α.

6-If α is a line then there exist at most n points on it.

Theorem 1. There exist exactly n^2 points.

Theorem 2. There exist exactly $n + 1$ lines through any point.

Theorem 3. There exist exactly n(n+1) lines.

Young*(n) System

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist exactly n points on it.

3- If α is a line then there exist at least a point p not on it.

4-There exists at least one line.

5-a- If α is a line and p be any point not on α, then there exists at least one line β through p and parallel to α.

5-b- If α is a line and p be any point not on α, then there exists at most one line β through p and parallel to α.

Theorem 1. There exist exactly n^2 points.

Theorem 2. There exist exactly $n+1$ lines through any point.

Theorem 3. There exist exactly n(n+1) lines.

Young**(n) System

1-If p and q are any two points then there exists exactly one line containing both p and .

2-If α is a line then there exist exactly n points on it.

3- If α is a line then there exist at least a point p not on it.

4-There exists at least one line.

5-a- If α is a line and p be any point not on α, then there exists at least one line β through p and parallel to α.

5-b- If α is a line and p be any point not on α, then there exists at most one line β through p and parallel to α.

Theorem 1. There exist exactly n^2 points.

Theorem 2. There exist exactly $n+1$ lines through any point.

Theorem 3. There exist exactly n(n+1) lines.

Fano (n) system

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist at least n points on it.

3- If α is a line then there exist at least a point P not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

6-If α is a line then there exist at most n points on it.

Theorem 1. There exist exactly $n^2 - n + 1$ points.

Theorem 2. There exist exactly n lines through any point.

Theorem 3. There exist exactly $n^2 - n + 1$ lines.

Fano* (n)system

1-a-If P and Q are any two points then there exist at least one line containing both P and Q.

1-b-If P and Q are any two points then there exist at most one line containing both P and Q.

2-If α is a line then there exist exactly n points on it.

3- If α is a line then there exist at least a point p not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

Theorem 1F*. There exist exactly $n^2 - n + 1$ points.

Theorem 2F*. There exist exactly n lines through any point.

Theorem 3F*. There exist exactly $n^2 - n + 1$ lines.

Fano** (n)system

1-If P and Q are any two points then there exists exactly one line containing both P and Q.

2-If α is a line then there exist exactly n points on it.

3- If α is a line then there exist at least a point p not on it.

4-There exists at least one line.

5-If α and β are any lines, then there exist at least one point p such that $p \in \alpha \cap \beta$.

Theorem 1F.** There exist exactly $n^2 - n + 1$ points.

Theorem 2F.** There exist exactly n lines through any point.

Theorem 3F.** There exist exactly $n^2 - n + 1$ lines.

Three Point Geometry

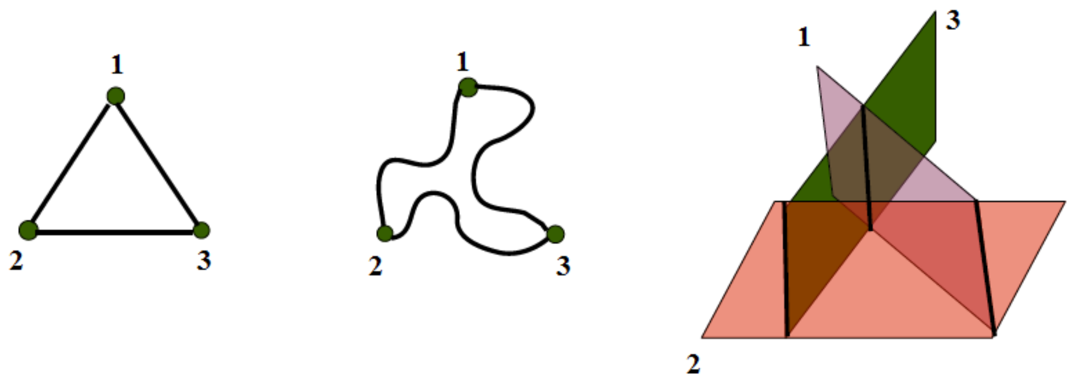
1. There exist exactly three distinct points.
2. Each two distinct points are on exactly one line.
3. Not all the points are on the same line.
4. Each two distinct lines are on at least one point.

Theorems

1. Each two distinct lines are on exactly one point.
2. There exist exactly three lines.

Proof 1. Let α and β be two lines. By A_4 there exists at least one point P on both α and β. Now it is enough to prove that point is unique. Assume, to the contrary, that distinct lines α and β, meet at points **P** and **Q**. This contradicts A_2, which says that the points **P** and **Q** lie on exactly one line. Thus, our assumption is false, and two distinct lines are on at most one point.

Proof 2. By A_1 there exist exactly three points say 1, 2, and 3. By A_2, 1 and 2 on exactly on line say α and by A_3 the point 3 not on α. By A_2, 1 and 3 on exactly on line say β and by A_3 the point 2 not on β. Note that $3 \notin \alpha$ and $2 \notin \beta$ so that α different from β Similarly, γ is a third line contains exactly two points 2 and 3. If there was a fourth line λ, then by A_2 it would contain two of 1, 2, 3. Which is contradiction with theorem 1.



Four Point Geometry

1. There exist exactly four points.

2. Each two distinct points have exactly one line that contains both of them.

3. Each line is on exactly two points.

Theorems H.W

1. There exist exactly six lines.
2. Each point has exactly 3 lines on it.

Five Point Geometry

1. There exist exactly five points.
2. Each two distinct points have exactly one line on both of them.
3. Each line has exactly two points.

Theorems H.W

1. There exist exactly ten lines.
2. Each point has exactly four lines on it.

Four-line Geometry

1. There exist exactly four lines.
2. Any two distinct lines have exactly one point on both of them.
3. Each point is on exactly two lines.

Theorems H.W

1. There exist exactly six points
2. Each line has exactly three points on it.