Department of Mathematics

College of Science

Salahaddin University-Erbil

Subject: Introduction to Number Theory

Questions Bank

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1. Prove that a number is triangular if and only if it is of the form $n(n+1)/2$, $n \geq 1$.

2. The integer n is a triangular number if and only if $8n+1$ is a perfect square.

3. The sum of any two consecutive triangular numbers is a perfect square.

4. If P_n denotes the nth pentagonal number, where $P_1 = 1$ and $P_n = P_{n-1} + (3n-2)$ for $n \geq 2$, then prove that $P_n = \frac{n(3n-1)}{2}$, $n \geq 1$.

5. Find three examples of triangular numbers that are sums of two other triangular numbers.

6. Show that 5|25,19|38 and 2 ∤ 98.

7. Use the division algorithm to find the quotient and the remainder when 76 is divided by 13.

8. Use the division algorithm to find the quotient and the remainder when -100 is divided by 13.

9. Show that if a, b, c and d are integers with a and c nonzero, such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.

10. Show that if a and b are positive integers and $a \mid b$, then $a \leq b$.

11. Prove that the sum of two even integers is even, the sum of two odd integers is even and the sum of an even integer and an odd integer is odd.

12. Show that the product of two even integers is even, the product of two odd integers is odd and the product of an even integer and an odd integer is even.

13. Show that if m is an integer then 3 divides $m^3 - m$.

14. Show that the square of every odd integer is of the form $8m + 1$.

15. Show that the square of any integer is of the form $3m$ or $3m + 1$ but not of the form $3m + 2$.

16. Show that if $ac \mid bc$, then $a \mid b$.

17. Show that if $a \mid b$ and $b \mid a$ then $a = \pm b$.

18. Convert $(7482)_{10}$ to base 6 notation.

19. Convert $(98156)_{10}$ to base 8 notation.

20. Convert $(101011101)_2$ to decimal notation.

21. Convert $(AB6C7D)_{16}$ to decimal notation.

22. Convert $(9A0B)_{16}$ to binary notation.

23. Find the greatest common divisor of 15 and 35.

24. Find the greatest common divisor of 100 and 104.

25. Find the greatest common divisor of -30 and 95.

26. Let m be a positive integer. Find the greatest common divisor of m and $m + 1$.

27. Let m be a positive integer, find the greatest common divisor of m and $m + 2$.

28. Show that if m and n are integers such that $(m, n) = 1$, then $(\mathrm{m} + \mathrm{n}, \mathrm{m} - \mathrm{n}) = 1$ or 2 .

29. Show that if m is a positive integer, then $3m + 2$ and $5m + 3$ are relatively prime.

30. Show that if a and b are relatively prime integers, then $(a + 2b, 2a + b) = 1$ or 3.

31. Show that if $a_1, a_2, \ldots, a_n$ are integers that are not all 0 and c is a positive integer, then $(ca_1, ca_2, \ldots, ca_n) = c(a_1, a_2, \ldots a_n)$.

32. Write a program (in Python) to compute the greatest common divisor gcd $(\boldsymbol{a}, \boldsymbol{b})$ of two integers $\boldsymbol{a}$ and $\boldsymbol{b}$. Your program should work even if one of a orb is zero.

33. Use the Euclidean algorithm to find the greatest common divisor of 412 and 32 and express it in terms of the two integers.

34. Use the Euclidean algorithm to find the greatest common divisor of 780 and 150 and express it in terms of the two integers.

35. Find the greatest common divisor of $70, 98, 108$

36. Let a and b be two positive even integers. Prove that $(a,b) = 2(a/2, b/2)$.

37. Show that if a and b are positive integers where a is even and b is odd, then $(a,b) = (a/2, b)$.

38. Find an upper bound for the number of steps in the Euclidean algorithm that is used to find the greatest common divisor of 38472 and 957748838.

39. Find an upper bound for the number of steps in the Euclidean algorithm that is used to find the greatest common divisor of 15 and 75. Verify your result by using the Euclidean algorithm to find the greatest common divisor of the two integers.

40. Use the Sieve of Eratosthenes to find all primes less than 100.

41. Use the Sieve of Eratosthenes to find all primes less than 200.

42. Show that no integer of the form $a^3 + 1$ is a prime except for $2 = 1^3 + 1$.

43. Show that if $2^n - 1$ is prime, then n is prime.

Hint: Use the identity $(a^{kl} - 1) = (a^k - 1)\left(a^{k(l-1)} + a^{k(l-2)} + \cdots + a^k + 1\right)$.

44. Show that the integer $Q_n = n! + 1$, where n is a positive integer, has a prime divisor greater than n. Conclude that there are infinitely many primes. Notice that this exercise is another proof of the infinitude of primes.

45. Find the smallest five consecutive composite integers.

46. Find one million consecutive composite integers.

47. Show that there are no prime triplets other than 3,5,7.

48. Find the prime factorization of 32, of 800 and of 289.

49. Find the prime factorization of 221122 and of 9 !!.

50. Show that all the powers of in the prime factorization of an integer a are even if and only if a is a perfect square.

51. Show that there are infinitely many primes of the form $6n + 5$.

52. Find the least common multiple of 14 and 15.

53. Find the least common multiple of 240 and 610.

54. Find the least common multiple and the greatest common divisor of $2^5 5^6 7^2 11$ and $2^3 5^8 7^2 13$

55. Show that every common multiple of two positive integers a and b is divisible by the least common multiple of a and b.

56. Show that if a and b are positive integers then the greatest common divisor of a and b divides their least common multiple. When are the least common multiple and the greatest common divisor equal to each other.

57. Show that $ab/(a,b) \mid m$ where $m = < a, b >$.

58. Either find all solutions or prove that there are no solutions for the diophantine equation $21x + 7y = 147$

59. Either find all solutions or prove that there are no solutions for the diophantine equation $2x + 13y = 31$

60. Either find all solutions or prove that there are no solutions for the diophantine equation $2x + 14y = 17.4$. A grocer orders apples and bananas at a total cost of \$8.4. If the apples cost 25 cents each and the bananas 5 cents each, how many of each type of fruit did he order.

61. Determine whether 3 and 99 are congruent modulo 7 or not.

62. Show that if x is an odd integer, then $x^2 \equiv 1(\text{mod } 8)$

63. Show that if a, b, m and n are integers such that m and n are positive, $n \mid m$ and $a \equiv b(\text{mod } m)$, then $a \equiv b(\text{mod } n)$.

64. Show that if $a_i \equiv b_i(\text{mod} m)$ for $i = 1,2, \dots, n$, where m is a positive integer and a_i, b_i are integers for $j = 1,2, \dots, n$, then $\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{n} b_i(\text{mod} m)$

65. For which n does the expression $1 + 2 + \cdots + (n - 1) \equiv 0(\text{mod} n)$ holds.

66. Give a reduced residue system modulo 12 .

67. Give a complete residue system modulo 13 consisting only of odd integers.

68. Find $\phi(8)$ and $\phi(101)$.

69. Find all solutions of $3x \equiv 6(\text{mod} 9)$.

70. Find all solutions of $3x \equiv 2(\text{mod} 7)$.

71. Find an inverse modulo 13 of 2 and of 11.

72. Show that if $\bar{a}$ is the inverse of a modulo m and $\bar{b}$ is the inverse of b modulo m, then $\bar{a}\bar{b}$ is the inverse of ab modulo m.

73. Find an integer that leaves a remainder of 2 when divided by either 3 or 5, but that is divisible by 4.

74. Find all integers that leave a remainder of 4 when divided by 11 and leaves a remainder of 3 when divided by 17.

75. Find all integers that leave a remainder of 1 when divided by 2, a remainder of 2 when divided by 3 and a remainder of 3 when divided by 5.

76. Show that $10! + 1$ is divisible by 11 .

77. What is the remainder when $5!\,25\,!$ is divided by 31 ?

78. What is the remainder when 5^{100} is divided by 7 ?

79. Show that if p is an odd prime, then $2(p-3)! \equiv -1(\text{mod}\ p)$.

80. Find a reduced residue system modulo 2^m, where m is a positive integer.

81. Show that if $a_1, a_2, \dots, a_{\phi(m)}$ is a reduced residue system modulo m, where m is a positive integer with $m \neq 2$, then $a_1 + a_2 + \cdots + a_{\phi(m)} \equiv 0(\text{mod}\ m)$.

82. Show that if a is an integer such that a is not divisible by 3 or such that a is divisible by 9 , then $a^7 \equiv a(\text{mod}\ 63)$.