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Root Systems and their applications

Research Project

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Abstract

In this work we study Root Systems and their applications. First we write basic definitions and results about vector spaces and inner product that we need in our work. Then we study simple Lie algebras and their root systems. At the end, we study the classification of irreducible root systems of semi simple Lie algebras by using Dynkin Diagrams and Cartan Matrices.

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Introduction

Linear algebra has in recent years become an essential part of the mathematical background required by mathematicians and mathematics teachers, engineers, computer scientists, physicists, economists, and statisticians, among others. This requirement reflects the importance and wide applications of the subject matter. Vector spaces play a key role in Lie theory and Lie groups. Lie theory has its roots in the work of Sophus Lie, who studied certain transformation groups that are now called Lie groups. His work led to the discovery of Lie algebras. By now, both Lie groups and Lie algebras have become essential to many parts of mathematics and theoretical physics.

A root system in mathematics is a configuration of vectors in a Euclidean space that meets specific geometrical requirements. The theory of Lie groups and Lie algebras, particularly the classification and representation theory of semisimple Lie algebras, both depend on the idea. Since Lie groups and Lie algebras have grown in significance in many areas of mathematics over the past century, the seeming specialness of root systems conceals the breadth of their applications. Additionally, the Dynkin diagram classification technique for root systems appears in areas of mathematics that don't directly relate to Lie theory. (such as singularity theory). In the context of spectral graph theory, root systems are also significant in and of themselves [1].

In this work we study root systems and their applications. This work consists of three chapters and is organized as follows. In chapter one we give basic definitions and results about vector spaces and algebras that we need in our work. In Chapter two we study inner product space and Lie algebras. At the last chapter, we study simple Lie algebras and their root systems. Furthermore, we study the classification of irreducible root systems of semi simple Lie algebras by using Dynkin Diagrams and Cartan Matrices.

Chapter One

Preliminary and Background

In this chapter we state basic definitions and results about ring and vector spaces that we need in our work. We gave many examples about these algebraic concepts.

Definition 1.1[8]:

A non-empty set G that is closed under a given operation '.' is called a group if the following axioms are satisfied.

1. If a, b, c $\in G$ then $a(bc) = (ab)\,c$.
2. There are exists an element e in G such that

(a) For any element a in G, $ea = ae = a$.

(b) For any element $a \in G$ there exists an element a^{-1} in G such that $a^{-1}a = aa^{-1} = e$.

A group, which contains only a finite number of elements, is called a finite group, otherwise it is termed as an infinite group. By the order of a finite group we mean the number of elements in the group

Example 1.2:

Let Q be the set of rationals. Q\{0} is a group under multiplication. This is an infinite group.

Example 1.3:

$Z_p = \{0, 1, 2, \dots , p - 1\}$, p a prime be the set of integers modulo p. Zp\{0} is a finite cyclic group of order p-1 under multiplication modulo p.

Definition 1.4[2]:

A non-empty set R is said to be an associative ring if in R are defined two binary operations '+' and '.' respectively such that

1. (R, +) is an additive abelian group and (R, .) is a semigroup.
2. a . (b + c) = a . b + a . c and (a + b) . c = a . c + b . c for all a, b, c $\in$ R (the two distributive laws).

Example 1.5:

Let Z be the set of integers. Then (Z,+,.) is a commutative ring with 1.

Example 1.6:

Let $Z_n = \{0, 1, 2, \dots, n-1\}$ be the ring of integers modulo n. Then Z_n is a ring with unit under modulo addition and multiplication.

Definition 1.7[2]:

A field is a set F which is closed under two operations + and × such that (F,+) is an abelian group, $(F-\{0\},\times)$ is an abelian group and the distributive law hold.

Example1.8:

R, the set of real numbers, and C, the set of complex numbers are both infinite fields with usual addition and multiplication.

Definition 1.9[8]:

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every u,v and w in V and every scalar (real number) c and d in F then V is called a **vector space**. First we list the condition for addition:

1. u+v is in V **Closure under addition**
2. u+v=v+u **Commutative property**
3. u+(v+w)=(u+v)+w **Associative property**
4. $u+(-1)u=0$ **Additive inverse**

Scalar Multiplication:

5. cu is inV **Closure under scalar multiplication**
6. c(u+v)=cu+cv **Distributive property**
7. (c+d)u=cu=du **Distributive property**
8. c(du)=(cd)u **Associative property**
9. 1(u)=u **Scalar identity**

Definition 1.10[8]:

A nonempty W subset of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V. If W is a nonempty subset of a vector space V then W is a subspace of V if and only if the following closure conditions hold.

1. If u and v are in w then u+v is in W.
2. If **u** is in W and c is any scalar, then c**u** is in W.

Example 1.11:

The set of all ordered -tuples of real numbers R^n with the standard operations is a vector space.

Example 1.12:

The set of polynomial K[x] is a vector space over K.

Example 1.13:

Let W be the set of singular matrices of order 2. Then W is not a subspace of $M_{2\times 2}$ (R) because W is not closed under addition. To see this, let $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ And then A and B are both singular (noninvertible), but their sum $A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is nonsingular.

Definition 1.13[8]:

A vector $u \in V$ is called a **linear combination** of the vectors $u_1,u_2,\ldots u_k$ in V if u can be written as $c_1u_1+c_2u_2+\ldots+c_ku_k$ where $c_1,c_2,\ldots,c_k$, are scalars.

Example 1.14:

(1,1,1) as a linear combination of vectors in the set S={(1,2,3),(0,1,2),(-1,0,1)}

Definition 1.15[8]:

Let A = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a collection of vectors from $\mathbf{R}^n$. If $r > 2$ and at least one of the vectors in A can be written as a linear combination of the others, then A is said to be **linearly dependent**. The motivation for this description is simple: At least one of the vectors depends (linearly) on the others. On the other hand, if no vector in A is said to be a **linearly independent** set.

Example 1.16:

The vectors (2, 5, 3), (1, 1, 1), and (4, −2, 0) are linearly independent.

Definition 1.17 [6](span):

Let $S = \{V_1, V_2, \ldots, V_k\}$ be a subset of a vector space V The set S is called a **spanning set** of V if every vector in V can be written as a linear combination of vectors in **S.** In such cases it is said that S **spans** V.

Example 1.18:

(a) The set $S=\{(1,0,0),(0,1,0),(0,0,1)\}$ spans R^3 because any vector $u=(u_1,u_2,u_3)$ in R^3 can be written as $u=u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)=(u_1,u_2,u_3)$.

(b) The set $S=\{1,X,X^{3\}}$ spans P2 because any polynomial function $P(X)=a+bx+c\ x^2$in P2 can be written as $P(X)=a(1)+b(x)+c(x^2) = a+bx+cx^2$

Definition 1.19[6]:

A bilinear form on a real vector space V is a function $f: V \times V \to R$ which assigns a number to each pair of elements of V in such a way that f is linear in each variable.

Theorem 1.20:

Every bilinear form on R^n has the form $< x, y > = x^tAy = \sum_{i,j} a_{ij}x_iy_j$ for some $n \times n$ matrix A and we also have $a_{ij} = < e_i, e_j >$ for all i, j.

Example 1.21:

Let A be an $m \times n$ matrix and let $B : R^m \times R^n \to R$ be defined by $B(x, y) = x^TAy$ for $x \in R^m$, $y \in R^n$. Then B is clearly a bilinear form. In particular, if $m = n$, $A = I_n$, the identity matrix, then it shows that the Euclidean inner product on R^n is a bilinear form. Generally for any inner product space V on the set of real numbers R, the function $B : V \times V \to R$ defined by $B(x, y) = < x, y >$ is a bilinear form on V.

Chapter Two

Inner Products and Lie Algebras

In this chapter we study inner product on vector spaces and basic definiens and results about Lie algebras. We give many examples to illustrate these algebraic concepts.

Definition 2.1[8]:

Let u, v, and w be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number (**u,v**) with each pair of vectors u and v and satisfies the following axioms.

1. $\langle u, v\rangle = \langle v, u\rangle$

2. $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

3. $c\langle u, v\rangle = \langle cu, w\rangle$

4,$\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff v=0.

Remark 2,2[8] :

A vector space with an inner product is called an inner product space. Whenever an inner product space is referred to, assume that the set of scalars is the set of real numbers.

Example 2.3:

In R^n, the dot product of two vectors **u**=$(u_1, u_2, \dots u_n)$ and **v**= $(v_1, v_2, \dots, v_n)$ is defined by **u · v** = $u_1v_1 + u_2v_2 + \dots + u_nv_n$. It is easy to check that the dot product in R^n satisfies the four axioms of an inner product.

The Euclidean inner product is not the only inner product that can be defined on R^n . Now we define more inner product on R^n .

Examples 2.4:

a) Define $\langle u, v\rangle := u_1v_1 + 2u_2v_2$ where u = (u_1 , u_2) and v = (v_1 , v_2) are in R^2. This function defines an inner product on R^2 due to the following properties:

ıe product of real numbers is commutative,

$\langle u, v\rangle = u_1v_1 + 2u_2v_2 = u_1v_1 + 2u_2v_2 = \langle v, u\rangle$

2. Let $w=(w_1,w_2)$ Then

$\langle u, v+w\rangle = u_1(v_1+w_1) + 2u_2(v_2+w_2)$

$= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2$

$= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)$

3. if c is any scalar, then

$c\langle u, v\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle cu, v\rangle$

4. The square of a real number is nonnegative, $\langle v, v\rangle = v_1^2 + 2v_2^2 \geq 0$

Moreover, this expression is equal to zero if and only if v=0 .

b) we can be generalized (a) as follows to get an inner product on R^n:

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{c_1u_1v_1} + \boldsymbol{c_2u_2v_2} + \dots + \boldsymbol{c_nu_nv_n} \quad \boldsymbol{c_i} \geq 0$$

The positive constants $c_1, \dots, c_2$ called **weights.** If any c_i is negative or 0, then this function does not define an inner product.

Gram-Schmidt Orthogonalizaition Process[8]

Suppose $\{v_1, v_2, \dots, v_n\}$ is a basis of an inner product space V . One can use this basis to construct an orthogonal basis $\{w_1, w_2, \dots, w_n\}$ of V as follows set

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

In other words, for k=2,3,…,n , we define

$$w_k = v_k - c_{k1}w_1 - \cdots - c_{k,k-1}w_{k-1}$$

Where $c_{ki} = \langle v_k, w_i\rangle / \langle w_i, w_i\rangle$ is component of v_k along w_i , thus $w_1, w_2, \dots, w_n$ form an arthogonal basis for V as claimed . Normalizing each w_i will then yield an orthogonal basis for V .

Definition 2.5[4]:

An algebra over a field F is a vector space A over F together with a bilinear map, $A \times A \rightarrow A, (x, y) \_\!\!\rightarrow xy$. We say that xy is the product of x and y. The algebra A is said to be associative if $(xy)z = x(yz)$ for all $x, y, z \in A$ and unital if there is an element 1_A in A such that $1_A x = x = x 1_A$ for all non-zero elements of A.

Example 2.6:

The space of n×n-matrices $M_n(R)$ with matrix addition and matrix multiplication form a R-algebra and the set of polynomial R[x] is an R-algebra.

Example 2.7: gl(V), the vector space of linear transformations of the vector space V, has the structure of a unital associative algebra where the product is given by the composition of maps. The identity transformation is the identity element in gl(V).

Usually one studies algebras where the product satisfies some further properties, for example, Lie algebras and Jordan algebras.

Definition 2.8[3]:

Let F be a field. A Lie algebra over F is an F-vector space L, together with a bilinear map, the Lie bracket $L \times L \rightarrow L, (x, y) \_\!\!\rightarrow [x, y]$,
satisfying the following properties:

$$[x, x] = 0 \text{ for all } x \in L, \tag{L1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in L. \tag{L2}$$

The Lie bracket [x, y] is often referred to as the commutator of x and y. Condition (L2) is known as the Jacobi identity. As the Lie bracket $[-,-]$ is bilinear, we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence condition (L1) implies $[x, y] = -[y, x]$ for all $x, y \in L$. (L1’)

If the field F does not have characteristic 2, then putting x = y in (L1’) shows that (L1’) implies (L1). Unless specifically stated otherwise, all Lie algebras in this book should be taken to be finite-dimensional.

Example 2.9:

Let F = **R**. The vector product (x, y) _→ x ∧ y defines the structure of a Lie algebra on $\mathbf{R}^3$. We denote this Lie algebra by $\mathbf{R}^3_\wedge$. Explicitly, if

$$x = (x_1, x_2, x_3) \text{ and } y = (y_1, y_2, y_3), \text{ then}$$

$$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

1. $[x,[y,z]]=[(x_1,x_2,x_3),(y_2z_3 - y_3z_2 , y_3z_1 - y_1z_3 , y_1z_2 - y_2z_1)]$

$= [x_2(y_1z_2 - y_2z_1) - x_3 (y_3z_1 - y_1z_3),x_3(y_2z_3-y_3z_2)-x1(y_1z_2-y_2z_1),x_1(y_3z_1-y_1z_3)-x_2(y_2z_3-y_3z_2)]$

$=((x_2y_1z_1-x_2y_2z_1)-(x_3y_3z_1-x_3y_1z_3),(x_3y_2z_3-x_3y_3z_2)-(x_1y_1z_2-x_1y_2z_1),(x_1y_3z_1-x_1y_1z_3)-(x_2y_2z_3-x_2y_3z_3))$.

Now we calculate $[(x_2y_3-x_3y_2,x_3y_1-x_1y_3,x_1y_2-x_2y_1),(z_1,z_2,z_3)]$

$=[(x_3y_1-x_1y_3)z3-(x_1y_2-x_2y_1)z_2,(x_1y_2-x_2y_1)z_1-(x_2y_3-x_3y_2)z_3,(x_2y_3-x_3y_2)z_2-(x_3y_1-x_1y_3)z1]$

$=((x_3y_1z_3-x_1y_3z_3)-(x_1y_2z_2-x_2y_1z_2),(x_1y_2z_1-x_2y_1z_1)-(x_2y_3z_3-x_3y_2z_3),(x_2y_3z_2-x_3y_2z_2)-(x_3y_1z_1-x_1y_3z_1)$

It is only remain to calculate

$[(y_1,y_2,y_3),(x_2z_3-x_3z_2,x_3z_1-x_1y_3,x_1z_2-x_2z_1)]$

$=[y_2(x_1z_2+x_2z_1)-y_3(x_3z_1-x_1y_3),y_3(x_2z_3-x_3z_2)-y_1(x_1z_2-x_2z_1),y_1(x_3z_1-x_1y_3)-y_2(x_2z_3-x_3z_2)]$.

We get that this product satisfies the Jacoby identity now we check bilinearity.

2. $[x+y,z]=[(x_1+y_1,x_2+y_2,x_3+y_3),(z_1,z_2,z_3)$

$=[(x_2+y_2)z_3-(x_3+y_3)z_2,(x_3+y_3)z_1-(x_1+y_1)z_3,(x_1+y_1)z_2-(x_2+y_2)z_1]$

$=[(x_2z_3+y_2z_3)-(x_3z_2+y_3z_2),(x_3z_1+y_3z_1)-(x_1z_3+y_1z_3),(x_1z_2+y_1z_2)-(x_2z_2+y_2z_1)]$

$[(x_1,x_2,x_3),(z_1,z_2,z_3)] =[x_2z_3-x_3z_2,x_3z_1-x_1z_3,x_1z_2-y_2z_1]$

$[(x_2z_3-x_3z_2)+(y_2z_3-y_3z_2),(x_3z_1-x_1z_3)+(y_3z_1-z_3),(x_1z_2-x_2z_1)+(y_3z_1-y_1z_3),(x_1z_2-x_2z_1)+(y_1z_2-y_3z_1)]$

3. We show that $[rx,z]=r[x,z]$.

$[r(x_1,x_2,x_3),(z_1,z_2,z_3)]=[(rx_1,rx_2,rx_3),(z_1,z_2,z_3)]=(rx_2z_3-rx_3z_2,rx_3z_1-rx_1z_3,rx_1z_2-rx_2z_1)$

$r[(x_1,x_2,x_3),(z_1,z_2,z_3)] =r(x_2z_3-x_3z_2,x_3z_1-x_1z_3,x_1z_2-x_2z_1)$

$=(rx_2z_3-rx_3z_2,rx_3z_1-rx_1z_3,rx_1z_2-rx_2z_1)$

Chapter Three

Root Systems

In this chapter we study simple Lie algebras and their root systems. Furthermore, we study the classification of irreducible root systems of semi simple Lie algebras by using Dynkin Diagrams and Cartan Matrices. We start by the definition of root system.

Definition 3.1[4]:

Let E be a finite-dimensional real vector space endowed with an inner product written $(-,-)$. Given a non-zero vector $v \in E$, let s_v be the reflection in the hyperplane normal to v. Thus s_v sends v to $-v$ and fixes all elements y such that $(y, v) = 0$. As an easy exercise, the reader may check that

$s_v(x) = x - \frac{2(x,v)}{(v,v)} v$ for all $x \in E$ and that sv preserves the inner product,

that is, $(s_v(x), s_v(y)) = (x, y)$ for all $x, y \in E$.

As it is a very useful convention, we shall write $< x, v >:= \frac{2(x,v)}{(v,v)}$,

noting that the symbol $< x, v >$ is only linear with respect to its first variable, x. with this notation, we can now define root systems.

Definition 3.2[4]:

A subset R of a real vector space E is a root system if it satisfies the following:

(R1) R is finite, it spans E, and it does not contain 0.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.

(R3) If $\alpha \in R$, then the reflection $s\alpha$ permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $< \alpha, \beta > \in \boldsymbol{Z}$.

The elements of R are called roots.

Examples 3.3:

1) The root space decomposition gives our main example. Let L be a complex semisimple Lie algebra, and suppose that Φ is the set of roots of L with respect to some fixed Cartan subalgebra H. Let E denote the real span of Φ. the

symmetric bilinear form on E induced by the Killing form (−,−) is an inner product. Then Φ is a root system in E, see [1].

2) We work in $\mathbf{R}^{l+1}$, with the Euclidean inner product.

Let ε_i be the vector in E with i-th entry 1 and all other entries zero. Then $R := \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq l + 1\}$ is a root system in E where

$E = \mathrm{Span}R = \{ \sum \alpha_i\varepsilon_i : \sum \alpha_i = 0\}$.

Proposition 3.4[5] :

Let V be a finite-dimentional inner-product space over R. For any x,y,v ϵ R^n with v ≠ 0 the reflection s_v preserves the inner product : That is $(s_v(x), s_v(y)) = (x, y)$.

Proof : We use that the inner product in a real vector space is bilinear and positive definite to expand our expression and get the desired equality.

$$(s_v(x)),(s_v(y))=\left(x - 2\frac{(x,v)}{(v,v)}\, v, y - 2\frac{(y,v)}{(v,v)}\, v\right)$$

$$= (x, y) + \left(x, -2\frac{(y,v)}{(v,v)}\, v\right) + \left(-2\frac{(x,v)}{(v,v)}\, v\, , y\right) + \left(-2\frac{(x,v)}{(v,v)}\, v\, , -2\frac{(y,v)}{(v,v)}\, v\right)$$

$$=(x, y) - 2\frac{(y,v)}{(v,v)}(x, v) - 2\frac{(x,v)}{(v,v)}(v, y) + 2\frac{(y,v)}{(v,v)} \cdot 2\frac{(x,v)}{(v,v)}(v, v)$$

$$=(x, y) - 4\frac{(x,v)(y,v)}{(v,v)} + 4\frac{(x,v)(y,v)}{(v,v)} = (x, y).$$

Lemma 3.5[4] (Finiteness Lemma): Suppose that R is a root system in the real inner-product space E. Let α, β ∈ R with β ≠ ±α. Then $< \alpha, \beta >< \beta, \alpha > \in \{0, 1, 2, 3\}$. The possibilities are as follows:

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

Given roots α and β, we would like to know when their sum and difference lie in R. Our table deduce the following:

Proposition 3.6[4]: Let $\alpha, \beta \in R$.

(a) If the angle between α and β is strictly obtuse, then $\alpha + \beta \in R$.

(b) If the angle between α and β is strictly acute and

$$(\beta, \beta) \geq (\alpha, \alpha), \text{ then } \alpha - \beta \in R.$$

Example 3.7:

We work on the root system called $A_2 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_2\}$ on R^3 where $\varepsilon_1 = (1,0,0), \varepsilon_2 = (0,1,0), \varepsilon_3 = (0,0,1)$. Set

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 = (1,0,0) - (0,1,0) = (1,-1,0)$$

$$\alpha_2 = \varepsilon_2 - \varepsilon_3 = (0,1,0) - (0,0,1) = (0,1,-1)$$

$$< \alpha_1, \alpha_2 >< \alpha_2, \alpha_1 >= \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)}$$

$$(\alpha_1, \alpha_1) = (1,-1,0), (1,-1,0) = 2$$

$$(\alpha_2, \alpha_2) = (0,1,-1), (0,1,-1) = 2$$

$$(\alpha_2, \alpha_1) = (0,1,-1), (1,-1,0) = -1$$

$$(\alpha_1, \alpha_2) = (1,-1,0), (0,1,-1) = -1$$

$$= \frac{2(-1)}{2} \times \frac{2(-1)}{2} = 1$$

Definition 3.8[4]:

The root system R is irreducible if R cannot be expressed as a disjoint union of two non-empty subsets $R_1 \cup R_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in R_1$ and $\beta \in R_2$.

Note that if such a decomposition exists, then R_1 and R_2 are root systems in their respective spans. The next lemma tells us that it will be enough to classify the irreducible root systems.

Lemma 3.9[4]:

Let R be a root system in the real vector space E. We may write R as a disjoint union $R = R_1 \cup R_2 \cup \ldots \cup R_k$, where each R_i is an irreducible root system in the space E_i spanned by R_i, and E is a direct sum of the orthogonal subspaces $E_1, \ldots, E_k$.

Definition 3.10[4]:

A subset B of R is a base for the root system R if

(B1) B is a vector space basis for E, and

(B2) every $\beta \in R$ can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with $k_\alpha \in \mathbf{Z}$, where all the non-zero coefficients k_α have the same sign..

Theorem 3.11[4]:

Every root system has a base.

Remark 3.12[4]:

A root system R will usually have many possible bases. For example, if B is a base then so is $\{-\alpha : \alpha \in B\}$. In particular, the terms “positive” and “negative” roots are always taken with reference to a fixed base B.

The Weyl Group of a Root System :

For each root $\alpha \in R$, we have defined a reflection $s\alpha$ which acts as an invertible linear map on E. We may therefore consider the group of invertible linear transformations of E generated by the reflections $s\alpha$ for $\alpha \in R$. This is known as the Weyl group of R and is denoted by W or W(R).

Lemma 3.13[4]: The Weyl group W associated to R is finite.

Lemma 3.14[4]: If $\alpha \in B$, then the reflection s_α permutes the set of positive roots other than α.

Proposition 3.15[4]:

Suppose that $\beta \in R$. There exists $g \in W$ and $\alpha \in B$ such that $\beta = g(\alpha)$.

Definition 3.16[4]:

Let R and R_ be root systems in the real inner-product spaces E and E’, respectively. We say that R and R’ are isomorphic if there is a vector space. isomorphism $\phi : E \to E'$ such that $\phi(R) = R'$, and for any two roots $\alpha, \beta \in R$, $(\alpha, \beta) = (\phi(\alpha), \phi(\beta))$.

Recall that if θ is the angle between roots α and β, then $4\cos 2\,\theta = (\alpha, \beta)\,(\beta,\alpha)$, so the condition says that ϕ should preserve angles between root vectors.

Definition 3.17:

Let B= $\{\alpha_1, \dots, \alpha_l\}$ be a base in a root system R . The Cartan matrix of R is defined on the l $\times$ l matrix with ij-th entry $A_{ij} = \langle \alpha_i , \alpha_j \rangle$. Since for any root β we have $A_{ij} = \langle \alpha_i , \alpha_j \rangle$. Therefore A=(A_{ij}) is the Cartan matrix of R with respect to B . It follows from Theorem that the Cartan matrix depends only on the ordering with our chosen base B and not on the base itself.

The Cartan matrix A has the following properties.

(i) $A_{ii}=2$ for all i .

(ii) $A_{ij} \in \{ 0, -1, -2, -3 \}$ if $i \neq j$.

(iii) If A_{ij}=-2 or -3 then A_{ji} =-1 .

(iv) $A_{ij} = 0$ if and only if $A_{ji} = 0$.

Definition 3.18:

Let A=(A_{ij}) be the Cartan matrix of R with respect to B= $\{\alpha_1, \dots, \alpha_l\}$ such that $A_{ij} = \langle \alpha_i , \alpha_j \rangle$ is not positive for $i \neq j$. We define the Dynkin diagram of A to be the graph on l vertices with labels $\alpha_1 , \alpha_2 , .. , \alpha_l$. Two vertices α_i and α_j are connected by $d_{\alpha_1 \alpha_j}$ j lines, where

$$d_{ij} = A_{ij}A_{ji} = \langle \alpha_i , \alpha_j \rangle \langle \alpha_j , \alpha_i \rangle \in \{ 0,1,2,3 \} .$$

If d ij >1 , which happens whenever α_i and α_j have different lengths and are not orthogonal we draw an arrow pointing from the longer root to the shorter root. Dynkin diagram of R is independent of the choice of base.

Proposition 3.19[4]:

Let R and R' be root systems in the real vector spaces E and E', respectively. If the Dynkin diagrams of R and R' are the same, then the root systems are isomorphic.

Proposition 3.20:

Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. If Φ is irreducible, then L is simple.

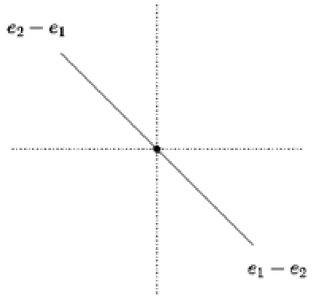
Examples 3.21

1) (A_1 root system):

Consider R^2 with the usual inner productgiven by dot product, and standard basis e_1, e_2. Let

$$\Phi = \{e_1 - e_2, e_2 - e_1\}$$

We can drow this as below . The dotted lines represent the e_1,e_2 axes.



Let E be of $(1, -1)$.Then Φ is root system in E .

$\langle e_1 - e_2 \rangle = \frac{2(e_1 - e_2, e_2 - e_1)}{(e_2 - e_1, e_2 - e_1)} = \frac{2(-1-1)}{(1+1)}$ = -2

This is called the root system of type A_1.

2) (A_2 root system). Consider R^3 with the usual inner product, given by dot product, and standard basis vectors e_1, e_2, e_3. Let

to each root. To verify the last condition regarding integrality, we just need to do some case checking. Let's just do one.

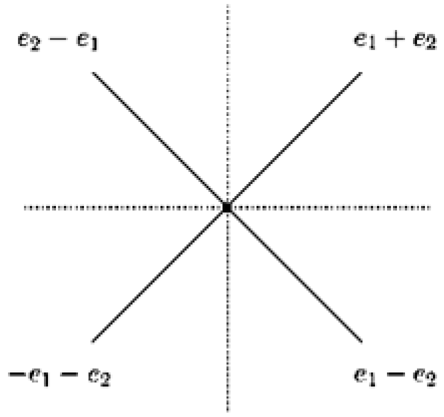
$\langle e_1 - e_2, e_2 - e_3 \rangle = \frac{2(\mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_2 - \mathrm{e}_3)}{(\mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_1 - \mathrm{e}_2)}$ = -1

This is called the $A_2\, root\ system. The\ 2\ refers\ to\ the\ dimension\ of\ span$ Of Φ.

3)($A_1 \times A_1$ root system). Consider R^2 with the usual inner product (dot product), with standard basis e_1, e_2. We have two copies of the A_1 root system, one given by

$\{e_1 - e_2, e_2 - e_1\}$ and the other given by $\{e_1 + e_2, -e_1 - e_2\}$.

Let Φ = $\{e_1 - e_2, e_2 - e_1, e_1 + e_1, -e_1 - e_1\}$



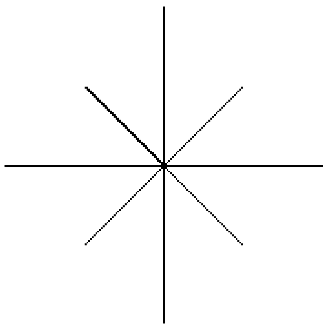
Furthermore, the two copies of A_1 here do not interact , in the sense that dot product or $(\langle , \rangle\, product)$ is zero between any vectors coming from different copies A_1.

$$\langle \mathrm{e}_{\mathbf{1}} + \boldsymbol{e}_{\mathbf{2}}, -\boldsymbol{e}_{\mathbf{1}} - \boldsymbol{e}_{\mathbf{2}} \rangle = \frac{2(\boldsymbol{e}_{\mathbf{1}} + \boldsymbol{e}_{\mathbf{2}}, -\boldsymbol{e}_{\mathbf{1}} - \boldsymbol{e}_{\mathbf{2}})}{(-\boldsymbol{e}_{\mathbf{1}} - \boldsymbol{e}_{\mathbf{2}}, -\boldsymbol{e}_{\mathbf{1}} - \boldsymbol{e}_{\mathbf{2}})} = \frac{2(-1-1)}{(1+1)} = -2$$

$$\langle e_1 + e_2, e_1 - e_1 \rangle = \frac{2(e_1 + e_2, e_1 - e_2)}{(e_1 - e_2, e_1 - e_2)} = 0$$

4) (C_2 root system): In R^2 as before, consider the root system

$$\Phi = \{\pm 2e_1, \pm 2e_1, \pm e_1 \pm e_2\}$$



This is the root system of type C_2.

Now we work on the root system of classical simple Lie algebras and for this part we mainly use [4]:

A_L) sl(L + 1, C)

(1) L = sl(L + 1,**C**) is the set of matrices of trace zero and size L + 1.

$\Phi = \{\pm (\varepsilon i - \varepsilon j) : 1 \le i < j \le l + 1\}$.

(2) the root system Φ has as a base $\{\alpha i : 1 \le i \le \_\}$, where $\alpha i = \varepsilon_i - \varepsilon_{i+1}$.

(3) The Dynkin diagram is

B_L) so(2L + 1,C) C):

1)Let L = gl_S(2L + 1,**C**) = $\{X \in gl(2\ell+1, C) :\ x^t\ S - \ S_x\}$ where

$S=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$. Then L=$\left\{\begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix} : p = -p^t \ and \ \ q = -q^t\right\}$.

Let εi ∈ H* be the map sending h to ai,

Root	ε_i	$-\varepsilon_i$	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	-$(\varepsilon_i + \varepsilon_j)$
eigenvector	b_i	c_i	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$

(2) B = $\{\alpha_i : 1 \le i < \ell\} \cup \{\beta_\ell\}$, is a bases where

$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ and \ \beta_\ell = \varepsilon_\ell \ , when \ 1 \le i < \ell,$

$$\varepsilon_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-1} + \beta_\ell ,$$

And that when $1 \le i < j \le \ell$,

$$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},$$

$$\varepsilon_i + \varepsilon_j = \alpha_i + \cdots \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{\ell-1} + \beta_\ell).$$

3) The Dynkin diagram of Φ is

.

As the Dynkin diagram is connected, Φ is irreducible and so L is simple.

The root system of so(2L + 1,**C**) is said to have type B_L.

C_L) so(2L, C)

1)Let L = gl_S(2L,**C**) where S = $\begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$. Then

L = $\left\{\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : p = -p^t \ and \ q = -q^t\right\}$.

Root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$
eigenvector	$m_{ij}\ (i \neq j)$	$p_{ij}\ (i < j)$	$q_{ij}\ (i < j)$

(2) B = $\{\alpha_I : \ 1 \le i < \ell\} \cup \{\beta_\ell\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$and\ \ \beta_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$. $when 1 \le i < j < \ell$,

$$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} ,$$

$\varepsilon_i + \varepsilon_j = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-2}\} + \{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{\ell-1} + \beta_\ell\}$.

(3) The Dynkin diagram of Φ is

As this diagram is connected, the Lie algebra is simple. When L = 3, the

D_L) sp(2L, C):

1)Let L = gl_S(2L,**C**)= $\left\{\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : p = p^t \ and\ q = q^t\right\}$., where S= $\begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$.

root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$	$2\varepsilon_i$	$-2\varepsilon_i$
eigenvector	$m_{ij}\ (i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$	p_{ii}	q_{ii}

(2) Let αi = εi − εi+1 for 1 ≤ i ≤ L − 1 as before, and let $\beta_L = 2\varepsilon_L$. We claim that $\{\alpha_1, \ldots, \alpha_{L-1}, \beta_L\}$ is a base for Φ. We have

$$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},$$

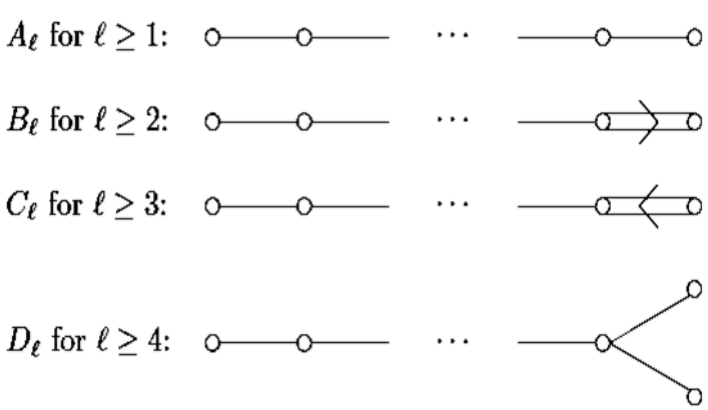
$$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{\ell-1}) + \beta_\ell,$$

$$2\varepsilon_i = 2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-1}) + \beta_\ell.$$

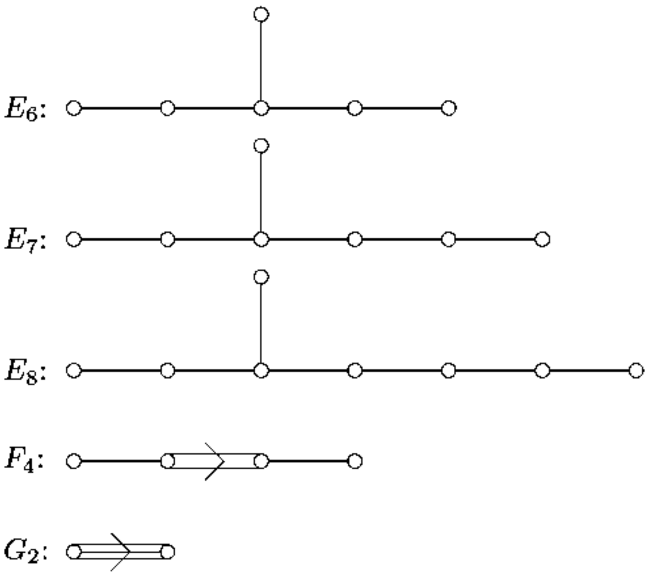
(3) The Dynkin diagram of Φ is

,

Theorem 3.25[4]:



Given an irreducible root system R, the diagram associated to R is either a member of one of the four families or one of the following five exceptional diagrams.



References

1.Erdmann, Karin. *Introducion to Lie Algebras.* London: Springer-Verlag, 2006.

2.Frank Ayres , Lloyd R .Jaisingh. *Abstruct Algebra.* Second Edition. Newyork: Mcgraw-Hill, 2004.

3.Humphreys, James E . *Indoduction to lie Algebras and Representation theory.* Third Edition. Newyork: Springer verlag, 1972.

4.Karin Erdmann,Thorsten Holm. *Algebras and Representation Theory .* UK: Springer, 2018.

5.Larson, Ron. *Elementary Linear Algebra 6th Edition.* New York: Houghton Mifflin Harcourt PublishingCompany, Richard Stratton, 2009.

6.Pierce, Richard S. *Assosiative Algebras .* Berlian: Springer-Verlag, 1982.

7.Ruiter, Joshua. "Semisimple Complex Lie Algebras." Research, Mathematic, 2016.

8.Seymour Lipschutz,Marc Lars Cipson. *Linear Algebra.* Fourth Edition. Newyork: Mcgraw ltill comanies, 2009.

پوختە

لەم کارەدا لێکۆلینەوە لە سیستەمەکانی ڕەگ و بەکارهێنانەکانی دەکەین، سەرەتا باسی پێناسە سەرەکیەکان دەربارەی فەزای ڤێکتەری و پرۆدەکتی ناوخۆیی و هەندێ لەئەنجامەکانیان دەکەین کە پێویستمانە لە کارەکەمان، پاشان جەبرە سادەکانی لی و سیستەمە ڕەگەکانیان دەخوێنین ، لە کۆتایەکەیدا باس لە پۆلینکردنی سیستەمە ڕەگە نیمچە جەبرە سادەی کەمنەکراوەی لی دەکەین لەرێگەی بەکار هێنانی دایەگرامی دینکینگی و ماتریکسی کارتانی.

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سیستەمەکانی ڕەگ و بەکارهێنانەکانی

پڕۆژەی دەرچوونە

پێشکەش بە بەشی ماتماتیك کراوە، وەك بەشێك لە پێداویستیەکانی بەدەستهێنانی بڕوانامەی بەکالۆریۆس لە زانستی ماتماتیك

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نیسان- ٢٠٢٣