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Semi-Simple Lie Algebras

Research Project

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Abstract

In this work we study semi-simple Lie algebras. First we write basic definitions and results about vector spaces, algebras and Lie algebras that we need in our work. Then we study semi-simple Lie algebras and killing form. We determine whether the Lie algebras are semi-simple by using the Killing form and Cartan's Criterion. Moreover we solve some examples to illustrate the reults.

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Introduction

Algebra is an algebraic structure consisting of a set together with operations of multiplication and addition and scalar multiplication by elements of a field and satisfying the axioms implied by "vector space" and "bilinear". The multiplication operation in algebra may or may not be associative, leading to the notions of associative algebras and non-associative algebras. The set of square matrices with entries in the field F is an associative algebra. A Lie algebra L over a field F is a vector space L over a field F , with a bilinear operation $[\ ,\ ] : L \times L \to L$ such that $[x, x] = 0$ for all x in L and $[x, [y, z]]+[y, [z, x]] + [z,[x, y]]=0$ for all $x, y, z \in L$.
Lie algebras were introduced by Marius Sophus Lie in the 1870s and they have applications in physics, differential geometry, Riemann geometry and Quantum mechanics. Many mathematicians around the world have made great achievements over Lie algebras. The Lie algebra L is simple if it has no ideals other than {0} and L and it is not abelian. A Lie algebra is called semisimple if it is a direct sum of simple Lie algebras.

In this work we study semi-simple Lie algebras. This work consists of three chapters. In chapter one we write basic definitions and results about vector spaces, algebras and Lie algebras that we need in our work. In chapter two we study basic definitions and results about semi-simple Lie algebras. At the last chapter we study Killing form and Cartan's Criterion and we use them to determine whether Lie algebras are semi-simple. In all the chapters we solve some examples to illustrate the concepts and results.

Chapter One

Preliminary and Background

In this chapter we provide the fundamental definitions and findings concerning ring, vector spaces and algebras that we need for our work and we gave many examples about these concepts.

Definition 1.1: (Lewis, 2017: 46)

A set G that is closed under a given operation '.' is called a group if the following axioms are satisfied.

1. The set G is non-empty.
2. If a, b, c $\in$ G then a(bc) = (ab) c.
3. There exists an element e in G such that

(a) For any element a in G, e a = a e = a.

(b) For any element a in G there exists an element a^{-1} in G such that $aa^{-1} = a^{-1}a = e$.

A group, which contains only a finite number of elements, is called a finite group, otherwise it is termed as an infinite group. By the order of a finite group we mean the number of elements in the group.

Definition 1.2: (LARSON, et al 2009:34)

We start by recalling the definition of a ring: A ring is a non-empty set R together with an addition $+ : R\times R \to R$, $(r + s) \to r + s$ and a multiplication $\cdot : R\times R \to R$, $(r, s) \to r \cdot s$ such that the following axioms are satisfied for all r, s, t $\in$ R:

1- (Associativity of +) $r + (s + t) = (r + s) + t$.

2- (Zero element) There exists an element $0_R \in R$ such that $r + 0_R = r = 0_R + r$

3- (Additive inverses) for every $r \in R$ there is an element $-r \in R$ such that $r + (-r) = 0_R$.

4- (Commutativity of +) $r + s = s + r$.

5- (Distributivity) $r(s + t) = s.r + r.t$ and $(r + s)t = r.t + s.t$.

6- (Associativity of $\cdot$) $r(s.t) = (rs)t$.

Definition1.3: (LARSON, et al 2009 :38)

1) The ring R is commutative if the multiplication is commutative and the ring R is said to have an identity 1 if $a \times 1 = 1 \times a = a$ for all $a \in R$.
2) A ring R with identity 1, where $1 \neq 0$ is called a division ring (or skew field) if every none zero element $a \in R$ has a multiplicative inverse, i.e. there is $b \in R$ such that a b = b a =1.

Examples 1.4:

1) The ring of integers $\mathbb{Z}$ under usual addition and multiplicative is a commutative ring with identity (Note that $\mathbb{Z} - \{0\}$under multiplication is not a group). Similarly the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers are commutative rings with identity (In fact they are fields).
2) The quotient group $\mathbb{Z} / n \mathbb{Z}$ is a commutative ring with identity under the operations of addition and multiplication modulo n.
3) : $Z_p = \{0, 1, 2, \dots, p - 1\}$, p a prime be the set of integers modulo p. $Z_p \backslash \{0\}$ is a group under multiplication modulo p. This is a finite cyclic group of order p-1.

Definition 1.5 : (Coder 2017: 29)

(V, +) is called is a vector space over a field F, if satisfies the following conditions, for all $u, v, w \in V$ and $c, d \in F$:

1) $u + v$ Is a vector in the plane closure under addition
2) $u + v = v + u$ Commutative property of addition
3) $(u + v) + w = u + (v + w)$ Associate property of addition
4) $(u + 0) = u$ Additive identity
5) $u + (-1) u = 0$ Additive inverse
6) $c u$ is a vector in the plane closure under scalar multiplication
7) $c (u + v) = c u + c v$ Distributive property of scalar mult.
8) $(c + d) u = c u + d u$ Distributive property of scalar mult.

9) $c(d\,u) = (c\,d)\,u$ Associate property of scalar mult.

10) $1(u) = u$ Multiplicative identity property

We call elements of V vectors and call elements of F scalars.

Examples 1.6:

1)Let F=R and V= R^2. The operations of addition and multiplication are define as follows $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$a(x_1, y_1) = (a x_1, a y_1)$,

for all $(x_1, y_1), (x_2, y_2) \in R^2$ and $a \in F$. Then R^2 is a vector space over R .

2) $p = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x + y + z = 0 \text{ and } x, y, z \in R \right\}$ is a vector space over set of real numbers R and "+" and "." are defined in this way

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 & + & x_2 \\ y_1 & + & y_2 \\ z_1 & + & z_2 \end{pmatrix} \text{ and } r\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}.$$

Definition 1.7:

A nonempty W subset of a vector space V over the ground field F is called a subspace of V if W is a vector space over F.

Theorem 1.8: (Coder 2017:37)

If W is a nonempty subset of a vector space V then W is a subspace of V if and only if the following closure conditions hold.

1. If u and v are in w then u+v is in W
2. If u is in *W* and *c* is any scalar, then *c*u is in W.

Definition 1.9:

A vector u in a vector space V is called a linear combination of the vectors $u_1, u_2, \ldots u_k$ in V if v can be written in the form $V = c_1u_1 + c_2u_2 + \ldots + c_ku_k$ where $c_1, c_2, \ldots, c_k$, are scalars.

Definition 1.10:

Let $S=\{V_1,V_2,\ldots,V_k\}$be a subset of a vector space V The set S is called a spanning set of V if every vector in V can be written as a linear combination of vectors in S In such cases it is said that S spans V.

Examples 1.11: (Coder 2017:39)

(a) The set S={(1,0,0),(0,1,0),(0,0,1)} spans because any vector $u=(u_1,u_2,u_3)$ in can be written as $u=u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)=(u_1,u_2,u_3)$.

(b) The set S={1,X), spans P2 because any polynomial function P(X)=a+bx+c in P2 can be written as P(X)=a(1)+b(x)+c = a+bx+c.

Definition 1.12 :

A set of vectors $V=\{v_1,v_2,v_3\}$ in a vector space V is called a basis for V if the following conditions are true.

1. *S* spans *V*.

2. *S* is linearly independent.:

Example1.13:

The vectors e_1=(1,0 . . . .,0), e_2=(0,1,. . . .,0) and e_n=(0,0,. . . . ,1) form a basis for R^n called the standard basis for R^n

Definition 1.14: (Chaim, et al 2012 :67)

Algebra over a field F is a vector space A over F together with a bilinear map, A×A→A, (x,y) → xy. The algebra A is said to be associative if (xy)z =x(yz) for all $x, y, z \in A$ and until if there is an element 1_A in A such that $1_A x = x = x\, 1_A$ for all non-zero elements of A.

Examples 1.15:

1)The set of polynomial K[x] is a K-algebra where K is a field.

2) The set of n by n matrices $M_{n\times n}$ (R) is an R-algebra over R.

3) Let H be the collection of elements of the form $a + bi + cj + dk$ where $a, b, c, k \in R$ are real numbers. Define addition component wise by

$(a + bi + cj + dk) + (a_1+b_1 i+c_1 j+d_1k) = ((a + a_1)+(b+b_1) i+(c+c_1)j+(d+d_1) k)$ and multiplication is defined by expanding $(a + bi + cj + dk)\ (a + b_1 i + c_1 j + d_1 k)$ and using the distributive law and simplifying using the relation $i^2= j^2= k^2=-1$, $ij=-j i = k$, $j k =- k j =i$, $ki =-i k =j$, $(1+i+2j)(j + k)=1(j + k)+ i (j+ k) +2j (j + k) =j+ k +ij+i k +2j^2+2j k =-2+2i+2 k$. It is not hard to check that H is an R-algebra which is called the real Hamilton Quaternion algebra. The Hamilton Quaternion's are non-commutative algebra with identity $1=1+0i+0j+0 k$. The inverse of non-zero elements are given by $(a + bi + cj + dk)^{-1}=(a + bi + cj + dk) / (a^2+ b^2 +c^2 +^2d)$

Definition 1.16: (Coder 2017: 42)

A Lie algebra L over a field F is a vector space L over a field F , with an operation [,] :$L \times L \rightarrow L$ that satisfiy the following conditions:

(L_1) [,] is bilinear.

(L_2) $[x,x] =0$ for all x in L .

(L_3) $[x,[y,z]] +[y,[z,x]] +[z,[x,y]] =0$ for all $x,y,z \in L$.

Proposition1.17:

Let I, J be ideals of a Lie algebra L. Then

$I + J := \{x + y : x \in I, y \in J\}$ is an ideal of L.

Proof. We need to show that I + J is a vector subspace of L and that for $a \in L$, $b \in I + J$, we have $[a, b] \in I + J$.

Let $v, w \in I + J$. Then $v = v_1 + v_2$ and $w = w_1 + w_2$ where $v_1, w_1 \in I$ and $v_2 , w_2 \in J$. Then $v + w = v_1 + w_2 + w_1 + w_2 =(v_1 + w_1)+(v_2 + + w_2) \in I + J$ because I, J are vector subspaces.

Let $\lambda \in F$. Then $\lambda v = \lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$. Since I, J are vector subspaces, $\lambda v_1 \in I$ and $\lambda v_2 \in J$. Thus $\lambda v \in I + J$.

Let $a \in L$, $b \in I + J$. Then $b = b_i + b_j$ so

$[a, b] = [a, b_1 + b_2] = [a, b_1] + [a, b_2]$

Since I, J are ideals of L, $[a, b_1] \in I$ and $[a, b_2]\in J$. Thus $[a, b] \in I + J$. ■

Definition1.18:

Let I, J be ideals of a Lie algebra L. Then we define

$$[I, J] := \mathrm{Span}\{[x, y]: x \in I, y \in J\}$$

Proposition 1.19:

$sl(2, C)' = [sl(2, C), sl(2, C)] = sl(2, C)$

Proposition 1.20:

Let L be a Lie algebra. Then L/Z(L) is isomorphic to a subalgebra of gl(L).

Proposition 1.21: (Drozd, et al 1994)

Let L be a Lie algebra over F , and let I be an ideal of L. We define a bracket on L/I by $[w + I, z + I] = [w, z] + I$. Then this bracket is bilinear.

Proof. Let $\lambda_1, \lambda_2 \in F$ and $v_1, v_2, w \in L$. Then

$[\lambda_1(v_1 + I) + \lambda_2(v_2 + I), w + I]$

$= [(\lambda_1 v_1 + I) + (\lambda_2 v_2 + I), w + I]$

$= [(\lambda_1 v_1 + \lambda_2 v_2) + I, w + I]$

$= [\lambda_1 v_1 + \lambda_2 v_2, w] + I$

$= ([\lambda_1 v_1, w] + [\lambda_2 v_2, w]) + I$

$= ([\lambda_1 v_1 w] + I) + ([\lambda_2 v_2, w] + I)$

$= (\lambda_1[v_1, w] + I) + (\lambda_2[v_2, w] + I).$

Therefore, the first component in the bracket is linear. Now we test the second component.

$[w + I, \lambda_1(v_1 + I) + \lambda_2(v_2 + I)]$

$= [w + I, (\lambda_1 v_1 + \lambda_2 v_2) + I]$

$= [w, \lambda_1 v_1 + \lambda_2 v_2] + I$

$= ([w, \lambda_1 v_1] + [w, \lambda_2 v_2]) + I$

$= ([w, \lambda_1 v_1] + I) + ([w, \lambda_2 v_2] + I)$

$= (\lambda_1[w, v_1] + I) + (\lambda_2[w, v_2] + I).$

Therefore, the second component is linear. Hence this bracket is bilinear on L/I. ■

Proposition 1.22.

Let L be a Lie algebra over F and let I be an ideal of L. Then the bracket on L/I satisfies $[x, x] = 0$, for $x \in L/I$.

Proof. Let $v \in L$, so $v + I \in L/I$. Then $[v + I, v + I] = [v, v] + I = 0 + I$ where $0 + I$ is the identity for L/I. ■

Proposition 1.23: (Edwin, et al 2007 :51)

Let L be a Lie algebra over F , and let I be an ideal of L. The bracket on L/I satisfies the Jacobi identity.

Proof. Let $u + I, v + I, w + I \in L/I$. Then

$[u + I, [v + I, w + I]] + [v + I, [w + I, u + I]] + [w + I, [u + I, v + I]]$

$= [u + I, [v, w] + I] + [v + I, [w, u] + I] + [w + I, [u, v] + I]$

$= ([u, [v, w]] + I) + ([v, [w, u]] + I) + ([w, [u, v]] + I)$

$= ([u, [v, w]] + [v, [w, u]] + [w, [u, v]]) + I$

$= 0 + I$ where $0 + I$ is the additive identity of L/I. ■

Proposition 1.24.

Let I be an ideal of Lie algebra L over F . Define $\pi : L \to L/I$ by $\pi(z) = z + I$.

Then π is a Lie algebra homomorphism.

Proof First we show that π is a linear map. Let $a \in F$ and $u, v \in L$

$\pi(au + v) = (au + v) + I = (au + I) + (v + I) = a(u + I) + (v + I) = a\pi(u) + \pi(v)$

so π is linear. Now we show that π preserves the bracket.

$\pi([u, v]) = [u, v] + I = [u + I, v + I] = [\pi(u), \pi(v)]$

Thus π is a Lie algebra homomorphism ■

Chapter Two

Semi-Simple Lie Algebra

A Lie algebra L over a field F is a vector space L over a field F , with a bilinear operation [,] : $L \times L \to L$ such that [x, x] = 0 for all x in L and [x, [y, z]]+[y, [z, x]] + [z,[x, y]]=0 for all x, y, z ∈ L. Lie algebras were introduced by Marius Sophus Lie in the 1870s and they have applications in physics, differential geometry, Riemann geometry and Quantum mechanics. Many mathematicians around the world have made great achievements over Lie algebras. The Lie algebra L is *simple* if it has no ideals other than {0} and L and it is not abelian. A Lie algebra is called semisimple if it is a direct sum of simple Lie algebras. The semisimple Lie algebras over the complex numbers were first classified by Wilhelm Killing (1888–90), though his proof lacked rigor. His proof was made rigorous by Élie Cartan (1894) in his Ph.D. thesis, who also classified semisimple real Lie algebras. (Erdmann, et al 2018 : 72)

Example 2.1:

The set of matrices of trace zero over a field F is simple lie algebra so it is a semisimple and denoted by $Sl_n(F)$.

Definition 2.2: (LARSON, et al 2009 : 68)

An associative algebra A over a field K is defined to be a vector space over K together with a K-bilinear multiplication A x A → A (where the image of (x,y) is written as xy) such that the associativity law holds:

▪ (x y) z = x (y z) for all x, y and z in A.

The bilinearity of the multiplication can be expressed as

▪ (x + y) z = x z + y z for all x, y, z in A,

▪ x (y + z) = x y + x z for all x, y, z in A,

▪ a (x y) = (a x) y = x (a y) for all x, y in A and a in K.

If A contains an identity element, i.e. an element 1 such that 1x = x1 = x for all x in A, then we call A an associative algebra with one or a unitary (or unital) associative algebra. Now we construct a Lie algebra from the associative algebra.

Theorem 2.3:

Let $(A,+,*)$ be an associative algebra over a field F. Define $[x, y] = x * y - y * x$ as the commutator where $x, y \in A$, then $(A,+, [\cdot, \cdot])$ is a Lie algebra over a field F.

Proof: We need to show that an associative algebra satisfies the properties of a Lie algebra under the above multiplication. First we show $[\cdot, \cdot])$ is abilinear map.

Let $x, y, z \in A$, $a, b \in F$. Then

$[ax + by, z] = (ax + by) * z - z * (ax + by)$

$= (ax) * z + (by) * z - z * (ax) - z * (by)$

$= a(x * z) - a(z * x) + b(y * z) - b(z * y)$

$= a(x * z - z * x) + b(y * z - z * y)$

$= a[x, z] + b[y, z]$.

Similarly it can be shown that, $[z, ax + by] = a[z, x] + b[z, y]$.

Next, we show skew symmetry. With $x, y \in A$,

$[x, x] = x * x - x * x = 0$ and $[x, y] = x * y - y * x= -y * x + x * y= -[y, x]$.

Finally, we show the Jacobi identity hold. With $x, y, z \in A$.

$[x, [y, z]]+[z, [x, y]] + [y, [z, x]]$

$=[x, y * z - z * y] + [z, x * y - y * x] + [y, z * x - x * z]$

$=x * (y * z - z * y) - (y * z - z * y) * x + z * (x * y - y * x)-$

$- (x * y - y * x) * z + y * (z * x - x * z) - (z * x - x * z) * y$

$=x * y * z - x * z * y - y * z * x + z * y * x + z * x * y - z * y * x-$

$- x * y * z + y * x * z + y * z * x - y * x * z - z * x * y + x * z * y$

$=x * y * z - x * y * z + z * y * x - z * y * x + z * x * y - z * x * y+$

$+ y * x * z - y * x * z + y * z * x - y * z * x + x * z * y - x * z * y =0$

Therefore $(A,+, [\cdot, \cdot])$ is a Lie algebra over a field F. ■

Theorem 2.4:

Let L be a complex Lie algebra. Then L is semisimple if and only if there are simple ideals $L_1, \ldots, L_r$ of L such that $L = L_1 \oplus L_2 \oplus \ldots \oplus L_r$.

Theorem 2.5:

If L is a semisimple Lie algebra and I is an ideal of L, then L/I is semisimple.

Definition 2.6: (Lewis, 2017: 55)

Let A be an algebra over a field F. A derivation of A is an F-linear map $D : A \to A$ such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$.

Remark 2.7:

Let DerA be the set of derivations of A. This set is closed under addition and scalar multiplication and contains the zero map. Hence DerA is a vector subspace of gl(A). Moreover, DerA is a Lie subalgebra of gl(A). If L is a finite-dimensional complex semisimple Lie algebra, then ad L = Der L.

Theorem 2.8:

Suppose that L is a finite dimensional semisimple Lie algebra over any subfield $F \subseteq C$. Then L can be expressed uniquely as a product of simple ideals.

Example 2.9;

Let $L=n(3,R)=\{\begin{pmatrix} 0 & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{pmatrix}; X_{12}, X_{13}, X_{23} \epsilon R\}$ be the set of strictly upper triangular matrices and $I=\{\begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}: X_{13}\ \epsilon R\}$. We show that I is ideal of L. let $x=\begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 & y_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \epsilon I$ and

$i=\begin{pmatrix} 0 & i & i_2 \\ 0 & 0 & i_3 \\ 0 & 0 & 0 \end{pmatrix} \epsilon$ L. Then

$x+y=\begin{pmatrix} 0 & 0 & X_{13}+y_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \epsilon I$ and $rx=\begin{pmatrix} 0 & 0 & rX_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \epsilon I$

[x,i]=- [i,x]=xi-ix

$=\begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & i_{12} & i_{13} \\ 0 & 0 & i_{23} \\ 0 & 0 & 0 \end{pmatrix}-\begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & i_{12} & i_{13} \\ 0 & 0 & i_{23} \\ 0 & 0 & 0 \end{pmatrix}$

$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \epsilon I$

Then I is ideal of L In fact [L,L]=Z(L).

Theorem 2.10:

1) 0 and L always ideal in L

2) If L is abelian then every subspace is ideal in L.

3)The kernal of a homomophism of Lie algebras is an ideal in a domain.

4)z(L)=(XϵL; [x,y]=0 for all yϵL)is ideal in L

Theorem 2.11;

If $L = L_1 \oplus L_2 \oplus \ldots \oplus L_r$ is a semisimple lie algebra Then [L,L]=L where Li simple ideals of L.

Corollary 2.12:

If L is a simple Lie algebra then [L,L] =L.

Theorem 2.13:

$SL_2(R) = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}$;a,b,cϵR is an ideal of gL_2 (R) = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$a,b,c,d ϵ R

Proof: Let $x=\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}$and $y=\begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} \epsilon SL_2(R)$ $i=\begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix} \epsilon gl_2(R)$

x+y = = $\begin{pmatrix} a_1+b_1 & a_2+b_2 \\ c_1+c_2 & -(a_1+a_2) \end{pmatrix} \epsilon SL_2(R)$

rx== $\begin{pmatrix} ra_1 & rb_1 \\ rc_1 & -ra_1 \end{pmatrix} \epsilon\ sL_2\ (R)$ and

[x,i]= $x_i - ix = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}\begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix} - \begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}$

$=\begin{pmatrix} a_1i_1 + b_1i_3 & a_1i_{2+b_1i_4} \\ c_1i_1 - a_1i_3 & c_1i_2 - c_1i_4 \end{pmatrix} - \begin{pmatrix} i_1\,a_1 + i_2c_1 & i_3\ a_1 + i_4\ c_1 \\ i_3a_1 + i_1c_1 & i_3b_1 - i_4a_1 \end{pmatrix} \epsilon$ SL_2 (R).

Then SL_2 (R) is an ideal of gl_2 (R)

We can genialize the above result as follows:

Example 2.14: (MAURICE 1946)

sl_n (R) is an ideal of gl_n (R) where gl_n (R) is the set of all square matrices of size n.

Example 2.15:

$I = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$; a, cϵR]and $L = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$;a,b,c ϵ F. We calim that I is a subalgebra of L ut it is not ideal. Let x,y ϵ I where $x = \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix}$.

$x - y = \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix} - \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} x = \begin{pmatrix} a_1 - a_2 & 0 \\ 0 & c_1 - c_2 \end{pmatrix} \epsilon\ I$

rx=r$\begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}=\begin{pmatrix} ra_1 & 0 \\ 0 & rc_1 \end{pmatrix} \epsilon\ I$

[x,y]=xy-yx =$\begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} - \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}$

$=\begin{pmatrix} a_1a_2 & 0 \\ 0 & c_{1c_2} \end{pmatrix}-\begin{pmatrix} a_2\ a_1 & 0 \\ 0 & c_2c_1 \end{pmatrix}=\begin{pmatrix} a_1a_2 - a_2\ a_1 & 0 \\ 0 & c_1\ c_2 - c_2c_1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \epsilon\ I$

Then I is a sub algebra of L , so it is only remain to show that I is not ideal of L.

Let xϵ Iand i ϵ I. where x=$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ and i=$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$. Then

[x,i]=xi-ix=$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} - \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$

$=\begin{pmatrix} xa & ay \\ 0 & cz \end{pmatrix}-\begin{pmatrix} xa & yc \\ 0 & cz \end{pmatrix}=\begin{pmatrix} 0 & ay - yc \\ 0 & 0 \end{pmatrix} \notin$ I.

Chapter three

The Killing form and Cartan's Criteria

The Killing Form is a symmetric bilinear form, which will be used in Cartan's criteria as a tool to help us assess solvability and semisimplicity of Lie Algebras. It is defined as follows: $K(x, y) := \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$ for $x, y \in L$. (Quarrington 2019:87).
In this chapter we study the criteria about semisimplicity of Lie Algebras by using the Killing form and we solve some examples to illustrate the method.

Definition 3.1: (Erdmann,et al 2006:45)
The killing form $k;L*L \rightarrow F$ is defined by $K(x,y)=\mathrm{Tr}\,(\mathrm{ad}_X\ \mathrm{ad}_Y)$
the killing form is clearly symmetric, i.e $K(X,Y)=K(Y,X)$

Definition 3.2: (Erdmann,et al 2006:28)
The Lie algebra L is said to be *solvable* if for some $m \geq 1$ we have $L^{(m)} = 0$ where
$L^{(1)} = [L,L]$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$ for $k \geq 2$.
Theorem (Cartan's First Criterion) 3.3 : (Erdmann,et al 2006:80)
The complex Lie algebra, L, is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$ and
$y \in [L,L]$
Theorem (Cartan's Second Criterion) 3.4: (Erdmann,et al 2006:82)
The complex Lie algebra, L, is semisimple if and only if the Killing form
κ of L is non-degenerate
Example 3.5:
Let L = sl(2, R). This has basis $X=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $, and$ $h=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$\mathrm{ad}x^{x}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0x+0y+0h$

$\mathrm{ad}x^{y}==\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}-\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=0x+0y+h$

$\text{ad}x^{h}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$=-2x+0y+0h

adx=$\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}$

$\text{ad}y^{x}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$-$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\1&1\end{pmatrix}$=0x+0y-h

$\text{ad}y^{y}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=0x+0y+0h$

$\text{ad}y^{h}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}$=0x+2y+0h

Thus we get ady=$\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix}$

$\text{ad}h^{x}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$=2x+0y+0h

$\text{ad}h^{y}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$=0x-2y+0h

$\text{ad}h^{h}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=0x+0y+0h$

Then adh=$\begin{pmatrix}2&0&0\\0-2&0\\0&0&0\end{pmatrix}$

k(x,x)=$\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}0&-2&0\\0&0&6\\0&0&0\end{pmatrix}$=0

k(x,Y)= $\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix}=\begin{pmatrix}2&0&0\\0&0&0\\0&0&2\end{pmatrix}$=4

k(x,h)= $\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}2&0&0\\0&-2&0\\0&0&0\end{pmatrix}$=0

k(y,x)=$\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix}\begin{pmatrix}0&0-2\\0&0&0\\0&1&0\end{pmatrix}$=4

k(y,y)= $\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix}=0$

k(y,h)= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$

k(h,x)= $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0$

k(h,y)= $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$=0

k(h,h)= $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8$

Example 3.6;

Let a= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ b= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ c= $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and d= $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

ada^a= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$=0x+0y+0z+0h

ada^b= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$=0x+y+0h+0z

ada^c= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$=0x+0y+h+0z

ada^d= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$=0x+0y+0h+z

Then ada= $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

adb^a= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$-$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$=$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$=0x+y+0h+z

adb^b= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$=0x+0y+0h+0z

adb^c= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$=$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$=x+0y+0h-z

adb^d= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$=0x+y+0h-z

Thus we get adb= $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix}$.

adc^a=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$=-x+0y+h+0z

adc^b=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & `1 \end{pmatrix}$=0x-y+0h+z

adc^c=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$=0x+0y+0h+0z

adc^d=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$=0x+0y-h+z

Then adc=$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

add^a=$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$=0x-y+h+0z

add^b=$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$=0x+0y+h-z

add^c=$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$=0x+0y+h-z

add^d=$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$=0x+0y+0z+0h

We get ad$_d$=$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$.

k(a,a)= $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ =2

k(a,b)=$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix}$=0

k(a,c)= $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$=0

k(a,d)=$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$=-2

$$k(b,a)=\begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}\ \begin{pmatrix} 0\ \ 0\ 0\ 0 \\ 0\ 1\ \ 0\ 0 \\ 0\ 0\ \ 1\ 0 \\ 0\ 0\ \ 0\ 1 \end{pmatrix}=0$$

$$k(b,b)=\begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}\begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}=0$$

$$k(b,c)=\begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}\begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}=4$$

$$k(b,d)=\begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}\begin{pmatrix} 0\ \ 0\ \ 0\ \ \ \ 0 \\ -1\ \ \ 0\ \ 0\ 0 \\ 1\ \ \ \ 1\ \ 1\ \ 0 \\ 0-1-1\ \ 0 \end{pmatrix}=0$$

$$k(c,a)=\begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}\begin{pmatrix} 0\ \ 0\ 0\ 0 \\ 0\ 1\ \ 0\ 0 \\ 0\ 0\ \ 1\ 0 \\ 0\ 0\ \ 0\ 1 \end{pmatrix}=0$$

$$k(c,b)=\begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}\ \begin{pmatrix} 0\ \ 0\ 1\ 0 \\ 1\ 0\ \ 0\ 1 \\ 0\ 0\ \ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}=4$$

$$k(c,c)=\begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}\ \begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}=0$$

$$k(c,d)=\begin{pmatrix} -1\ \ 0\ 0\ 0 \\ 0-1\ \ 0\ 0 \\ 1\ 0\ \ 0-1 \\ 0\ 1\ \ 0\ \ \ 1 \end{pmatrix}\begin{pmatrix} 0\ \ 0\ \ 0\ \ \ \ 0 \\ -1\ \ \ 0\ \ 0\ 0 \\ 1\ \ \ \ 1\ \ 1\ \ 0 \\ 0-1-1\ \ 0 \end{pmatrix}=0$$

$$k(d,a)=\begin{pmatrix} 0\ \ 0\ \ 0\ \ \ \ 0 \\ -1\ \ \ 0\ \ 0\ 0 \\ 1\ \ \ \ 1\ \ 1\ \ 0 \\ 0-1-1\ \ 0 \end{pmatrix}\begin{pmatrix} 0\ \ 0\ 0\ 0 \\ 0\ 1\ \ 0\ 0 \\ 0\ 0\ \ 1\ 0 \\ 0\ 0\ \ 0\ 1 \end{pmatrix}=-2$$

$$k(d.b)=\begin{pmatrix} 0\ 0\ 0\ \ 0 \\ -1\ \ 0\ 0\ 0 \\ 1\ \ \ 1\ 1\ 0 \\ 0-1-1\ 0 \end{pmatrix}\begin{pmatrix} 0\ 0\ 1\ 0 \\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \\ 1\ 0\ -1-1 \end{pmatrix}=0$$

$$k(d,c)=\begin{pmatrix} 0\ 0\ 0\ \ 0 \\ -1\ \ 0\ 0\ 0 \\ 1\ \ \ 1\ 1\ 0 \\ 0-1-1\ 0 \end{pmatrix}\begin{pmatrix} -1\ 0\ 0\ 0 \\ 0-1\ 0\ 0 \\ 1\ 0\ 0-1 \\ 0\ 1\ 0\ \ 1 \end{pmatrix}=0$$

$$k(d,d)=\begin{pmatrix} 0\ 0\ 0\ \ 0 \\ -1\ \ 0\ 0\ 0 \\ 1\ \ \ 1\ 1\ 0 \\ 0-1-1\ 0 \end{pmatrix}\begin{pmatrix} 0\ 0\ 0\ \ 0 \\ -1\ \ 0\ 0\ 0 \\ 1\ \ \ 1\ 1\ 0 \\ 0-1-1\ 0 \end{pmatrix}=2$$

$k(gl(2,c)=\begin{pmatrix} 2\ \ 0\ 0-2 \\ 0\ \ 0\ 4\ 0 \\ 0\ 4\ \ 0\ \ 0 \\ -2\ 0\ \ 0\ \ 2 \end{pmatrix}=4\neq 0$ we get this algebra is semisimple (Ruiter, 2016:64).

Example 3.7 (Spiege , et al 2010: 77)

Let’s look at the Lie algebra sl(3), of trace free 3×3 real matrices with the following multiplication table

20 sl(3)	e1	e2	e3	e4	e5	e6	e7	e8
e1	0	0	e3	2e4	-e5	e6	-2e7	-e8
e2	0	0	-e3	e4	e5	2e6	-e7	-2e8
e3	-e3	e3	0	0	e1-e2	e4	-e8	0
e4	-2e4	-e4	0	0	-e6	0	e1	e3
e5	e5	-e5	-e1+e2	e6	0	0	0	-e7
e6	-e6	-2e6	-e4	0	0	0	e5	e2
e7	2e7	e7	e8	-e1	0	-e5	0	0
e8	e8	2e8	0	-e3	e7	-e2	0	0

then the adjoint representation of the basis elements are calculated to be

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and it is degenerate.

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زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

جەبرە لیە نیمچە سادەكان

پرۆژەی دەرچوون

پێشكەش بە بەشی ماتماتیك كراوە،وەك بەشێك لە پێداویستیەكانی بەدەستهێنانی بڕوانامەی بەكالۆریۆس لە زانستی ماتماتیك

ئامادەكراوە لەلایەن:
بێگەرد حمدامین عبدالله

بەسەرپەرشتی:
د.هۆگر محمد یاسین

٢٠٢٣-نیسان

پوختە:

لەم پرۆژەدا لێکۆلینەوە لە جەبرە لیە نیمچە سادەکان و خاسیەتەکانی دەکەین. وەلەرێگەی کیلین فۆرم نیمچە سادەیەکەی دیاری دەکەین