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Associative Algebra with Idempotent Elements

Research Project

Submitted to the department of (Mathematic) in partial fulfillment of the requirements for the degree of BSc. in (forth)

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Abstract

In this work we study associative algebras with idempotent elements. First we write basic definitions and results about vector spaces and algebras that we need in our work. Then we study Lie algebras and Algebras with idempotent elements and some properties of these kind of algebras. Moreover, we study classification of associative algebras and we study pierce decomposition of associative algebras.

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Introduction

Algebra is an algebraic structure consisting of a set together with operations of multiplication and addition and scalar multiplication by elements of a field and satisfying the axioms implied by "vector space" and "bilinear". Algebraic multiplication may or may not be an associative operation, giving rise to the concepts of associative algebras and non-associative algebras. The spaces of n by n-matrices with coefficients in some field K and the standard matrix operations are examples of associative algebras that are already present in elementary linear algebra. Polynomials over a particular field offer another illustration, but there are plenty others. A ring that also doubles as a vector space over a field K, in general, is what is meant by an associative algebra A when scalars commute with all of its members.

In this work we study associative algebras with idempotent elements. This work consists of three chapters and is organized as follows. In chapter one we give basic definitions and results about vector spaces and algebras that we need in our work. We illustrate these definitions and results by many examples. In chapter two we study Algebras and special type of it which is called Lie algebras. Then we study some properties of Algebras and Lie algebras and the derivation of these algebras. In the last chapter we study the classification of associative algebras and pierce decomposition of them.

Chapter One

Preliminary and Background

In this chapter we state basic definitions and results about ring and vector spaces that we need in our work. We gave many examples about these algebraic concepts.

Definition of group 1.1: [1]

A set G that is closed under a given operation '.' is called a group if the following axioms are satisfied.

1. The set G is non-empty.

2. If $a, b, c \in G$ then $a(bc) = (ab)c$.

3. There are exists an element e in G such that

(a) For any element a in G, $e a = a e = a$.

(b) For any element a in G there exists an element a^{-1} in G such that $aa^{-1} = a^{-1} a = e$.

A group, which contains only a finite number of elements, is called a finite group, otherwise it is termed as an infinite group. By the order of a finite group we mean the number of elements in the group

Example 1.2:

1) Let Q be the set of rational numbers.Then $Q\backslash\{0\}$ is a group under multiplication which is an infinite group.

2) Let p be a prime number and $Z_p = \{0, 1, 2, \dots , p - 1\}$ be the set of integers modulo p. Then $Z_p\backslash\{0\}$ is a group under multiplication modulo p which is a finite cyclic group of order p-1.

Definition 1.3: [2]

We start by recalling the definition of a ring: A ring is a non-empty set R together with an addition$+ : R\times R \to R, (r, s) \to r+s$ and a multiplication $\cdot : R\times R \to R, (r, s) \to r \cdot s$ such that the following axioms are satisfied for all $r, s, t \in R$

R1- (Associativity of +) $r + (s + t) = (r + s) + t$.

R2-(Zero element) There exists an element $0_R \in R$ such that $r+0_R = r = 0_R +r$.

R3- (Additive inverses) For every $r \in R$ there is an element $-r \in R$ such that

$r + (-r) = 0_R$.

R4- (Commutativity of +) $r + s = s + r$.

R5- (Distributivity) $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(r + s) \cdot t = r \cdot t + s \cdot t$.

R6- (Associativity of .) $r \cdot (s \cdot t) = (r \cdot s) \cdot t$.

R7- (Identity element) There is an element $1_R \in R \setminus \{0\}$ such that $1_R \cdot r = r = r \cdot 1_R$.

Moreover, a ring R is called commutative if $r \cdot s = s \cdot r$ for all $r, s \in R$. As usual, the multiplication in a ring is often just written as rs instead of $r \cdot s$; we will follow this convention from now on. Note that axioms (R1)–(R4) say that (R,+) is an abelian group. We assume by Axiom (R7) that all rings have an identity element; usually we will just write 1 for 1_R. Axiom (R7) also implies that 1_R is not the zero element. Now we list some common examples of rings.

Example 1.4:

(1) The integers Z form a ring. Every field is also a ring, such as the rational numbers Q, the real numbers R, the complex numbers C, or the residue classes Z_p of integers modulo p where p is a prime number.

(2) The set of integers Z is a commutative ring with 1.

3) Let $Z_n = \{0, 1, 2, \dots , n - 1)$ be the ring of integers modulo n. Z_n is a ring under modulo addition and multiplication. Z_n is a commutative ring with unit.

4) The $n \times n$-matrices $M_n(K)$, with entries in a field K, form a ring with respect to matrix addition and matrix multiplication.

5) The ring K[X] of polynomials over a field K where X is a variable. Similarly, the ring of polynomials in two or more variables, such as K[X, Y].

Note that examples (4) and (5) are not just rings but also vector spaces. There are many more rings which are vector spaces, and this has led to the definition of a vector space.

Definition 1.5: [2]

$(V, +)$ is called is a vector space over a field K, if satisfies the following conditions, for all $u, v, w \in V$ and $c, d \in K$:

1) $u + v$ Is a vector in the plane closure under addition

2) $u + v = v + u$ Commutative property of addition

3) $(u + v) + w = u + (v + w)$ Associate property of addition

4) $(u + 0) = u$ Additive identity

5) $u + (-1)\,u = 0$ Additive inverse

6) $c\,u$ is a vector in the plane closure under scalar multiplication

7) $c\,(u + v) = c\,u + c\,v$ Distributive property of scalar mult.

8) $(c + d)\,u = c\,u + d\,u$ Distributive property of scalar mult.

9) $c\,(d\,u) = (c\,d)\,u$ Associate property of scalar mult.

10) $1(u) = u$ Multiplicative identity property

We call elements of V vectors and call elements of K scalars.

Example 1.6:

1) The set of polynomial K[x] is a vector space over the field K.

2) The set of n by n Matrices $M_{n\times n}$ (R) is a vector space over R. for example if n=2,

$M_{2\times2}(R) = \left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \,\middle|\, x, y, z, w \in R\right\}$ is a vector space of 2 by 2 Matrices over set of real numbers R and "+" and "." are defined in this way:

$$\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix} + \begin{pmatrix} a2 & b2 \\ c2 & d2 \end{pmatrix} = \begin{pmatrix} a1+a2 & b1+b2 \\ c1+c2 & d1+d2 \end{pmatrix} \text{ and } r\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix} = \begin{pmatrix} ra1 & rb1 \\ rc1 & rd1 \end{pmatrix}$$

Now we show that $M_{2\times2}$(R) is a vector space over the field R:

1-let A,B, C $\in M_{2\times2}$(R) where A=$\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix}$, B=$\begin{pmatrix} a2 & b2 \\ c2 & d2 \end{pmatrix}$, and C=$\begin{pmatrix} a3 & b3 \\ c3 & d3 \end{pmatrix}$.

$$A+(B+C)= \begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix} + \left(\begin{pmatrix} a2 & b2 \\ c2 & d2 \end{pmatrix} + \begin{pmatrix} a3 & b3 \\ c3 & d3 \end{pmatrix}\right)$$

$$=\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix} + \left(\begin{pmatrix} a2+a3 & b2+b3 \\ c2+c3 & d2+d3 \end{pmatrix}\right)=\begin{pmatrix} a1+(a2+a3) & b1+(b2+b3) \\ c1+(c2+c3) & d1+(d2+d3) \end{pmatrix}$$

$$=\begin{pmatrix} (a1+a2)+a3 & (b1+b2)+b3 \\ (c1+c2)+c3 & (d1+d2)+d3 \end{pmatrix}=\begin{pmatrix} a1+a2 & b1+b2 \\ c1+c2 & d1+d2 \end{pmatrix}+\begin{pmatrix} a3 & b3 \\ c3 & d3 \end{pmatrix}$$

$$=\left(\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix}+\begin{pmatrix} a2 & b2 \\ c2 & d2 \end{pmatrix}\right)+ \begin{pmatrix} a3 & b3 \\ c3 & d3 \end{pmatrix}=(A+B)+C$$

2- We show that A+0=0+A. Let A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$A+0=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=A$$

3- Let A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, -A=$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Then

$$A+(-A)= \begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} =\begin{pmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4- Let A=$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and B=$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$Then

A+B=$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_2+c_2 & d_1+d_2 \end{pmatrix}$ =B+A

5- k(A+B)=k$\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_2+c_2 & d_1+d_2 \end{pmatrix}$

=$\begin{pmatrix} \mathrm{k}(a_1+a_2) & \mathrm{k}(b_1+b_2) \\ \mathrm{k}(c_2+c_2) & \mathrm{k}(d_1+d_2) \end{pmatrix}$

=$\begin{pmatrix} \mathrm{k}\,a_1+\mathrm{k}\,a_2 & \mathrm{k}\,b_1+\mathrm{k}\,b_2 \\ kc_2+kc_2 & kd_1+kd_2 \end{pmatrix}$

= $\begin{pmatrix} \mathrm{k}\,a_1 & \mathrm{k}b_1 \\ kc_1 & \mathrm{k}\,d_1 \end{pmatrix} + \begin{pmatrix} ka_2 & \mathrm{k}b_2 \\ kc_2 & \mathrm{k}\,d_2 \end{pmatrix}$=

k$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ + k$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$=kA+kB

6- (k+r).A=(k+r). $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{(k+r)\ a} & \mathrm{(k+r)\ b} \\ \mathrm{(k+r)c} & \mathrm{(k+r)d} \end{pmatrix} = \begin{pmatrix} \mathrm{ka+ra} & \mathrm{kb+rb} \\ \mathrm{kc+rc} & \mathrm{kd+rd} \end{pmatrix}$

== $\begin{pmatrix} \mathrm{ka} & \mathrm{kb} \\ \mathrm{kc} & \mathrm{k\,d} \end{pmatrix} + \begin{pmatrix} \mathrm{ra} & \mathrm{rb} \\ \mathrm{rc} & \mathrm{rd} \end{pmatrix} = \mathrm{k}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} + \mathrm{r}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$ =k.A+r.A

7- k(r.A)= k(r$\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$)=k($\begin{pmatrix} \mathrm{r\,a} & \mathrm{r\,b} \\ \mathrm{rc} & \mathrm{r\,d} \end{pmatrix}$)= $\begin{pmatrix} \mathrm{k(r\,a)} & \mathrm{k(\,r\,b)} \\ \mathrm{k(rc)} & \mathrm{k(r\,d)} \end{pmatrix} = \begin{pmatrix} \mathrm{(kr)\ a} & \mathrm{(\,k\,r)\ b} \\ \mathrm{(kr)c} & \mathrm{(\,kr)\ d)} \end{pmatrix}$=kr $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$

8- 1. $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{1.a} & \mathrm{1.b} \\ \mathrm{1.c} & \mathrm{1.d} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$=A

Definition 1.7: [3]

A nonempty W subset of a vector space V is called a subspace of V if W is a vector space under the operations of addition and scalar multiplication defined in V. If W is a nonempty subset of a vector space V then W is a subspace of V if and only if the following closure conditions hold.

1. If u and v are in w then u+v is in W

2. If u is in W and c is any scalar, then cu is in W.

Example 1.8:

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ (C) with the standard operations. Because W is not closed under addition. To see this, let A and B be A=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and B=$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and then A and B are both singular (noninvertible), but their sum A+B=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular. Thus W is not closed under addition, and we get it is not a subspace of $M_{2\times2}$ (R).

Definition 1.9: [3]

A vector u in a vector space V is called a linear combination of the vectors $u_1,u_2,\ldots.u_k$ in V if v can be written in the form $V=c_1u_1+c_2u_2+. \ . \ . +c_ku_k$ where $c_1,c_2,. . . ,c_k$, are scalars.

Example 1.10:

(1,1,1) is a linear combination of vectors in the set S={(1,2,3),(0,1,2),(-1,0,1)}

Definition 1.11: [3]

Let $S =\{V_1,V_2,\ldots,V_k\}$be a subset of a vector space V. The set S is called a spanning set of V if every vector in V can be written as a linear combination of vectors in S In such cases it is said that S spans V.

Example 1.12 :

(a) The set S={(1,0,0),(0,1,0),(0,0,1)} spans R^3 because any vector $u=(u_1,u_2,u_3)$ in R^3 can be written as $u=u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)=(u_1,u_2,u_3)$

(b) The set $S=\{1,X, X^{3\}}$ spans P_2 because any polynomial function $P(X)=a+bx+c\,x^2$in P_2 can be written as $P(X)=a(1)+b(x)+c(x^2) =a+bx+cx^2$

Now we review linear Dependent and linear Independent:

Definition 1.13: [3]

A set of vectors $S=\{V_1,V_2,\ldots,V_K\}$ in a vector space V is called linearly independent if the vector equation $c_1v_1+c_2v_2+\ldots..+c_kv_k=0$ has only the trivial solution $c_1=0,c_2=0,\ldots..c_k=0$, If there are also nontrivial solutions, then S is called linearly dependent.

Example 1.14:

(a) The set S={(1,2),(2,4)} in R^2 is linearly dependent because -2(1,0)+(2,4)=(0,0)

(b) The set S={(1,0),(0,1),(-2,5)} in R^2 is linearly dependent because 2(1,0)-5(0,1)+(-2,5)=(0,0)

(c) The set S={(0,0),(1,2)} in $R^2$is linearly dependent because 1(0,0)+0(1,2)=(0,0)

(d) S={(1,2,3),(0,1,2),(-2,0,1)} is linearly independent or linearly dependent inR^3.

Definition 1.15: [3]

A set of vectors V={v_1,v_2,v_3} in a vector space V is called a basis for V if the following conditions are true.

1. S spans V.   2. S is linearly independent.:

Example 1.16:

S={(1,0,0),(0,1,0),(0,0,1)} is a basis for R^3 and generally e_1=(1,0 . . . .,0), e_2=(0,1,. . . .,0), …, e_n=(0,0,. . . . ,1) form a basis for R^n called the standard basis for R^n

Definition 1.17: [3]

If a vector space V has a basis consisting of vectors, then the number n is called the dimension of V denoted by dim(V)=n. If V consists of the zero vector alone, the dimension of V is defined as zero .

Example 1.18:

(a) W={(d,c-d,c): c and d are real number } is a two dimensional subspace of R^3

(b) W={(2b,b,0): b is a real number} is a one dimensional subspace of R^3.

Chapter two
Algebras and Lie algebras

In this chapter we study algebras with idempotent elements and the main properties of them. First we study Algebras and a special type of it which is called Lie algebras. Then we study some properties of Algebras and Lie algebras and the derivation of their algebras. Moreover we gave many examples about these algebraic structures.

Definition 2.1: [2]

Algebra over a field F is a vector space A over F together with a bilinear map, $A\times A\rightarrow A$, $(x,y) \rightarrow xy$. We say that xy is the product of x and y.

Usually one study algebras where the product satisfies some further properties. The algebra A is said to be associative if $(xy)z = x(yz)$ for all $x, y, z \in A$ and until if there is an element 1_A in A such that $1_A x = x = x\, 1_A$ for all non-zero elements of A.

Example 2.2:

1) The set of polynomial K[x] is a K-algebra.

2) The space of n×n-matrices $M_n(K)$ with matrix addition and matrix multiplication form a K-algebra. It has dimension n^2; the matrix units E_{ij} for $1 \leq i, j \leq n$ form a K-basis. Here E_{ij} is the matrix which has entry 1 at position (i, j), and all other entries are 0. This algebra is not commutative for $n \geq 2$. For example we have $E_{11}E_{12} = E_{12}$ but $E_{12}E_{11} = 0$.

4) $H(R) = \{a+b_i+c_j+d_k\ ; a, b, c, d \in R\}$ are four dimensional algebra and it is called quaternion algebra (Historically, it is one of the first examples of algebra.

5) If V is a vector space over the field K, then the linear transformations

Of the space V form also algebra E (V). This algebra is finite dimensional

6) Consider the n-dimensional vector space of all n-tuples $(\alpha_1, \alpha_2, \dots, \alpha_n), \alpha_1 \in$ k with coordinate wise addition and Scala multiplication. By defining the multiplication Coordinate wise

$$(\alpha_1, \alpha_2, \dots, \alpha_n)(\beta_1, \beta_2, \dots, \beta_n) = (\alpha_1\beta_1, \alpha_2\beta_2, \dots \alpha_n\beta_n),$$

We obtain an algebra over the field K which will be denoted by K^n.

5) Let $A_1, A_2, \dots, A_n$ be algebra over the field K consider their Cartesian product A.i.e. the set all sequences ($a_1, a_2, \dots, a_n), a_i \in A_i$ and define the operations Coordinate wise: $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$, $\alpha(a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$,and$(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, a_n)(a_1 b_1, a_2 b_2, \dots, a_n b_n)$ Clearly, in this way A becomes algebra over K which is called the direct product of the algebra $A_1, A_2, \dots, A_n$ and is denoted by $A_1 \times A_2 \times \dots \times A_n$, Or $\prod_{i=1}^{n} A_i$.The algebra $A_1, A_2, \dots, A_n$are said to be direct factors of the algebrae.

(7) The field K is a commutative K-algebra, of dimension 1.

(8) The field C is also an algebra over R, of dimension 2, with R-vector space basis {1, i}, where i2 = −1. More generally, if K is a subfield of a larger field L, then L is an algebra over K where addition and (scalar) multiplication are given by the addition and multiplication in the field L.

Remark 2.3:

(1) The condition relating scalar multiplication and ring multiplication roughly says that scalars commute with everything. Due to A is a vector space over K we have for all a, b ∈ A and λ,μ ∈ K :

(i) $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$;

(ii) $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$;

(iii) $(\lambda\mu) \cdot a = \lambda \cdot (\mu \cdot a)$;

(iv) $1_K \cdot a = a$.

(2) Since A is a vector space, and 1_A is a non-zero vector, it follows that the map $\lambda \to \lambda \cdot 1_A$ from K to A is injective. We use this map to identify K as a subset of A. Similar to the convention for ring multiplication, for scalar multiplication we will usually also just write λa instead of $\lambda \cdot a$.

(3) The dimension of a K-algebra A is the dimension of A as a K-vector space. The K-algebra A is finite-dimensional if A is finite-dimensional as a K-vector space.

(4) The algebra is commutative if it is commutative as a ring y, z ∈ L.

Jordan algebra 2.4: [9]

Jordan algebra is a nonassociative algebra over a field whose multiplication satisfies the following axioms:

1. xy=yx (commutative law)
2. (xy)(xx)=x(y(xx)) (Jordan identity).

The product of two elements x and y in a Jordan algebra is also denoted $x \circ y$, particularly to avoid confusion with the product of a related associative algebra.

Example 2.5:

The set of self-adjoint real, complex, or quaternionic matrices with multiplication $\frac{xy+yx}{2}$ form a special Jordan algebra.

Definition 2.6: [6]

A vector space L over a field F, with an operation $L\times L\rightarrow L$, denoted

$(x, y) \rightarrow [xy]$ and called the bracket or commutator of r and y, is called a Lie algebra over F if the following axioms are satisfies:

(LI) The bracket operation is bilinear

(L2) [xxl = 0 for all x in L.

(L3) $[x[yz]]+[y[zx]]+[z[xy]] = 0$ $(x, y, z \in L)$.

Axiom (L3) is called the Jacobi identity. Notice that (LI) and (L2), applied

to [x+y, x+y], imply anticommutativity: (L2') [xy] = - [yx]. Conversely,

if char F + 2, it is clear that (L2') will imply (L2

We say that two Lie algebras L, L' over F are isomorphic if there exists a vector space isomorphism : $L\rightarrow L'$ satisfying $\emptyset[x] = [(\emptyset(x)(\emptyset(y)]$ for all x, y in L (and then $\emptyset$ is called an isomorphism of Lie algebras). Similarly, it is obvious how to define the notion of (Lie) subalgebra of L: A subspace K of L is called a subalgebra if $[xy] \in K$ whenever x, y K; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra Fx, with trivial multiplication, because of L2

Proposition 2.7:[10]

Let L be a Lie algebra, and let $v \in L$. Then $[v, 0] = [0, v] = 0$.

Proof. By bilinearity of the bracket,

$[v, 0] = [v, v - v] = [v, v] - [v, v] = 0$

$[0, v] = [v - v, v] = [v, v] - [v, v] = 0$

(ii) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent over F.

Lemma 2.8:

Let L be a Lie algebra over F. Let $x, y \in L$ and $a \in F$.

Then $a[x, y] = [ax, y] = [x, ay]$.

Examples 2.9:[6]

(1) Let F = R. The vector product $(x, y) \to x \wedge y$ defines the structure of a Lie algebra on R^3. We denote this Lie algebra by R^3 Explicitly, if

$x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then

$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

Example 2.10:

Convince yourself that $\wedge$ is bilinear. Then check that the Jacobi identity holds. Hint: If $x \cdot y$ denotes the dot product of the vectors $x, y \in$, R^3 then $x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z$ for all $x, y, z \in .R^3$

Lemma 2.11: [10]

Let $u, v, w \in R^3$. Then $u\times(v\times w) = (u\cdot w)v - (u\cdot v)w$.

Proof. $u\times(v\times w) = u\times(v^2w^3 - v^3w^2 , v^3w^1 - v^1w^3 , v^1w^2 - v^2w^1)$

$=(u^2(v^1w^2-v^2w^1) - u^3(v^3w^1-v^1w^3),$

$u^3(v^2w^3-v^3w^2) - u^1(v^1w^2-v^2w^1),$

$u^1(v^3w^1-v^1w^3) - u^2(v^2w^3-v^3w^2))$

$=(v^1u^2w^2-v^2u^2w^1-v^3u^3w^1+v^1u^3w^3,$

$v^2u^3w^3-v^3u^3w^2-u^1v^1w^2+v^2u^1w^1,$

$v^3u^1w^1-v^1u^1w^3-u^2v^2w^3+v^3u^2w^2)$

$= u^2v^1w^2-v^2u^2w^1-v^3u^3w^1+v^1u^3w^3+u^1v^1w^1-u^1v^1w^1$

$,v^2u^3w^3 - v^3u^3w^2-v^1u^1w^2 +u^1v^2w^1 +v^2u^2w^2 - v^2u^2w^1$

, $v^3u^1w^1 - v^1u^1w^3 - v^2u^2w^3 + v^3u^2w^2 + v^3u^3w^3 - v^3u^3w^3$

$=(v^1(u^1w + u^2w^2 + u^3w^3) - w^1(u^1v^1 + u^2v^2 + u^3v^3),$

$v^2(u^1w^1+u^2w^2+ u^3w^3) - w^2(u^1v^1 + u^2v^2+ u^3v^3),$

$v^3(u^1w^1+ u^2w^2+u^3w^3) - w^3(u^1v^1 + u^2 v^2 + u^3v^3)$

$=(v^1(u.w),v^2(u.w),v^3(u.w))-(w^1(u.v),w^2(u.w),w^3(u.w)) =(u.w)v-(u.v)w$

(2) Any vector space V has a Lie bracket defined by $[x, y] = 0$ for all $x, y \in V$.
This is the abelian Lie algebra structure on V . In particular, the field F
may be regarded as a 1-dimensional abelian Lie algebra.

Proposition 2.12 : [10]

The Jacobi identity holds for the cross product of vectors in R^3 .

Proof. Using the above proposition,

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$

$= (x \cdot z)y - (x \cdot y)z + (y \cdot x)z - (y \cdot z)x + (z \cdot y)x - (z \cdot x)y$

$= (x \cdot z)y - (z \cdot x)y + (y \cdot x)z - (x \cdot y)z + (z \cdot y)x - (y \cdot z)x$

$= 0 + 0 + 0 = 0$

(3) Suppose that V is a finite-dimensional vector space over F. Write gl(V) for the set of all linear maps from V to V . This is again a vector space over F, and it becomes a Lie algebra, known as the general linear algebra, if we define the Lie bracket $[-,-]$ by $[x, y] := x \circ y - y \circ x$ for $x, y \in gl(V)$, where $\circ$ denotes the composition of maps.

Proposition 2.13 :[10]

Let V be a finite-dimensional vector space over F and let gl(V) be the set of all linear maps from V to V . We define a Lie bracket on this space by

$[x, y] = x \circ y - y \circ x$ where $\circ$ denotes map composition. We claim that the Jacobi identity holds for this bracket operator

Proof.

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$

$= (x \circ y \circ z - z \circ y \circ x) + (y \circ z \circ x - x \circ z \circ y) + (z \circ x \circ y - y \circ x \circ z)$

$= (x \circ y \circ z - x \circ z \circ y) + (y \circ z \circ x - y \circ x \circ z) + (z \circ x \circ y - z \circ y \circ x)$

$= x \circ [y, z] + y \circ [z, x] + z \circ [x, y]$ $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$

$= (x \circ y \circ z - y \circ x \circ z) + (z \circ x \circ y - x \circ z \circ y) + (y \circ z \circ x - z \circ y \circ x)$

$= [x, y] \circ z + [z, x] \circ y + [y, z] \circ x$

Thus we reach $x \circ [y, z] + y \circ [z, x] + z \circ [x, y]$

$= [x, y] \circ z + [z, x] \circ y + [y, z] \circ x$ Now we can subtract to have one side equal zero,

$0 = x \circ [y, z] - [y, z] \circ x + y \circ [z, x] - [z, x] \circ y + z \circ [x, y] - [x, y] \circ z$

$= [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ which is precisely the Jacobi identity

Now we show that the associative algebra with (A,+,.) with (A,+,[ ]) where $[x,y]=xy-yx$ become lie algebra:

1- $[x,x]=x.x-x.x=x^2-x^2=0$

2-$[x,[y,z]]=[[x,y],z]+[y,[x,z]]$

RHS$=[x,[y,z]=[x,(yz-zy)] =x(yz-zy)-(yz-zy)x$

$=x(yz)-x(zy)-(yz)x+(zy)x$

LHS$[[x,y],z]+[y,[x,z] =[(xy-yx),z]+[y,(xz-zx)]$

$=(xy-yx).z-z(xy-yx)+y(xz-zx)-(xz-zx)y$

$=(xy)z-(yx)z-z(xy)+z(yx)+y(xz)-y(zx)-(xz)y+(zx)y$

$=(xy)z+z(yx)-y(zx)-(xz)y$

Thus $[x,[y,z]]=[[x,y],z]+[y,[x,z]]$. ■

Remark 2.14: [10]

Let L be a Lie algebra. Show that the Lie bracket is associative, that is,

$[x, [y, z]] = [[x, y], z]$ for all $x, y, z \in L$, if and only if for all $a, b \in L$ the commutator $[a, b]$ lies in $Z(L)$.

If A is an associative algebra over F, then we define a new bilinear operation $[-,-]$ on A by $[a, b] = ab - ba$ for all $a, b \in A$.

Then A together with $[-,-]$ is a Lie algebra; this is not hard to prove. The Lie algebras gl(V) and gl(n, F) are special cases of this construction.

Quotient Algebras 2.15:[8]

If I is an ideal of the Lie algebra L, then I is in particular a subspace of L, and so we may consider the cosets $z + I = \{z + x : x \in I\}$ for $z \in L$ and the quotient vector space $L/I = \{z + I : z \in L\}$.

Now we define that a Lie bracket on L/I by $[w + I, z + I] = [w, z] + I$ for $w, z \in L$. Here the bracket on the right-hand side is the Lie bracket in L. To be sure that the Lie bracket on L/I is well-defined, we must check that $[w, z] + I$ depends only on the cosets containing w and z and not on the particular coset representatives w and z. Suppose $w + I = w' + I$ and $z + I = z' + I$. Then $w - w' \in I$ and $z - z' \in I$. By bilinearity of the Lie bracket in L,

$[w', z'] = [w' + (w - w'), z' + (z - z')]$

$= [w, z] + [w - w', z'] + [w', z - z'] + [w - w', z - z'],$

where the final three summands all belong to I.

Therefore $[w' + I, z' + I] = [w, z] + I$, as we needed. It now follows from part (i) of the exercise below that L/I is a Lie algebra. It is called the quotient or factor algebra of L by I.

Proposition 2.16: [10]

Let L be a Lie algebra over F, and let I be an ideal of L. The the bracket on L/I satisfies the Jacobi identity.

Proof. Let $u + I, v + I, w + I \in L/I$. Then

$[u + I, [v + I, w + I]] + [v + I, [w + I, u + I]] + [w + I, [u + I, v + I]] = [u + I, [v, w] + I] + [v + I, [w, u] + I] + [w + I, [u, v] + I] =([u,[v,w]]+I)+([v,[w,u]]+I)+([w,[u,v]]+I) =([u,[v,w]]+[v,[w,u]]+[w,[u,v]])+I =0+I$ Where $0+I$ is the additive identity of L/I. ■

Now we show that that the linear transformation $\pi : L \to L/I$ which takes an element $z \in L$ to its coset $z + I$ is a homomorphism of Lie algebras.

Proposition 2.17:[10]

Let I be an ideal of Lie algebra L over F. Define $\pi : L \to L/I$ by $\pi(z) = z + I$. Then π is a Lie algebra homomorphism.

Proof. First we show that π is a linear map. Let $a \in F$ and $u, v \in L$

$\pi(au + v) = (au + v) + I = (au + I) + (v + I) = a(u + I) + (v + I) = a\pi(u) + \pi(v)$ so π is linear. Now we show that π preserves the bracket.

$\pi([u, v]) = [u, v] + I = [u + I, v + I] = [\pi(u), \pi(v)]$. ■

Thus π is a Lie algebra homomorphism.

Definition 2.18: [2]

A K-algebra A is called semisimple if A is semisimple as an A module. We have already seen some semisimple algebras or it is a direct some of simple algebras.

Example 2.19:

Every matrix algebra $M_n(K)$ is a semisimple algebra.

Definition 2.20: [7]

An idempotent element or simply idempotent of a ring is an element a such that $a^2 = a$.[1] That is, the element is idempotent under the ring's multiplication. Inductively then, one can also conclude that $a = a^2 = a^3 = a^4 = \ldots = a^n$ for any positive integer n. For example, an idempotent element of a matrix ring is precisely an idempotent matrix.

We can check this for the integers mod 6, R = Z/6Z. Since 6 has two prime factors (2 and 3) it should have 2^2 idempotents.

$0^2 \equiv 0 \equiv 0 \pmod 6$, $1^2 \equiv 1 \equiv 1 \pmod 6$ and $2^2 \equiv 4 \equiv 4 \pmod 6$

$3^2 \equiv 9 \equiv 3 \pmod 6$, $4^2 \equiv 16 \equiv 4 \pmod 6$ and $5^2 \equiv 25 \equiv 1 \pmod 6$

From these computations, 0, 1, 3, and 4 are idempotents of this ring, while 2 and 5 are not. This also demonstrates the decomposition properties described below: because $3 + 4 = 1 \pmod 6$, there is a ring decomposition $3Z/6Z \oplus 4Z/6Z$. In 3Z/6Z the identity is 3+6Z and in 4Z/6Z the identity is 4+6Z

Definition 2.21: [8]

If L1 and L2 are Lie algebras over a field F, then we say that a map $\phi : L1 \rightarrow L2$ is a homomorphism if ϕ is a linear map and $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L1$.

Notice that in this equation the first Lie bracket is taken in L1 and the second Lie bracket is taken in L2. We say that ϕ is an isomorphism if ϕ is also bijective.

An extremely important homomorphism is the adjoint homomorphism . If L is a Lie algebra, we define ad : $L \to gl(L)$ by $(\mathrm{ad}\,x)(y) := [x, y]$ for $x, y \in L$. It follows from the bilinearity of the Lie bracket that the map ad x is linear for each $x \in L$. For the same reason, the map $x \_\!\!\to \mathrm{ad}\,x$ is itself linear. So to show that ad is a homomorphism, all we need to check is that $\mathrm{ad}([x, y]) = \mathrm{ad}\,x \circ \mathrm{ad}\,y - \mathrm{ad}\,y \circ \mathrm{ad}\,x$ for all $x, y \in L$ this turns out to be equivalent to the Jacobi identity. The kernel of ad is the centre of L.

Example 2.22: [10]

Show that if $\phi : L_1 \to L_2$ is a homomorphism, then the kernel of ϕ, $\ker \phi$, is an ideal of L_1, and the image of ϕ, $\mathrm{im}\phi$, is a Lie subalgebra of L_2.

Proposition 2.23: [10]

Let L_1, L_2 be Lie algebras and let $\varphi : L_1 \to L_2$ be a homomorphism. Then $\ker \varphi$ is an ideal of L_1.

Proof: We need to show that for $x \in L_1$, $y \in \ker \varphi = \{v \in L1 : \varphi(v) = 0\}$, we have $[x, y] \in \ker \varphi$. Let $x \in L_1$, $y \in \ker \varphi$. Then

$\varphi([x, y]) = [\varphi(x), \varphi(y)] = [\varphi(x), 0] = 0.$ ■

Proposition 2.24: [10]

Let L be a Lie algebra such that the bracket is associative. Then for $x, y \in L$, $[x, y] \in Z(L)$.

Proof. Let $x, y, z \in L$. We need to show that $[[x, y], z] = 0$.

Using anti-communitivity, linearity, and associativity we get

$[z, [x, y]] = -[[x, y], z] = -[-[y, x], z] = [[y, x], z]$

$= [y, [x, z]] = [y, -[z, x]] = -[y, [z, x]]$

Then using the Jacobi identity and substituting $-[y, [z, x]]$ for $[z, [x, y]]$ $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ $[x, [y, z]] + [y, [z, x]] - [y, [z, x]] = 0$

$[x, [y, z]]=0$ and $[[x, y], z] = 0.$ ■

Theorem 2.25:(Isomorphism theorems) :[8]

(a) Let $\phi : L1 \to L2$ be a homomorphism of Lie algebras. Then $\ker\phi$ is an ideal of L1 and im ϕ is a subalgebra of L2, and

$L1/\ker\phi \cong \mathrm{im}\,\phi$.

(b) If I and J are ideals of a Lie algebra, then $(I + J)/J \cong I/(I \cap J)$.

(c) Suppose that I and J are ideals of a Lie algebra L such that $I \subseteq J$.

Then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.

Definition2.26 : [2]

Let A and B be K-algebras. A map $\varphi : A \to B$ is a K-algebra homomorphism (or homomorphism of K-algebras) if

(i) φ is a K-linear map of vector spaces,

(ii) $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$,

(iii) $\varphi(1_A) = 1_B$.

The map $\varphi : A \to B$ is a K-algebra isomorphism if it is a K-algebra homomorphism and is in addition bijective. If so, then the K-algebras A and B are said to be isomorphic, and one writes $A \cong B$. Note that the inverse of an algebra isomorphism is also an algebra isomorphism.

Remark 2.27:

(1) To check condition (ii) of Definition 1.22, it suffices to take for a, b any two elements in some fixed basis. Then it follows for arbitrary elements of A as long as φ is K-linear.

(2) Note that the definition of an algebra homomorphism requires more than just being a homomorphism of the underlying rings. Indeed, a ring homomorphism between K-algebras is in general not a K-algebra homomorphism.

Definition 2.28: [8]

Let A be an algebra over a field F. A derivation of A is an F-linear map $D : A \to A$ such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$.

Remark 2.29:

Let DerA be the set of derivations of A. This set is closed under addition and scalar multiplication and contains the zero map. Hence DerA is a vector subspace of gl(A). Moreover, DerA is a Lie subalgebra of gl(A), for by part (i) of the following exercise, if D and E are derivations then so is [D,E].

Theorem: [10]

Let D and E be derivations of an algebra A. Then

$[D,E] = D \circ E - E \circ D$ is also a derivation of A

Proof. We need to show that

$[D, E](xy) = x[D, E](y) + [D, E](x)y$. First we compute

$D \circ E(xy)$ and $E \circ D(xy)$. $D \circ E(xy)$

$= D(xE(y) + E(x)y) = D(xE(y)) + D(E(x)y)$

$= xD \circ E(y) + D(x)E(y) + D \circ E(x)y + E(x)D(y)$ $E \circ D(xy)$

$= xE \circ D(y) + E(x)D(y) + D(x)E(y) + E \circ D(x)y$

Now that we've done that we can easily compute

$[D, E](xy)$. $[D, E](xy) = (D \circ E - E \circ D)(xy)$

$= D \circ E(xy) - E \circ D(xy)$

$= xD \circ E(y) + D \circ E(x)y - xE \circ D(y) - E \circ D(x)y$

$= x(D \circ E(y) - E \circ D(y)) + (D \circ E(x) - E \circ D(x))y$

$= x[D, E](y) - [D, E](x)y$. ■

Example 2.31 :[8]

(1) Let A =R^∞ R be the vector space of all infinitely differentiable functions $R \to R$. For $f, g \in A$, we define the product fg by pointwise multiplication: $(fg)(x) = f(x)g(x)$. With this definition, A is an associative algebra. The usual derivative, $Df = f'$, is a derivation of A since by the product rule $D(fg) = (fg)' = f'g + fg' = (Df)g + f(Dg)$.

(2) Let L be a Lie algebra and let $x \in L$. The map $adx : L \to L$ is a derivation of L since by the Jacobi identity. Then

$(ad\, x)[y, z] = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [(adx)y, z] + [y, (ad\, x)z]$

Peirce decomposition:

For an Algebra A containing an idempotent e there exist left, right and two-sided Peirce decompositions, which are defined by

A=Ae + A (1−e) and A=eA + (1−e) A

A=e Ae + eA (1−e) + (1−e) Ae + (1−e) A(1−e)

Respectively. If has no identity, then one puts, by definition,

A (1−e) = { x − x e: x ∈ A}and (1−e)Ae { x − x e: x ∈ A},

(1−e) A (1−e) = { x − e x − x e +e x e: x ∈ A}.

The sets (1−e) A and eA (1−e) are defined analogously. Therefore, in a two-sided Peirce decomposition an element x ∈ A can be represented as

x=ee + (e x − e xe) + (e x − e x e) + (x −e x − x e + e x e),

In a left decomposition as x= xe + (x − xe)

And in right decomposition as x=ex+ (x − xe)

There is also Peirce decomposition with respect to an orthogonal system of idempotent $\{e_1 \dots e_n\}$where $\sum_i e_i = 1$: A=$\sum_{ij} A_j$ e e_i.

Chapter Three

Classification of associative algebras

In this chapter we review and highlight the main theorems about the classification of associative algebra. We show that every simple K -algebra is isomorphic to an algebra of the form $M_n(D)$, where D is a division algebra and every semisimple algebra is isomorphic to a direct product of matrix algebras over division algebras. We illustrate this algebraic concepts by solving some examples about them.

Definition 3.1:

A right module over a K-algebra A, or a right A-module, is a vector space M over the field K whose elements can be multiplied by the elements of the algebra, i. e. to every pair (m, a), m $\in$M, a $\in$ A, there corresponds a uniquely determined element ma $\in$ M such that the following axioms are satisfied:

1) $(m_l + m_2)a = m_la + m_2a$;

2) $m(a_l + a_2) = ma_l + ma_2$;

3) (am)a = m(aa) = a(ma) where a $\in$K;

4) m(ab) = (ma)b;

5) $m_1 = m$.

Example 3.2: R^2 is $M_{2\times2}(R)$- module via the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} av + bw \\ cv + dw \end{pmatrix}.$$

It is not hard to check that this action satisfies the above five conditions . We can generalize this action and we get that R^n is $M_{n\times n}(R)$-module.

Proposition 3.3:[10]

Let M be a module over an algebra $A = A_1 \times A_2 \times \ldots \times A_s$ and $1 = e_l + e_2 + ... + e_s$ be the corresponding central decomposition of the identity of A. Then $M = Me_1 \oplus ... \oplus Me_s$, where $Me_i$are modules over A_i.

Remark 3.4: [11]

There is a bijective correspondence between the decompositions of an A-module M into a direct sum of sub modules and the decompositions of the identity of the algebra E $=E_A$ (M).We have already attached to every decomposition of the module M a decomposition of the identity of the algebra E. Now, let $1 = e_l + e_2 + ... + e_s$ be a decomposition of the identity of the algebra E. Put $M_i = I_m e_i$. Then, for every element $m \in M$, $m = (e_l + e_2 + ... + e_s)m = e_l m + e_2 m + ... + e_s m$, where $e_i m \in$ Mi. If $m = m_l + m_2 + ... + m_s$ is a decomposition of the element m in the form of the sum of the elements mi $\in$ Mi , then $m_i = e_i x_i$ for some $x_i \in M$. Then, $e_i m=\sum_{j=1}^{s} e_i m_j = m_i$

Remark 3.5:[11]

A module M is indecomposable if and only if there are no non-trivial (i. e. different from 0 and 1) idempotents in the algebra E_A (M). If e is a non-trivial idempotent, then f = 1 - e is also a non-trivial idempotent which is orthogonal to e, and thus 1 = e + f is a decomposition of the identity.

Theorem 3.6: [11]

There is a bijective correspondence between

1) the decompositions of the algebra A into a direct product of algebras;

2) the decompositions of A into a direct sum of ideals.

Remark 3.7: [11]

There is a bijective correspondence between the direct product decompositions of the algebra A and those e_i of decomposition of the identity $1 = e_l + e_2 + ... + e_s$ because $e_i A e_j = 0$ for $i \neq j$.

Theorem 3.8 (Schur): [11]

If U and V are simple A-modules, then every nonzero homomorphism $f : U \to V$ is an isomorphism.

Definition 3.9: [11]

A module M is called semisimple if it is isomorphic to a direct sum of simple modules.

Proposition 3.10:[11]

The following conditions are equivalent:

1) the module M is semisimple

2)$M=\sum_{i=1}^{m}$ Miwhere Mi are simple submodules of M.

3) every submodule $N\subset M$ has a complement.

4) every simple submodule $N\subset M$ has a complement

Corollary 3.11: [11]

Every submodule and every factor module of a semisimple module is semisimple.

Lemma 3.12 (Brauer): [11]

If I is a minimal right ideal of an algebra A, then either $I^2 = 0$, or $I = eA$, where e is an idempotent

Theorem 3.13: [11] The following conditions for an algebra A are equivalent:

1) A is semisimple;

2) every right ideal of A is of the form eA, where e is an idempotent;

3) every non-zero ideal of A contains a non-zero idempotent;

4) A has no non-zero nilpotent ideals;

5) A has no non-zero nilpotent right ideals.

Theorem 3.14: [11]

1) A commutative algebra is semisimple if and only if it contains no nilpotent elements

2) The center of a semisimple algebra is semisimple.

3) Every vector space over a division algebra D is isomorphic to nD (direct sum of n copies of the regular module). The number n is determined uniquely.

4) The module V over the algebra $A = M_n(D)$ is simple and the algebra $M_n(D)$ is semisimple.

Proposition 3.15: [11]

1) Every module over the algebra A = $M_n(D)$ is semisimple.
2) Every simple A-module is isomorphic to V, and the regular A-module is isomorphic to nV.

Theorem 3.16: [11]

1) A commutative semisimple algebra is isomorphic to a direct product of fields. Conversely, a direct product of fields is a semisimple algebra.

2) If K is algebraically closed, then every commutative semi simple K -algebra is isomorphic to k^n.

Theorem 3.17: [11] (Wedderburn-Artin).

Every semisimple algebra is isomorphic to a direct product of matrix algebras over division algebras. Moreover, a direct product of matrix algebras over division algebras is a semisimple algebra.

Theorem 3.18: [11] (Molien).

If K is algebraically closed, then every semisimpie K -algebra is isomorphic to the algebra of the form $M_{n1}(K) \times M_{n2}(K) \times \ldots \times M_{ns}(K)$.

Theorem 3.19: [11]

1)Every simple K -algebra is isomorphic to an algebra of the form $M_n(D)$, where D is a division algebra.

2) Every simple algebra over an algebraically closed field K is isomorphic to M_n (K) for some n.

Proposition 3.20: [2]

Let K be a field. Then we have the following:

(a) Every 1-dimensional K-algebra is isomorphic to K.

(b) Every 2-dimensional K-algebra is commutative.

(c) Up to isomorphism, there are precisely three 2-dimensional algebras over R. Any 2-dimensional algebra over R is isomorphic to precisely one of

$R[X]/(X^2)$ or $R[X]/(X^2 - 1)$ and $R[X]/(X^2 + 1)$.

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Salahaddin University-Erbil

جەبری یەکتربەستن کە دانەی ئایدێمپۆتێنتیان تێدایە

پڕۆژەی دەرچوون

پێشکەش بە بەشی ماتماتیك کراوە،وەك بەشێك لە پێداویستیەکانی بەدەستهێنانی بڕوانامەی بەکالۆریۆس لە زانستی ماتماتیك

ئامادەکراوە لەلایەن:

هێلین لوقمان برایم

بەسەرپەرشتی:

د.هۆگر محمد یاسین

٢٠٢٣-نیسان

پوخته

لەم پرۆژەدا لێکۆلینەوە لە و ئەلجێبرایانە دەکەنین کەخاسیەتی یەکتربەستنیان هەیە و دانەی بێهێزیان تێدایە، لەسەرەتا باسی فەزای ڤێکتەرو ئەلجێبراو و لی ئەلجێبرا دەکەین ئەو پێناسەو ئەنجامانەیان باس دەکەین کە پێویستمانە لە لێکۆلینەوەکە. پاشان ئەو ئەلجێبراینە پۆلێن دەکەین کە سیفەتی یەکتربەستنیان هەیە و وەپلەبەندی پێرس پاس دەکەین.