Chapter 1

Mathematical Logic

Foundation of Mathematics, First Stage- Mathematics Department

Lecturer: Dr Hogir Mohammed Yaseen, 2022-2023

To be able to understand mathematics and mathematical arguments, it is necessary to have a solid understanding of logic and the way in which known facts can be combined to prove new facts. In this chapter we study basic mathematical logic that consists of statements, truth-values, and logical operations (NOT, AND, OR, IF ANDONLY IF and IF... THEN) interms of truth tables. Moreover, we take a careful look at the rules of logic and the way in which mathematical arguments are constructed. Logical statements

Definition 1.1. A statement is a sentence which can be classified as true or false without ambiguity. The truth or falsity of the statement is known as the truth value. We use letters p, q, r... to denote the statements. For a sentence to be a statement, it is not necessary that we actually know whether it is true or false, but it must be clear that it is one or the other.

Ambiguity: Something that is not clear because it has more than one possible meaning.

Example 1.2. Consider the following sentences:

1. "11 is an even number" is a statement with truth value "false".
2. "Every even number greater than 2 is the sum of two primes" is a statement, whose truth value is not known...yet.
3. "2 is an even number. " is a statement with truth value "True".

Logical Connectives In studying mathematical logic we shall not be concerned with the truth value of any particular simple statement. What will be important is how the truth value of a compound statement is determined from the truth values of its simpler parts. To obtain such compound statements it is necessary to introduce the concept of a connective.

Definition 1.3. A sentential connective is a logic symbol representing an operator that combines statements into a new statement. Statements with connectives are called compound statements. Statements without connectives are known as atomic statements. The sentential connectives are "not", "and", "or", "if ...then", and "if and only if ". The respective operators for these connectives are negation, conjunction, disjunction, implication and equivalence respectively.

Definition 1.4. A truth table of a logical formula shows the conditions under which the logical formula is true and those under which is false.The truth table of a connective is an alternative way of defining a connective, since these are defined in terms of the truth value of the resulting compound statement, given the truth value of its components.

Type of connectives : Negation , Conjunction, Disjunction, Conditional and Biconditional.

Definition 1.5. The negation (denial) of a statement is another statement which has opposite meaning for the statement. If "p" is a statement, then negation of p is written as $\sim p$ and read "not p". When p is true, then $\sim p$ is false and viceversa.

Example 1.6. p: Yesterday is Monday. $\sim p$: Yesterday is not Monday.

The truth table for the negation of statement where T stands for true and F for false.

p	*~p*
T	*F*
F	*T*

Exercise 1.7. Suppose that p is a false statement.

1) What is the truth-value of the compound statement $\sim p$?

2) What is the truth-value of the compound statement $\sim (\sim p)$?

Definition 1.8. Let p and q be two statements. The statement p∧q is called the conjunction of p and q, and read as "p and q". p∧q has true value of both p and q are true, otherwise it is false. The truth table for the conjunction of two statements:

	p	*q*	*p ∧ q*
1	*T*	*T*	*T*
2	*T*	*F*	*F*
3	*F*	*T*	*F*
4	*F*	*F*	*F*

Example 1.9. Let p and q be two statements as follows:

1. p : 2 is an odd number. False

2. q : 2 is a prime number. True.

3. p∧q: 2 is an odd number and 2 is a prime number. False. (case 3)

Definition 1.10. Let p and q be two statements. The statement p∨q is called the Disjunction of p and q, and read as "p or q". p∨q is true when at least one of the two statements is true, and is false when both are false. The truth table for the Disjunction of two statements:

	p	*q*	*p ∨ q*
1	T	T	T
2	T	F	T
3	F	T	T
4	F	F	F

Note that the inclusive disjunction doesn't complete the list of disjunctions used in everyday life. In fact, we also have the exclusive disjunction, which is true when either p or q is true, but not when both are true. In logic the only use for the connective or is for the inclusive meaning.

Definition 1.11. Let p and q be two statements. The statement $p \to q$ is called conditional and it is read as "If p then q". $p \to q$ has false value if p is true and q is false, otherwise it is true. In this implication $p \to q$, p is called hypothesis or premise and q is conclusion or consequence. The truth table for conditional

	P	Q	p→ q
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

Exercise 1.12. Suppose that p is a false statement, and q is a true statement. What is the truth-value of the compound statement $(\sim p)$→q? What is the truth-value of the compound statement p→ (∼ q)? What is the truth-value of the compound statement p→q? What is the truth-value of the compound statement ∼(p→ (∼q))?

Exercise 1.13. Suppose that the compound statement p→ q is a true statement. In order for p to be true, what must the truth-value of q be? Suppose that the compound statement p→q is a true statement. Which truth value of p assures us that q is true?

Exercise 1.14. Suppose that the compound statement p→q is false. What are the truth-values of p and of q?

Exercise 1.15. Suppose that p is a false statement, and q is a statement whose truth-value is presently unknown. Suppose that the compound statement $(\sim p)$→q is true. What is the truth-value of the statement q? What is the truth-value of the statement q, if you are given that $(\sim p)$→q is false? What is the truth-value of the compound statement ∼(q→ (∼q))?

Definition 1.16. Let p and q be two statements. The statementp↔q is called bi conditional and it is read as "p if and only if q" or "p iff q". p↔q is true if both p and q are true or false. p↔q is false if p and q are not equal. The truth table for Bi conditional

	P	Q	$p \leftrightarrow q$
1	T	T	T
2	T	F	F
3	F	T	F
4	F	F	T

Example 1.17. Let p and q be two statements as follows: p: a is a prime number and q : a^2 is a prime number. Then

1. p→q: if a is a prime number, then a^2 is a prime number.

2. p↔q :a is a prime number, if and only if a^2 is a prime number.

Exercise 1.18. Suppose that p is a false statement, and q is a true statement. What is the truth-value of the compound statement $(\sim p)$↔q? What is the truth-value of the compound statement p↔(∼q)? What is the truth-value of the compound statement p↔q? What is the truth-value of the compound statement ∼(p↔ (∼q))?

Remark 1.19. The number of cases of truth value of the true table = 2^n where n is the number of simple statements.

p
T
F

p	q
T	T
T	F
F	T
F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Problem 1.20. Construct the truth table for the compound statement $(\sim(p\vee q))\wedge(p\vee r)$.

Solution. To find $(\sim(p\vee q))\wedge(p\vee r)$ we need to find $(p\vee q))$, $\sim(p\vee q)$ and$(p\vee r)$:

p	q	r	$p\vee q$	$\sim(p\vee q)$	$(p\vee r)$	$(\sim(p\vee q))\wedge(p\vee r)$
T	T	T	T	F	T	F
T	T	F	T	F	T	F
T	F	T	T	F	T	F
T	F	F	T	F	T	F
F	T	T	T	F	T	F
F	T	F	T	F	F	F
F	F	T	F	T	T	T
F	F	F	F	T	F	F

Problem 1.21. Construct the truth table for the compound statement $((\sim p)\vee q)\leftrightarrow(p\to q)$.

Solution. First we need to find $(\sim p)\vee q$ and $p\to q$:

p	q	$\sim p$	$\sim q$	$(\sim p)\vee q$	$p\to q$	$((\sim p)\vee q)\leftrightarrow(p\to q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Definition 1.22. If two or more statements p and q have the same truth value in each logical possibilities,then p is said to be logical equivalent to q and denoted by $p\equiv q$. If p and q have not the same truth value in at least one logical possibility we say p is not logical equivalent to q and denoted by $p\not\equiv q$.

Problem 1.23. Show that $\sim(\sim p\vee q)\equiv p\wedge\sim q$.

Solution. First find the truth table of the statement $\sim(\sim p\vee q)$ and $p\wedge\sim q$. To find $\sim(\sim p\vee q)$, we need to find $(\sim p\vee q)$

p	q	$\sim p$	$(\sim p\vee q)$	$\sim(\sim p\vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

Now we are going to find $p\wedge\sim q$:

p	q	$\sim q$	$p\wedge\sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Since $\sim(\sim p \vee$q) and $p \wedge \sim q$ have the same truth value in each logical possibilities, then $\sim(\sim p \vee$q)$\equiv p \wedge \sim q$.

Problem 1.24. Show that $(\sim p) \vee q \equiv$ p$\rightarrow$q.

Solution. We need to find $\sim p$, $(\sim p) \vee q$ and p$\rightarrow$q

p	q	$\sim p$	$(\sim p) \vee q$	p$\rightarrow$q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since $\sim(\sim p \vee$q) and $p \wedge \sim q$ have the same truth value in each logical possibilities, then $\sim(\sim p \vee$q)$\equiv p \wedge \sim q$.

Definition 1.25. A statement is said to be a tautology if it has only true value.

Example 1.26. $p \vee \sim p$ is a tautology.

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Definition 1.27. A statement is said to be contradiction if it has only false value.

Example 1.28. $p \wedge \sim p$ is a contradiction

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

Remark 1.29. The negation of tautology is a contradiction and the negation of contradiction is a tautology. For example, $p \vee \sim p$ is a tautology then $\sim(p \vee \sim p)$ is a contradiction and $p \wedge \sim p$ is a contradiction then$\sim(p \wedge \sim p)$ is a tautology.

1. Verify(show) that the statement $p \vee \sim(p \wedge$q) is tautology.

2. Verify that the statement $(p \wedge$q)$\wedge \sim(p \vee$q) is contradiction.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$P \vee \sim (P \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercise 1.30. Construct the truth table for the compound statement $((\sim p) \vee$q)$\leftrightarrow$(p$\rightarrow$q). What does the truth table tell you about the two statements$(\sim p) \vee$q and p$\rightarrow$q?

Definition 1.31. If two or more statementsp and q have the same truth value in each logical possibilities,then p is said to be logical equivalent to q and denoted by p$\equiv$q. If p and q have not the same truth value in at least one logical possibility we say p is not logical equivalent to q and denoted by $p \not\equiv$Q.

Remark 1.32. Let p and q be two statements then the below truth table show the following:

1) $p \equiv p$
2) $p \equiv p \vee p$
3) $p \equiv p \wedge p$
4. $(p \to q) \equiv (\sim p \vee q)$ column 7 and column 8
5. $(p \to q) \equiv (\sim q \to \sim p)$ column 8 and column 9
6. $(p \leftrightarrow q) \equiv (q \leftrightarrow p)$ column 12 and column 13
7. $(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$ column 12 and column 11
8. $(p \to q) \not\equiv (q \to p)$ column 8 and column 10

Column / Row	1	2	3	4	5	6	7	8	9	10	11	12	13
	p	**q**	**$p \vee p$**	**$p \wedge p$**	**$\sim p$**	**$\sim q$**	**$\sim p \vee q$**	**$p \to q$**	**$\sim q \to \sim p$**	**$q \to p$**	**$(p \to q) \wedge (q \to p)$**	**$p \leftrightarrow q$**	**$q \leftrightarrow p$**
1	T	T	T	T	F	F	T	T	T	T	T	T	T
2	T	F	T	F	F	T	F	F	F	T	F	F	F
3	F	T	T	F	T	F	T	T	T	F	F	F	F
4	F	F	F	F	T	T	T	T	T	T	T	T	T

Exercise 1.33. Let p, q and r be any three statements if p≡q, then

1. $\sim p \equiv \sim q$.
2. $p \wedge r \equiv q \wedge r$.
3. $r \wedge p \equiv r \wedge q$
4. $p \vee r \equiv q \vee r$.
5. $r \vee p \equiv r \vee q$
6. $p \to r \equiv q \to r$.
7. $r \to p \equiv r \to q$
8. $p \leftrightarrow r \equiv q \leftrightarrow r$.
9. $r \leftrightarrow p \equiv r \leftrightarrow q$.

A1: Suppose that p≡q to prove∼p≡∼q. We have two cases.

Case 1: Let ∼p be a true statement, then p is false statement [by definition of ∼] Since ≡q , then q is also false statement, so that∼q is true [by definition of ∼].

Case 2: ∼p is false statement, then p is true statement [by definition of ∼] Since p≡q, then q is also true statement, so that∼q is false statement[by definition of ∼] Therefore, ∼p≡∼q [by the definition of ≡.

Laws of the Algebra of statements:

1. Idempotent Laws (i)$p \vee p \equiv p$(ii)$p \wedge p \equiv p$
2. Associative Laws (i)(p∨q)∨r≡p∨(q∨r) (ii) (p∧q)∧r≡p∧(q∧r)
3. Commutative Laws (i)p∨q≡q∨p(ii)p∧q≡q∧p
4. De-Morgan's Laws (i)∼(p∨q)≡∼p∧∼q(ii)∼(p∧q)≡∼p∨∼q
5. Distributive Laws (i)p∨(q∧r)≡(p∨q)∧(p∨r)(ii)p∧(q∨r)≡(p∧q)∨(p∧r)
6. Complement laws (i) p∨∼p≡T (ii) p∧∼p≡F (iii)∼∼p≡p(iv)∼T≡F,∼F≡T
7. Identity laws (i) p∨F≡p (ii) p∧T≡p (iii) p∧F≡F (iv) p∨T≡T.

Problem 1.34. Prove that (p∨q)∧∼p≡∼p∧q

Solution 1.35. L.H.S=(p∨q)∧∼p≡∼p∧(p∨q) (by commutative Laws)

≡(∼p∧p)∨(∼p∧q) (by distributive Laws)

≡F∨(∼p∧q) (by complement laws) ≡∼p∧q (by Identity laws)

Problem 1.36. Prove that [q∨(p∧∼q)]∨(∼p∧q) is a tautology

Solution 1.37. L.H.S=[q∨(p∧∼q)]∨(∼p∧∼q)
≡[(q∨p)∧(q∨∼q)]∨(∼p∧∼q)(by distributive law)
≡[(q∨p)∧T]∨(∼p∧∼q) (by complement law) ≡(q∨p)∨(∼p∧∼q) (by Identity law)
≡(p∨q)∨(∼p∧∼q) (by commutative law) ≡(p∨q)∨(∼(p∨q)) (by De-Morgan's Law)
≡T (by Complement law) This means that [q∨(p∧∼q)]∨(∼p∧∼q) is a tautology.

Exercise 1.38. Let p, q and r be three statements then prove the following:
p→(q→r)≡(p∧q)→r p→(q→r)≡∼p∨(q→r) p→(q→r)
≡p→(∼q∨r). [∼(p∨q)]⟷[(∼ p)∧(∼q)] is tautology.
Therefore ∼[∼(p∧q)⟷((∼ p)∨(∼q))] is contradiction.

Logical Implication:

Definition 1.39. Let p and q be two statements (simples or compounds) if the condition statement p→q is tautology, then is called an implication and denoted by p⇒q. Definition: Let p and q be two statements (simples or compounds) if the Bi condition statement p↔q is tautology, then is called (p equivalent to q) and denoted by p⇔q. Remark:p⇔q if and only if p≡q or we say that If p↔q is a tautology, then p≡q. Theorem: For any two statements p and q , p⇒q if and only if ∼p∨q is tautology. $p$roof: H.W.

Problem 1.40. Prove that p→q⟺∼q→∼p $p$roof: p→q⟺(∼p∨q) [by p→q≡∼p∨q] ⟺ q∨∼p [by commutative laws] ⟺∼(∼q)∨∼p [by Complement laws] ⟺∼q→∼p [by p→q≡∼p∨q]

Definition 1.41. Let A be any set and let p(x) be a statement of a variable x , then the statement p(x) in a variable x defined on the set A is called and open sentence if p(a) is a true or false statement for all $a \in A$. The set of solution is the set of all elements a in the set A if p(a) is a true. If we denote the set of solution by S.S, then $S.S = a \in A; p(a) istrue$.

Example 1.42. Let $A = 1, 5, 7$ and $p(x) : 3 + x > 6p(1) : 3 + 1 > 6$ false statement. $p(5) : 3 + 5 > 6$ true statement. $p(7) : 3 + 7 > 6$ true statement. Then p(x) is an open sentence in a variable x defined on the set A.

Remark 1.43. In above example if $B = 1, 2, 3, a$. Then p(x) is not open sentence in a variable x defined on the set B.

Definition 1.44. Let p(x) be an open sentence in a variable x defined on the set A, then the statement there exists x,x in A,p(x) is called existential quantifier and denoted by ∃ x,x∈A,p(x).

Example 1.45. Let $A = 5, 10, 15$ and p(x): x is prime number. p(x) is an open sentence inx defined on A. Since, $5 \in A$ and p(5) is true, then the statement (∃ x,x∈A,p(x) is true) (∃ 5,5∈A,5 is prime number) is an existential quantifier. Remark: the statement (∃ x,x∈A,p(x) is true if the set of solutionis non empty set. That is S.S={a∈A;p(a) is true }≠∅.

Definition 1.46. Let p(x) be an open sentence in a variable x defined on the set A, then the statement for all x,x in A,p(x) is called universal quantifier and denoted by ∀ x,x∈A,p(x). Remark: the statement (∀ x,x∈A,p(x) is true if and only if $S.S = a \in A; p(a) istrue$.

Example 1.47. Let $A = 1, 3, 5, 7$ and p(x) : x is odd number Then the statement ∀ x,x∈A,p(x)is true is a universal quantifier. Remark: ∀ x,p(x) is a shorthand of ∀ x,x∈A,p(x).

Theorem 1.48. *Let p(x) be anopen sentence in x defined on the setA, then*

1. $\sim (\forall\, x, p(x)) \equiv \exists\, x, \sim p(x)$
2. $\sim (\exists\, x, p(x)) \equiv \forall x, \sim p(x)$
3. $\forall\, x, p(x) \equiv \sim (\exists\, x, \sim p(x))$
4. $\exists\, x, p(x) \equiv \sim (\forall x, \sim p(x))$

Proof. (1) To prove this theorem we have two cases.

Case 1: Suppose that ∼(∀ x,p(x)) is true we have to prove that ∃ x,∼p(x) is true. Suppose that∼(∀ x,p(x)) is true then∀ x,p(x) is false. This means that there exists an element say b∈A such that p(b)is false, then there exists an element say b∈A such that ∼p(b)is true. This means that∃ x,∼p(x) is true.

Case 2: Suppose that∼(∀ x,p(x)) is false then∀ x,p(x) is true. This means that for all elements b∈A such that p(b)is true, then for all elements b∈A such that ∼p(b)is false. Then there exists an element b∈A such that ∼p(b)is false. This means that∃ x,∼p(x) is false. By case 1 and case 2 we can decide ∼(∀ x,p(x))≡∃ x,∼p(x). □

Exercise 1.49. Let p(x) andq(x) be two open sentence in x defined on the set A. Then prove or disprove the following:

1. $\forall\, x, (p(x) \wedge q(x)) \equiv \forall\, x, p(x) \wedge \forall\, x, q(x)$
2. $\forall x, (p(x) \vee q(x)) \equiv \forall x, p(x) \vee \forall x, q(x).$
3. $\forall x, (p(x) \rightarrow q(x)) \equiv \forall x, p(x) \rightarrow \forall x, q(x$
4. $\forall x, (p(x) \leftrightarrow q(x)) \equiv \forall x, p(x) \leftrightarrow \forall x, q(x$
5. $\exists x, (p(x) \wedge q(x)) \equiv \exists x, p(x) \wedge \exists x, q(x).$
6. $\exists x, (p(x) \vee q(x)) \equiv \exists x, p(x) \vee \exists x, q(x).$
7. $\exists x, (p(x) \rightarrow q(x)) \equiv \exists x, p(x) \rightarrow \exists x, q(x)$
8. $\exists x, (p(x) \leftrightarrow q(x)) \equiv \exists x, p(x) \leftrightarrow \exists x, q(x$

$\boldsymbol{A_1}$ ***Proof*** : We have two cases.

Case 1: Suppose that $\forall\, x, (p(x) \wedge q(x))$is true, then for each $a \in A, p(a) \wedge q(a)$is true. Then for each $a \in A, p(a)$is true and $q(a)$is true.

Then $\forall\, x, p(x) \wedge \forall\, x, q(x)$ is true.

Case 2: Suppose that $\forall\, x, (p(x) \wedge q(x))$ is false, then there exist $a \in A \ni p(a) \wedge q(a)$is false. Then there exist $a \in A \ni p(a)$is false or $q(a)$ is false.

Then $\forall\, x,\ p(x)$ is false or $\forall\, x,\ q(x)$ is false. Thus, $\forall\, x, p(x) \wedge \forall\, x, q(x)$ is false. By case1 and case 2 we get the result.

$\boldsymbol{A_2}$ **: Disprove:** Let $A = \{1,2,3,4\}$, $p(x): x$ is odd number.

Then $q(x): x$ is even number.

$\forall x, (p(x) \vee q(x))$ means that every element of the set A is odd or even.

So that $\forall x, (p(x) \vee q(x))$ is true. $\forall x, p(x) \vee \forall x, q(x)$ means that every element of the set A is odd number and every element of the set A is even number.

So that $\forall x, p(x) \vee \forall x, q(x)$ is false.

This means that L.H.S≡T but R.H.S≡F which is contradiction.

$\boldsymbol{A_3}$ **: Disprove.** Let $A = \{1,2,4\}$, $p(x): (x-1)(x-2) = 0$ and

$q(x): (x-1)(x-3) = 0$. Then

$p(1): (1-1)(2-1) = 0, p(2): (2-1)(2-2) = 0$, and

$p(4): (4-1)(4-2) \neq 0$, then $\forall x, p(x)$ is false.

$q(1): (1-1)(1-3) = 0, q(2): (2-1)(2-3) \neq 0, q(4): (4-1)(4-3) \neq 0$, then $\forall x, q(x)$ is false. Therefore, $\forall x, p(x) \rightarrow \forall x, q(x)$ is true.

$p(1) \rightarrow q(1)$ is true, $p(2) \rightarrow q(2)$ is false,

$p(3) \rightarrow q(3)$ is true, then $\forall x, (p(x) \rightarrow q(x))$ is false.

This means that L.H.S≡F but R.H.S≡T which is contradiction. Therefore, $\forall x, (p(x) \rightarrow q(x)) \not\equiv \forall x, p(x) \rightarrow \forall x, q(x)$.

A₄: Disprove. Let $A = \{1,2\}$, $p(x): x - 1 = 0$. $q(x): x - 2 = 0$.

$p(1): 1 - 1 = 0$, $p(2): 2 - 1 \neq 0$. Then $\forall x, p(x)$ is false.

$q(1): (1 - 2) \neq 0$, $q(2): 2 - 2 = 0$. Then $\forall x, q(x)$ is false.

Therefore, $\forall x, p(x) \leftrightarrow \forall x, q(x)$ is true.

$p(1) \leftrightarrow q(1)$ is false, $p(2) \leftrightarrow q(2)$ is false. Then $\forall x, p(x) \leftrightarrow q(x)$ is false.

This means that L.H.S≡F but R.H.S≡T which is contradiction.

Therefore, $\forall x, (p(x) \leftrightarrow q(x)) \not\equiv \forall x, p(x) \leftrightarrow \forall x, q(x)$.

***A*₅ : Disprove:** Let $A = \{1,2\}$, $p(x): x$ is odd number and $q(x): x$ is even number.

$\exists x, \big(p(x) \wedge q(x)\big)$ means that there exists an element of the set A is odd and even. So that $\exists x, \big(p(x) \wedge q(x)\big)$ is false.

$\exists x, p(x) \wedge \exists x, q(x)$ means that there exists an element of the set A is odd and an element of the set A is even.

So that $\exists x, p(x) \wedge \exists x, q(x)$ is true. This means that L.H.S≡F but R.H.S≡T which is contradiction. Therefore, $\exists x, (p(x) \wedge q(x)) \not\equiv \exists x, p(x) \wedge \exists x, q(x)$.

A*₆ *Proof : We have two cases.

Case 1: Suppose that $\exists x, \big(p(x) \vee q(x)\big)$ is true, then there exists $a \in A$ such that $p(a) \vee q(a)$ is true. Then there exist $a \in A$ such that $p(a)$is true or $q(a)$ is true. Then $\exists\, x,\ p(x)$ is true or $\exists\, x,\ q(x)$ is true.

Thus $\exists x, p(x) \wedge \exists\, x, q(x)$ is true.

Case 2: Suppose that $\exists\, x, \big(p(x) \vee q(x)\big)$ is false, then for each $a \in A, p(a) \vee q(a)$is false. Then for each $a \in A, p(a)$ is false and $q(a)$is false.

Then $\exists\, x, p(x)$ or $\exists\, x, q(x)$ is false.

Then $\exists\, x, p(x) \vee \exists\, x, q(x)$ is false. By case1 and case 2 we get the result.

A_7 *Proof* : Disprove. Let $A = \{1, 2\}$, $p(x): (x-1)(x-3) = 0$ and

$q(x): x - 3 = 0$. Then

$p(1): (1-1)(1-3) = 0, p(2): (2-1)(2-3) \neq 0$, then $\exists x, p(x)$ is true.

$q(1): 1 - 3 \neq 0, q(2): 2 - 3 \neq 0,\ \exists\, x, q(x)$ is false.

Therefore, $\exists x, p(x) \rightarrow \exists x, q(x)$ is false. So,

$p(1) \rightarrow q(1)$ is false, $p(2) \rightarrow q(2)$ is true, then $\exists x, (p(x) \rightarrow q(x))$ is true.

This means that L.H.S≡T but R.H.S≡F which is contradiction. Therefore, $\exists x, \big(p(x) \rightarrow q(x)\big) \not\equiv\ \exists x, p(x) \rightarrow \exists x, q(x)$.

A_8 Proof : Disprove. Let $A = \{1, 2\}, p(x): x - 1 = 0.\ q(x): x - 2 = 0$.

$p(1): (1-1) = 0, p(2): 2 - 1 \neq 0$, then $\exists x, p(x)$ is true.

$q(1): 1 - 2 \neq 0, q(2): 2 - 2 = 0,\ \exists\, x, q(x)$ is true.

Therefore, $\exists x, p(x) \leftrightarrow \exists x, q(x)$ is true.

$p(1) \leftrightarrow q(1)$ is false, $p(2) \leftrightarrow q(2)$ is false, then $\exists x, (p(x) \leftrightarrow q(x))$ is false.

This means that L.H.S≡F but R.H.S≡T which is contradiction.

Therefore, $\exists x, \big(p(x) \leftrightarrow q(x)\big) \not\equiv\ \exists x, p(x) \rightarrow \exists x, q(x)$.

Exercise 1.50. Show, by constructing its truth table, that $(\sim(p \vee q)) \leftrightarrow (\sim p) \wedge (\sim q)$ is a tautology.

Exercise 1.51. Construct the truth table for the compound statement $(q \rightarrow p) \leftrightarrow p \rightarrow q)$. What does the truth table tell you about the two statements $q \rightarrow p$ and $p \rightarrow q$?

Exercise 1.52. Construct the truth table for the compound statement $(\sim q \rightarrow \sim p) \leftrightarrow p \rightarrow q)$. What does the truth table tell you about the two statements $\sim q \rightarrow \sim p$ and $p \rightarrow q$?

Exercise 1.53. Construct the truth table for the compound statement $((p \vee q) \vee \mathrm{r}) \leftrightarrow (p \vee (q \vee \mathrm{r})))$. What does the truth table tell you about the two statements $(p \vee q) \vee \mathrm{r}$ and $p \vee (q \vee \mathrm{r})$?

Exercise 1.54. Construct the truth table for the compound statement $((p \rightarrow q) \wedge (q \rightarrow \mathrm{r})) \rightarrow (p \rightarrow \mathrm{r})$.

Exercise 1.55. Construct the truth table for the compound statement $((p \wedge q) \vee (p \wedge \mathrm{r})) \leftrightarrow (p \wedge (q \vee \mathrm{r}))$.

Chapter Two **Set Theory**

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. The notion of set is taken as "undefined", "primitive", or "basic", so we don't try to *define* what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Similar (but informal) words: collection, group, aggregate.

Description: A *set* is a collection of objects which are called the *members* or *elements* of that set. If we have a set we say that some objects *belong* (or *do not belong*) to this set, *are* (or *are not*) *in* the set. A set is any collection of objects, for example, set of numbers. The objects of a set are called the elements of the set. There are some main ways to specify a set:

a) by listing all its members (*list notation*);

Examples: $\{2, 4, 6\}$, $\{1, 2, \dots, 100\}$, $\{a, b, c, d\}$, {Hewa, Aram, Ahmed, Awat}

b) by stating a property of its elements (*predicate notation*);

Examples: General form : $\{ x \mid P(x)\}$, where P is some condition or property.

i. $\{x : x$ is a natural number and $x < 8\}$, Reading: "the set of all x such that x is a natural number and is less than 8".

ii. $\{ x \mid x$ is a letter of Kurdish alphabet$\}$

c) by defining a set of rules which generates (defines) its members (*recursive rules*).

Example: The set A of odd numbers greater than 2:

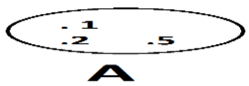
i. $3 \in A$

ii. if $x \in A$, then $x + 2 \in A$

iii. nothing else belongs to A.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules i and ii.

d) There is another way (Diagram) to show sets is called Venn diagram. For example:



This means that the set A contains three elements these are 1, 2 and 5 or $A = \{1,2,5\}$.

Remark:

1. We usually use capital lettersA, B, C, etc., to denote sets.
2. The notation $x \in A$ means x is an element of A. But $x \notin A$ means x is not an element of A. Example: $1 \in \{1, 2, 6\}$, $2 \in \{1, 2, 6\}$, $6 \in \{1, 2, 6\}$ but $4 \notin \{1, 3, 6\}$.
3. A finite set is a set containing only finite number of elements.

   For example $A = \{1,2,5\}$ is a finite set contains three elements.
4. A set with infinitely many elements is called an infinite set. For example, The set of all positive integers, or $\mathbb{N} = \{1,2,3, \dots, \}$.
5. If the set is finite, its number of elements is represented by $|A|$ or $o(A)$.

   Example, if $A = \{1, 2, 3, 4, 5\}$ then $|A|=5$ or $o(A) = 5$.
6. Let a and b be two elements of the set of real numbers where $a < b$, then

   $[a, b]= \{x: x \in \mathbb{R}, a \leq x \leq b\}$.

Empty Set: A set with no elements is called *empty set* (or null set, or void set), and is represented by $\emptyset$ or { }.

Example: Let $\mathbb{N}$ be the set of all natural numbers. Then the set $\{x \mid x \in \mathbb{N},\ x^2 = 6\}$ is an empty set because there is no natural number whose square is 6.

SUBSET:

A is a subset of a set B or A is contained in B, if every element of a set A is also a member of set B. Then A is called a subset of B and denote by $A \subseteq B$. Every set is a subset of itself.

Definition: A set A is a proper subset of a set B if A is a subset of B and A is not equal to B and denoted by $A \subset \boldsymbol{B}$ $i.e. (A \subset \boldsymbol{B}) \leftrightarrow (A \subseteq B \ \wedge \ A \neq B)$.

Remark: $A \subseteq B$ means "B is superset of A or B contains A". If A is not a subset of B, we write $A \not\subset B$. That means there is at least one element in A that is not a member of B.

Definition:

Two sets are comparable if one of the sets is a subset of the other set, i.e. $A \subseteq B$ or $B \subseteq A$.

Theorem:

Let A, B and C be sets. If A is a subset of B and B is a subset of C, then A is a subset of C.

Proof: Let $x \in A$. Since $A \subseteq B$ then $x \in B$. [by the definition of $\subseteq$]

Since $B \subseteq C$ then every element of B, which includes x, is a member of C.

Then $x \in C$. Therefore, $A \subseteq C$.

Equality of Sets: Let A and B be two sets. Then we called that set A is equal to the set B (coincide) if every element of a set A is an element of a set B and every element of a set B is an element of a set A. That is A is a subset of B and B is a subset of A. The equality of sets A and B is denoted by A = B. That is (A=B) if and only if (A⊆B ∧ B⊆A).

Example: The set {a, b, c, d} is equal to the set {c, a, d, b}, i.e. {a, b, c, d} = {c, a, d, b}. The set {a, b, c} is different from the set {a, b}, i.e. {a, b, c} ≠{a, b}.

Example: Let ℕ be the set of positive integers.

If $A = \{x \mid x \in \mathbb{N}, x < 4\}$ and $B = \{1, 2, 3\}$. Then $A = B$.

Remark:

1. Every set is equal to itself, i.e. $(A = B) \Leftrightarrow (B = A)$
2. If $A = B$ and $B = C$ then $A = C$.

Universal Set:

Sometimes we are interested only in the subsets of one set, and other sets have no meaning for our consideration. In such a case we call this set the universal set.

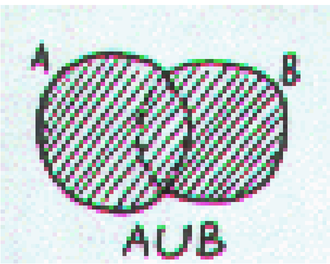
Example:

Let $A = \{1,2,3\}$, $A_0=\emptyset$, $A_1=\{1\}$, $A_2=\{2\}$, $A_3=\{3\}$, $A_4=\{1,2\}$, $A_5=\{1,3\}$, $A_6=\{2,3\}$, then the set A is super set for the sets A_0,A_1,A_2, A_3,A_4, A_5,A_6 and A so that can be called A is a universal set.

<u>**ALGEBRA OF SETS**</u>

Basic Operation: As we have introduced meaning of the terms set, subset, null set and universal set, we can learn how to build new sets using the sets we already know. The way we do it is called set operations. The set operations are: union, intersection, difference, Symmetric Difference and complement.

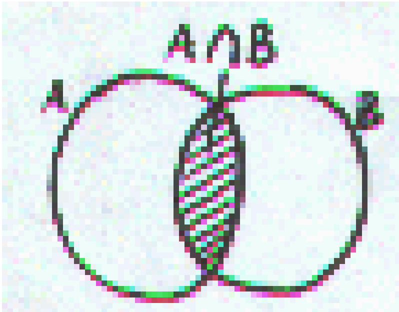
Union: The union of sets A and B is the set of all elements which belong to A or B or to both. It is denoted by $A \cup B$, $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$. $A \cup B$ contains all elements of set A and all elements of set B, but no other elements.



Intersection:

The intersection of sets A and B is the set of elements that are common to sets A and B. It is denoted by $A \cap B$ and is also a subset of U. $A \cap B = \{x \in U |\ x \in A \text{ and } x \in B\}$.

Remark: 1) $A \cap B$ Consists of those and only those elements of U that are in A and in B at the same time.



2) $\{x \mid x \in A \text{ and } x \in B\}$ is a shorthand for $\{x \mid x \in U, x \in A \text{ and } x \in B\}$.

Exercise: Let A and B be two sets.

1. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
3. If $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$

Difference: The difference of sets A and B is the set of elements which belong to A, but do not belong to B. It is denoted by $A - B$ or $A \setminus B$. $A \setminus B = \{x \mid x \in U, x \in A \text{ and } x \notin B\}$

Symmetric Difference: The ***Symmetric Difference*** of sets A and B is the set of elements which belong to A, but do not belong to B or the set of elements which belong to B, but do not belong to A . It is denoted by

$A \triangle B = \{x \in U \mid x \in A - B \text{ or } x \in B - A\} = (A - B) \cup (B - A)$

Complement Set:

Let U be the universal set and $A \subseteq U$. Then the set of all elements in U which are not in A called complement set and denoted by A^C or A. $A^C = \{x \mid x \in U, \text{ and } x \notin A\}$.

Remark: Let A be a set and U is a universal set.

1. Then $A^c = U - A$.
2. $A \cup A^c = U$
3. $A \cap A^c = \emptyset$.
4. If $x \in A$ then $x \notin A^c$.
5. If $x \in A^c$ then $x \notin A$.

Theorem: Let A and B be two sets and U is a universal set. Then $A - B = A \cap B^c$.

Proof: Let $x \in A - B$ iff $x \in A$ and $x \notin B$ iff $x \in A$ and $x \in B^c$ iff $x \in A \cap B^c$.

Disjoint Sets: If two sets A and B have no common elements. *i.e.* no element of A is in B and no element of B is in A, then A and B are disjoint.

Remark:

1. If A and B are disjoint. Then $A \cap B = \emptyset$.
2. Suppose A and B are not comparable. If they are disjoint, they can be represented by the diagram on the left. If they are not disjoint, they can be represented by the diagram on the right.

Example: If $A = \{1,2\}, B = \{2,3\}$, $C = \{1, 3, 5\}$ and $U = \{0,1,2,3,4, 5\}$ then

1. $A \cup B = \{1, 2, 3\}$
2. $A \cap B = \{2\}$
3. $A - B = \{1\}$
4. $B - A = \{3\}$
5. $A \triangle B = (A - B) \cup (B - A)=\{1, 3\}$
6. , $A^c = \{0, 3, 4, 5\}$

Find each of the following: 7. $(A \cup B)^C$ 8. $(A \cap B)^c$ 9. $A \cup C$ 10. $C \cap B$ 11. $A - C$ 12. $C - A$ 13. $C \triangle B$ 14. C^c

Example: Show that $A \subseteq B$ iff $B^c \subseteq A^c$

Suppose that $A \subseteq B$ to prove $B^c \subseteq A^c$. Let $x \in B^c$ iff $x \notin B$ iff $x \notin A$ (B is a super set of the set A) iff $x \in A^c$. Therefore, $B^c \subseteq A^c$

Properties of Sets:

Let A , B and C are sets and U is a universal set. Then

1. Associative Laws

   i. $(A \cap B) \cap C = A \cap (B \cap C)$, ii. $(A \cup B) \cup C = A \cup (B \cup C)$

2. Commutative Laws

   i. $A \cap B = B \cap A$, ii. $A \cup B = B \cup A$

3. Distributive Laws

   i. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

ii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. Identity Laws i. $A \cup \emptyset = A$ ii. $A \cap U = A$
5. Complement Laws i. $A \cap A^c = \emptyset$ ii. $A \cup A^c = U$
6. Idempotent Laws i. $A \cap A = A$ ii. $A \cup A = A$
7. Bound Laws i. $A \cap \emptyset = \emptyset$ ii. $A \cup U = U$
8. Absorption Laws: i. $A \cup (A \cap B) = A$ ii. $A \cap (A \cup B) = A$
9. Involution Law: $(A^C)^C = A$
10. i. $\emptyset^c = U$ ii. $U^c = \emptyset$
11. DeMorgan's Laws i. $(A \cap B)^C = A^C \cup B^C$, ii. $(A \cup B)^C = A^C \cap B^C$

$Proof\ 1 - i$ Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$[by the definition $\cap$]

$\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ [by the definition $\cap$]

$\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ [by associative law of $\wedge$]

$\Leftrightarrow x \in A \wedge (x \in B \cap C)$ [by the definition $\cap$]

$\Leftrightarrow x \in A \cap (B \cap C)$ [by the definition $\cap$]

Therefore, $(A \cap B) \cap C = A \cap (B \cap C)$.

$Proof\ 1 - ii$. H.W

P$roof\ 2 - i$: Let $x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$. [by the definition of $\cap$]

$\Leftrightarrow (x \in B) \wedge (x \in A)$ [by commutative laws of $\wedge$]

$\Leftrightarrow x \in B \cap A$ [by the definition of $\cap$]

Therefore, $A \cap B = B \cap A$.

$Proof\ 2 - ii$: Let $x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$. [by the definition of union]

$\Leftrightarrow (x \in B) \vee (x \in A)$ [by by commutative laws of $\vee$]

$\Leftrightarrow x \in B \cup A$ [by the definition of union]

Therefore, $A \cup B = B \cup A$.

$Proof\ 3 - i$: To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C) \Leftrightarrow [(x \in A) \wedge (x \in B \cup C)]$. [by the definition of $\cap$]

$\Leftrightarrow (x \in A) \wedge [(x \in B) \vee (x \in C)]$ [by the definition of $\cup$]

$\Leftrightarrow [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]$ [by $p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$]

$\Leftrightarrow (x \in A \cap B) \vee (x \in A \cap C)$ [by the definition of $\cap$]

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ [by the definition of $\cup$]

$\Leftrightarrow$ Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$Proof\ 3 - ii$: H.W

$Proof$ 4: Identity Laws H.W

$Proof$ 5: Complement Laws H.W

$Proof\ 6 - i$: Let $x \in A \cap A \Leftrightarrow (x \in A) \wedge (x \in A)$. [by the definition of ∩]

$\Leftrightarrow (x \in A)$ [by Idempotent Laws of ∧ ($p \wedge p \equiv p$)]

$\Leftrightarrow$ Therefore, $A \cap A = A$.

$Proof\ 6 - ii$: H.W

$proof$ 7:- Bound Laws H.W

$proof$ 8:- Absorption Laws H.W

$Proof$ 9: Involution Law $(A^C)^C = A$,

Let $x \in (A^C)^C \Leftrightarrow x \notin A^C$ [by the definition of complement]

$\Leftrightarrow x \in A$. Therefore, $(A^C)^C = A$.

$Proof\ 11 - i$ De Morgan Laws $(A \cap B)^C = A^C \cup B^C$

To prove that $(A \cap B)^C = A^C \cup B^C$.

Let $x \in (A \cap B)^c \Leftrightarrow x \notin (A \cap B)$ [by the definition of complement]

$\Leftrightarrow x \notin A \vee x \notin B$. [by the definition of ∩]

$\Leftrightarrow x \in A^c \vee x \in B^c$ [by the definition of complement]

$\Leftrightarrow x \in A^C \cup B^C$ [by the definition of ∪]. Therefore, $(A \cap B)^C = A^C \cup B^C$.

Example: Let A and B be two sets and U is a universal set then $A \cup (A \cup B^c)^c = A \cup B$.

Proof: L.H.S=$A \cup (A \cup B^c)^c = A \cup (A^c \cap (B^c)^c)$ (By DeMorgan's)

$A \cup (A^c \cap B)$ (By involution law)

$= (A \cup A^c) \cap (A \cup B)$ (By distributive laws)

$= U \cap (A \cup B)$ (By Complement Laws)

$=(A \cup B)$ (By Identity Laws) =R.H.S

Exercises : Let A and B be two sets and U is a universal set then prove the following:

1. $A \cap (A^c \cup B) = A \cap B$
2. Simplify $A \cap (A \cup B^c)^c$

Definition:-

If every element of a set A is a set. Then A is called ***family of sets***.

Example:-

Let **A={**{1,2}, {3,5},{4}**}**

Definition (Power Set): Let **A** be any sets. Then the set of all subset of A is called a ***power set*** of A, and denoted by P(A). That is P(A) = {B| B⊆A}.

Example: If A = {1, 2}, then p(A) = { {1,2}, {1}, {2}, ∅}.

Theorem:

Let A be a set if o(A) =n , then o(p(A))=2^n where n∈ ℕ.

Theorem:

Let A and B be two sets. Then

1. A⊆ B if and only if P(A) ⊆ P(B).

2. P(A∩ B) = P(A) ∩ P(B) H.W

3. P(A) ∪ P(B)⊆ P(A∪ B) H.W

Proof 1: Suppose that A⊆ B to prove P(A) ⊆ P(B) (viceversa)

Let D ∈ P(A) ⇔D ⊆ A [by definition of power set]

⇔ D ⊆ B [since A⊆ B]

⇔ D ∈ P(A) [by definition of power set]

⇔ P (A) ⊆ P (B).

Exercise: By an example show that P(A∪ B)⊈ P(A) ∪ P(B) H.W

Index family of sets: عائلة المجموعات المرقمة

Let F be a family of sets, and I be any set such that for each $i \in I$, there exist a unique A_i in F, then I is called **index set** , and i ∈ I is called the index of A, and F is called the **index family** of sets and denoted by F= $\{A_i\}_{i\in I}$.

Example: Let A_1={1}, A_2={2}, A_3={3}, A_4={1,2}, A_5={1,3}, A_6={2,3}, then F={ A_1, A_2, A_3, A_4, A_5, A_6} is a family of sets, and I={1, 2, 3, 4, 5, 6} is **index set.**

Example: Let $A_a=\{1, x, y\}$, $A_b=\{2, z\}$, $A_c=\{3\}$, $A_d=\{1,2\}$, then F={A_a , A_b , A_c ,A_d , A_e} is a family of sets, and I={a, b, c, d} is **index set.**

<u>Generalized Union and Intersection:-</u>

Definition:

Let $\{A_i\}_{i\in I}$ be an index family of sets then the union of sets A_i consists of all elements which are belongs to A_i for some $i \in I$, that is $\bigcup_{i\in I} A_i = \{ x; x\in A_i \text{ for some } i \in I\}$.

Definition:

Let $\{B_i\}_{i\in I}$ be an index family of sets then the intersection of sets B_j consists of all elements in sets B_j for all $j \in J$, that is $\bigcap_{j\in J} B_j= \{y; y \in B_j \text{ for all } j \in J \}$.

Theorem:- Let $\{A_i\}_{i\in I}$ be an indexed family of sets then

1- If $A_i \subseteq B, \ \forall i \in I, \quad then \quad \bigcup_{i\in I} A_i \subseteq B$

2- If $B \subseteq A_i, \ \forall i \in I, \quad then \quad B \subseteq \bigcap_{i\in I} A_i$

Proof:- 1) Suppose that $A_i \subseteq B \ \ \forall i \in I$ we have to prove that $\bigcup_{i\in I} A_i \subseteq B$.

Let $x \in \bigcup_{i\in I} A_i$ $then \ \exists i \in I \ such \ that \ x \in A_i$ {by definition of generalization of union}, then $x \in B$ since $A_i \subseteq B \ \ \forall i \in I$. Therefore, $\bigcup_{i\in I} A_i \subseteq B$.

2) Suppose that $B \subseteq A_i, \ \forall i \in I$ we have to prove that $B \subseteq \bigcap_{i\in I} A_i$

Let $y \in B \ then \ y \in A_i \ , \ \forall i \in I$ [since $B \subseteq A_i \ ,\forall i \in I$] then $y \in \bigcap_{i\in I} A_i$ [by the definition of generalization of intersection], therefore $B \subseteq \bigcap_{i\in I} A_i$.

Theorem (Generalized Demorgan's theorem)

Let $\{A_i\}_{i\in I}$ be an indexed family of sets then

1- $(\bigcup_{i\in I} A_i)^c = (\bigcap_{i\in I} {A_i}^c)$

2- $(\bigcap_{i\in I} A_i)^c = (\bigcup_{i\in I} {A_i}^c)$

***Proof* 1**: Let $x \in (\bigcup_{i\in I} A_i)^c$

$\Leftrightarrow x \notin \bigcup_{i\in I} A_i$ (by the definition of complement)

$\Leftrightarrow x \notin A_i \ \forall i \in I$ (by the definition of union)

$\Leftrightarrow x \in {A_i}^c \ \ \forall i \in I$ (by the definition of complement)

$\Leftrightarrow x \in \bigcap {A_i}^c \ \forall i \in I$ (by definition of $\bigcap$)

Therefore, $(\bigcup_{i\in I} A_i)^c = (\bigcap_{i\in I} {A_i}^c)$.

Proof 2: H.W

Chapter Three Relation

Ordered Pairs and Cartesian Product:

Definition:

Let A and B be two sets. Then the ***Cartesian Product*** of A and B is defined to be the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. That is $A\times B = \{ (a, b) , a \in A \wedge b \in B \}$.

Example: Let $A=\{1, 2\}$ and $B=\{a, b, c\}$. Then

$A\times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B\times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

Exercise: If $A =\{1, 3, 5, 7\}$ and $B=\{-2, -9, 6\}$, $c=\{x, y, z\}$then find the following :

1.$(A\cup B)\times C$ 2.$(A\times C)\cup (B\times C)$ 3.$(A\cup B)\times (B\cup C)$

Remark: Let A and B be two sets. Then

1. In general $A\times B \neq B\times A$;
2. If $(a, b) \in A\times B$ then $a \in A$ and $b \in B$;
3. If $(a, b) \notin A\times B$ then $a \notin A$ or $b \notin B$;
4. $A\times B = \emptyset$ iff $A= \emptyset$ or $B= \emptyset$;
5. $A\times B = B\times A$ iff $A=B$.

Theorem: Let A, B, C and D be three sets then:

1.$A\times (B\cap C) = (A\times B)\cap(A\times C)$,

2. $A\times (B\cup C) = (A\times B)\cup(A\times C)$,

3.$A\times(B - C) = (A\times B) - (A\times C)$,

Proof 1: Let $(a, b) \in A\times (B\cap C)$

$\Leftrightarrow a\in A \wedge b \in (B\cap C)$ [By the definition of Cartesian product]

$\Leftrightarrow a\in A \wedge (b \in B \wedge b\in C)$ [By the definition of intersection]

$\Leftrightarrow (a\in A \wedge a\in A) \wedge (b \in B \wedge b\in C)$ [By $p\wedge p \equiv p$]

$\Leftrightarrow (a\in A \wedge b \in B) \wedge (a \in A \wedge b\in C)$ [By $p\wedge q \equiv q\wedge p$ and $p\wedge(q\wedge r)\equiv (p\wedge q)\wedge r$]

$\Leftrightarrow (a, b)\in A\times B \wedge (a, b) \in A\times C$ [By the definition of Cartesian product]

$\Leftrightarrow (a, b)\in (A\times B \cap A\times C)$ [By the definition of intersection]

Therefore, $A\times (B\cap C) = (A\times B)\cap(A\times C)$.

Proof 2: H.W

Proof 3: To prove A×(B - C) = (A×B) - (A×C) we have to prove A×(B - C) ⊆(A×B) - (A×C) and (A×B) - (A×C)⊆ A×(B - C). First we prove that A×(B - C) ⊆(A×B) - (A×C).

Let (a, b) ∈A×(B - C) ⟹ (a∈ A) ∧ (b ∈ (B - C)) [By the definition of Cartesian product]

⟹a∈ A ∧ (b ∈ B ∧ b ∉ C) [By the definition of deference]

⟹(a∈ A ∧a∈ A) ∧ (b ∈ B ∧ b ∉ C) [By p∧p≡p]

⟹ a∈ A ∧ [a∈ A ∧ (b ∈ B ∧ b ∉ C)] [By associative law]

⟹a∈ A ∧ [(a ∈ A ∧ b ∈ B) ∧ b ∉ C] [By associative law]

⟹a∈ A ∧ [(b ∈ B ∧ a ∈ A) ∧ b ∉ C] [By commutative law]

⟹a∈ A ∧ [b ∈ B ∧ (a ∈ A ∧ b ∉ C)] [By associative law]

⟹(a∈ A ∧ b ∈ B) ∧ (a∈ A ∧ b ∉ C) [By associative law]

⟹ (a, b) ∈ A×B ∧ (a, b)∉A×C [By the definition of Cartesian product]

⟹ (a, b) ∈ A×B – (A×C). Therefore, A×(B - C) ⊆(A×B) - (A×C).

Conversely: H.W

Definition: A relation R from a set A to a set B is a subset of Cartesian product A×B. That is R⊆A×B.

Example: Let A={a, b} and B={1, 2, 3}. Then R={(a, 2), (b, 3)} is a relation from A to B.

Remark:

1. A relation R from a set A to a set B is a set and its element of the form (a, b).
2. If R is a relation from A to A, then R is a relation on A.

Definition: Let A and B two sets and R be relation from A to B then the inverse of the relation R is denoted by $R^{-1}$and defined to be the set of all order pairs (b, a) of B×A, where (a, b)∈R. That is R^{-1}={(b, a) ; (a, b)∈R}.

Remark:

1. R⊆ A×B If and only if R^{-1} ⊆ B× A.
2. (a, b) ∈R If and only if (b, a) ∈ R^{-1}

Example: If $A = \{1,2\}, B = \{2,3\}$ then A×B = {(1, 2), (1, 3), (2, 2), (2, 3)} and

1. R_1=$\{(a, b) \in A \times B; a = b\} = \{(2, 2)\}$, R_1^{-1}= $\{(2, 2)\}$
2. R_2=$\{(a, b) \in A \times B; a < b\} = \{(2,3)\}$, R_2^{-1}= $\{(3, 2)\}$
3. R_3=$\{(a, b) \in A \times B; a \leq b\}$ =R_1∪ R_2=$\{(2, 2), (2, 3)\}$, R_3^{-1}= $\{(2, 2), (3,2)\}$
4. R_4=$\{(a, b) \in A \times B; a \geq b\}$ = R_3^{-1}, R_4^{-1}= R_3

Definition: let R be a relation on a set A. Then

1. R is reflexive relation if (a, a)∈R , ∀ a∈A
2. R is symmetric relation if (a, b)∈R, then (b, a)∈R , ∀(a, b)∈R
3. R is transitive relation if (a, b)∈R and (b, c)∈R, then (a, c)∈R, ∀(a, b), (b,c)∈R
4. R is equivalent relation if and only if R is reflexive, symmetric and transitive relation.

Example: Let A ={1, 2, 3} and consider the following relation on A:

R_1= {(1, 1), (1, 3), (2, 3), (2, 2), (3, 3)}

R_2={(1, 1), (1, 3), (3, 1)}

R_3={(1, 1), (2, 2), (3, 3)}

R_4={(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}

$R_1 \subseteq$ A×A, R_1 is reflexive relation, is not symmetric relation, since (1, 3))∈ R_1 but (3, 1) ∉ $R_1$and R_1 is transitive relation, therefore, R_1 is not equivalent relation, since is not symmetric relation. R_2, R_4, and R_4 H.W

Exercise: Let R be a relation on a set A. Then prove the following:

1. $(R^{-1})^{-1}$= R
2. R is reflexive relation if and only if R^{-1} is reflexive relation.
3. R is symmetric relation if and only if R^{-1} is symmetric relation.
4. R is transitive relation if and only if R^{-1} is transitive relation.
5. R is equivalent relation if and only if R^{-1} is equivalent relation.

Proof 1: H.W

Proof 2: Suppose that R is reflexive relation on A, we have to prove that R^{-1} is also reflexive relation on A.

R is reflexive iff ∀ x ∈ A , (x, x) ∈ R [By the definition of reflexive]

Iff ∀x∈ A , (x, x) ∈ R^{-1} [By the definition of inverse of the relations].

Iff $R^{-1}$is reflexive relation on A. [By the definition of reflexive]

Therefore R is reflexive relation If and only if R^{-1} is reflexive relation.

Proof 3: H.W

Proof 4: Suppose that R is transitive relation to prove $R^{-1}$is transitive relation.

Let (a, b), (b, c) ∈ R^{-1} then (b, a), (c, b) ∈ R [By the definition of inverse of the relation].

Then (c, b), (b, a) ∈ R then (c, a) ∈ R [Since R is transitive]

Then (a, c) $\in R^{-1}$ [By the definition of inverse of the relation].

Therefore, $R^{-1}$is transitive. Conversely: H.W

Proof 5: H.W

Theorem: Let R be a relation on A. Then R is symmetric if and only if $\mathrm{R} = \mathrm{R}^{-1}$.

Proof: Suppose that R is symmetric to prove $\mathrm{R} = \mathrm{R}^{-1}$.

Let (x, y)$\in R$, then (y, x)$\in R$[since R is symmetric], then (x, y)$\in R^{-1}$[By the definition of inverse of the relation]. Therefore, $R \subseteq R^{-1}$.

Let (a, b)$\in R^{-1}$, then (b, a)$\in R$[by the definition of inverse of the relation]

Then (a, b)$\in R$ [since R is symmetric]. Therefore, $R^{-1} \subseteq R$, hence $\mathrm{R} = \mathrm{R}^{-1}$.

Conversely: Suppose that $\mathrm{R} = \mathrm{R}^{-1}$ to prove R is symmetric.

Let (x, y)$\in R$, then (x, y)$\in R^{-1}$[since $\mathrm{R} = \mathrm{R}^{-1}$],then (y, x)$\in R$[by the definition of inverse of the relations] Therefore, R is symmetric.

Remark: Let R and S be two relations on set A. Then:

1) R∩S={(x, y); (x, y)∈R ∧(x, y)∈S}
2) R∪S={(x, y); (x, y)∈R ∨(x, y)∈S}
3) R−S={(x, y); (x, y)∈R ∧(x, y)∉S}
4) R△S= R∪S- R∩S.

Theorem: Let R and S be two relations onset A. Then:

1. $(R \cap S)^{-1} = \mathrm{R}^{-1} \cap S^{-1}$ 2. $(R \cup S)^{-1} = \mathrm{R}^{-1} \cup S^{-1}$

Proof 1: H.W

Proof 2: Let (x, y)$\in (R \cup S)^{-1}$

⇔ (y, x)$\in R \cup S$ [By the definition of inverse of relation]

⇔ (y, x)∈R ∨ (y, x)∈S [By the definition of union]

⇔ (x, y)$\in \mathrm{R}^{-1}$ ∨ (x, y)$\in S^{-1}$ [By the definition of inverse of relation]

⇔(x, y)$\in \mathrm{R}^{-1} \cup S^{-1}$ [By the definition of union]

Therefore, $(R \cup S)^{-1} = \mathrm{R}^{-1} \cup S^{-1}$.

Exercise: Let S and T be two reflexive relations on a set A. Are the following relations reflexive, symmetric or transitive?

1. S∩T 2. S∪T 3. S-T 4. SΔT

Definition: Let R be a relation from A to B a domain of relation R is the set of all elements a∈A, such that (a, b) ∈R for some b∈B. That is Domain R={a∈A ; (a, b) ∈R for some b∈B }.

Definition: Let R be a relation from A to B. Then Rang of relation R is the set of all elements b∈B, such that (a, b) ∈R for some a∈A, that is

Rang R={b∈B ; (a, b) ∈R for some a∈A}.

Example: Let A ={1, 2, 3} and B={a, b, c, d} consider the relations

R_1={(1, a), (1, b), (3, c)}, R_2={(1, a), (2, b)}. Then

domain of $R_1 = \{1, 3\}$ and Domain of $R_2 = \{1,2\}$.

Rang of $R_1 = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ and Rang of $R_2 = \{a, b\}$.

Remark: Dom R is a shorthand of Domain R and Ran R is a shorthand of Rang R.

Theorem: Let R be a relation from A to B. Then

1) Dom R= Ran R^{-1}. 2) Ran R= Dom R^{-1}

Proof 1: let x ∈Dom R then there exists y ∈B such that (x, y) ∈ R [By the definition of domain] ⇒(y, x) ∈ R^{-1}[By the definition of inverse of the relation]

⇒x∈ Ran R^{-1} [By the definition of inverse of the relation]

Therefore, Dom R ⊆ Ran R^{-1}.

CONVERSLY: Suppose that s∈ Ran R^{-1}, then there exists t ∈B such that

(t, s) ∈ R^{-1}[By the definition of range]

Then (s, t) ∈R [By the definition of inverse of the relation]

Then s ∈ Dom R [By the definition of domain]

Therefore, Ran R^{-1} ⊆ Dom R. Hence Dom R = Ran R^{-1}.

Definition: Let A be a set. Then the relation $\mathrm{I_A}$ on A is called identity relation if $\forall(\mathrm{x}, \mathrm{y}) \in \mathrm{A} \times \mathrm{A}, (\mathrm{x}, \mathrm{y}) \in \mathrm{I_A}$ then $x = y$.

Example: Let A={1, 2, 3}, then consider the following relations on A

1. R_1={(1, 1), (2, 2), (3, 3)} is an identity relation
2. R_2={(1, 1),(2, 3), (2, 2), (3, 3)} is not identity relation, since (2, 3))∈ $R_2$and 2≠ 3.
3. R_3={(1, 1), (2, 2) } is not identity relation because (3, 3) ∉ R_3

Definition: Let R be an equivalence relation on a set A and let x be an element in A. The equivalence class of x is the set of all elements y in A such that x has a relation with y in R. The equivalence class of x is denoted by [x]. That is [x]={y∈ A; (x, y)∈R}.

Example: let A={1, 2, 3, 4} and R={(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3,4), (4, 3)}.

Then [1]={1, 2}, [2]={1, 2}, [3]={ 3, 4}, [4]={3, 4}

Theorem (equivalence theorem)**:** Let R be an equivalence relation on a set A and let a, b ∈A. Then

Then

1) a∈[a]

2) if b∈[a] , then [a]=[b]

3) [a]=[b] if and only if (a, b) ∈R

4) If [a]∩[b]≠ ∅, then [a]=[b]

Proof 1: By the definition of equivalence class [a]={y∈ A; (a, y)∈R}.

Since R is reflexive (a, a)∈R, ∀a∈A, therefore a∈[a].

Proof 2: Suppose that b∈[a] to prove [a]=[b](that is [a]⊆ [b] and [b] ⊆[a]).

Let x∈[a], then (a, x)∈R [by the definition of equivalence class]

Since b∈[a], then (a, b)∈R [by the definition of equivalence class]

then (b, a) ∈R[Since R is symmetric]

So that (b, a), (a, x) ∈R then (b, x)∈ R [Since R is transitive]

Then x∈ [b] [by the definition of equivalence class].

So that if x∈[a] then x∈ [b]. Thus [a]⊆[b]

Let x∈[b], then (b, x)∈R [by the definition of equivalence class]

Since b∈[a], then (a, b)∈R [by the definition of equivalence class]

So that (a, b), (b, x) ∈R then (a, x)∈ R [Since R is transitive]

Then x∈ [a] [by the definition of equivalence class].

So that if x∈[b] then x∈ [a]. Thus [b]⊆[a]. Therefore, [a]=[b].

Proof 3: Suppose [a] = [b] to prove (a, b) ∈R.

From (1), a∈[a] then a∈[b] (since [a]=[b])

Then (b, a)∈R[by the definition of equivalence class]

Then (a, b)∈R [Since R is symmetric]

Conversely: Suppose (a, b)∈R to prove [a]=[b].

If (a, b)∈R then b∈[a] [by the definition of equivalence class] then [a]=[b] [By part 2]

Proof 4: If [a]∩[b]≠ ∅, then [a]=[b]

If [a]∩[b]≠ ∅, then there exist x∈A such that x∈ [a]∩[b] then

$x \in$ [a] and $x \in$[b] [By the definition of intersection]

then [a]=[x] and [b]=[x] [By part 2] then [a]=[b]

Definition: Let$\{\boldsymbol{A_i}\}_{\boldsymbol{i \in I}}$be a family of nonempty subsets of a set A. Then $\{\boldsymbol{A_i}\}_{\boldsymbol{i \in I}}$ is called ***partition*** for A if $\{\boldsymbol{A_i}\}_{\boldsymbol{i \in I}}$ satisfy the following conditions:-

1) $\boldsymbol{A_i \cap A_j = \emptyset}$ $\forall$i, j$\in$ ***I* and *i* ≠ *j* ,** 2) A=$\bigcup_{\boldsymbol{i \in I}} \boldsymbol{A_i}$.

Example: Let A={1,2,3,4,5} and F_1={{1},{2,3},{4,5} } and F_2= {{1,2,3},{4}, {5}, {1}}. Then F_1 is partition for A but F_2 is not partition for A because

$\{1,2,3\} \cap \{1\} \neq \emptyset$ and $\{1,2,3\} \neq \{1\}$

Theorem: Let R be an equivalence relation on a set A and let $\{[a]\}_{\boldsymbol{a \in A}}$be a family of equivalence class with respect to R. Then $\{[a]\}_{\boldsymbol{a \in A}}$ is a partition for A.

Example: A= {1, 2, 3} and R=$I_A \cup \{(2,3), (3,2)\}$ then [1]={1}, [2]={2,3}, [3]={2,3} then A= [1]∪ [2]∪ [3]. The family {[1], [2], [3]}={{1}, {2, 3}} is a partition for A.

Theorem: Let $\{\boldsymbol{A_i}\}_{\boldsymbol{i \in I}}$ be a partition of a nonempty set A. Then there exists an equivalence relation R on A, such that the family of equivalence class with respect to R is equal to $\{\boldsymbol{A_i}\}_{\boldsymbol{i \in I}}$.

Definition: Let R be a relation on A. Then R is called anti-symmetric relation on A, if

$\forall x, y \in A$, (x, y)∧(y, x) ∈ R then x=y.

Example: Let A={1, 2, 3, 4}, then consider the following relations on A

1) R_1={(1, 1), (1, 2), (2, 1)}
2) R_2={(1, 1),(1, 2), (2, 2), (4, 4)}
3) R_3={(1, 1), (2, 2), (3, 3)}
4) R_4={(1, 2), (2, 1), (2, 3)}

$R_1$is symmetric but not anti-symmetric, because (1, 2) and (2, 1)∈ R_1 and $1 \neq 2$. R_2 is not symmetric, because (1, 2)∈ R_2but (2, 1)∉ R_2 but R_2 anti-symmetric. R_3 both symmetric and anti-symmetric. R_4 neither symmetric since (2, 3) ∈ R_4 but (3, 2))∉ R_4nor anti-symmetric because (1, 2) and (2, 1)∈ R_4 and $1 \neq 2$.

Theorem:

Let R be a relation on a set A. Then R is anti-symmetric if and only if $R \cap R^{-1} \subseteq I_A$

Proof: Suppose that R is anti-symmetric to prove $R \cap R^{-1} \subseteq I_A$.

Let (x, y)∈ $R \cap R^{-1}$. Then (x, y) ∈R ∧(x, y) ∈ R^{-1} [By the definition of intersection]

Then (x, y) ∈R ∧(y, x) ∈ R [By the definition of inverse of the relations]

Then x= y [By assumption]. Then (x, y) $\in I_A$ [By definition of I_A], therefore $R \cap R^{-1} \subseteq I_A$

*Conversely: S*uppose that $R \cap R^{-1} \subseteq I_A$ to prove R is anti- symmetric.

Let (x, y), (y, x)$\in$ R. Then (x, y), (y, x) $\in$ R $\wedge$ (y, x), (x, y)) $\in R^{-1}$[by the definition of inverse of the relations]. Then (x, y), (y, x) $\in R \cap R^{-1}$[By the definition of intersection]

Then (x, y) $\wedge$ (y, x) $\in I_A$ [Since $R \cap R^{-1} \subseteq I_A$]

Then x=y [by the definition of I_A]. Therefore, R is anti-symmetric.

Definition: Let R be a relation on a set A. Then R is called ***partially ordered relation*** on A, if R is reflexive anti- symmetric and transitive relation on the set A.

Example: Let $\mathbb{Z}$ be a set of integers, then consider the following relations on $\mathbb{Z}$:

1) $R_1=\{(x, y) \in \mathbb{Z} \times \mathbb{Z}; x \leq y \}$,
2) $R_2=\{(x, y) \in \mathbb{Z} \times \mathbb{Z};; x \geq y\}$
3) $R_3=\{(x, y) \in \mathbb{Z} \times \mathbb{Z}; x>y \}$,
4) $R_4=\{(x, y) \in \mathbb{Z} \times \mathbb{Z}; x^2=y^2\}$

R_1 is reflexive relation, anti-symmetric and transitive, therefore R_1 is partially ordered relation. R_2 is reflexive relation, anti-symmetric and transitive, therefore R_2 is partially ordered relation. R_3 is not reflexive relation, therefore R_3 is not partially ordered relation. R_4 is not partially ordered relation, since R_4 is not anti-symmetric.

Exercise: Let R be a partially ordered relation on a set A, then prove that $\mathbf{R^{-1}}$ is also partially ordered relation on A.

Definition: Let R be a relation on a set A, if R is a partially ordered relation on A, then the order pair (A, R) is called ***partially ordered set***.

Example: Let A={1, 2, 3} and let R ={(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)}, since R is partially ordered relation, then (A, R) is partially ordered set.

Definition: Let (A, R) be a partially ordered set, an element a$\in$A is called ***least element*** of a set A with respect to a relation R if and only if (a, x)$\in$ ***R***, $\forall$ x$\in$A.

Definition: Let (A, R) be a partially ordered set an element b$\in$A is called ***greatest element*** of a set A with respect to a relation R if and only if (x, b)$\in$R, $\forall$x$\in$A.

Example: Let A= {3, 6, 9} and then consider the following relations on A.

$R_1=\{(x, y) \in A\times A; x\leq y\}$, $R_2= \{(x, y) \in A\times A; x\geq y \}$, $R_3= \{(x, y) \in A\times A;$ y divisible by x$\}$

$\boldsymbol{R_1}$={(3, 3), (6, 6), (9, 9), (3, 6), (3, 9), (6, 9)} R_1 is a partially ordered relation on A. 3 is a least element of A with respect to R_1 since (3, x) $\in R_1$, $\forall$ x$\in$ A. 9 is a ***greatest element*** of A with respect to R_1 since (x, 9) $\in R_1$, $\forall$ x$\in$ A.

$\boldsymbol{R_2}$={(3, 3), (6, 6), (9, 9), (9, 6),(9, 3), (6, 3)} is a partially ordered relation on A. 9 is a least element of A with respect to R_2. Since (9, x) ∈R, ∀x∈ A. 3 is a ***greatest element*** of A with respect to R_2 since (x, 3) ∈R_1, ∀ x∈ A. $\boldsymbol{R_3}$={(3, 3), (6, 6), (9, 9), (3, 6), (3, 9)} R_3 is a partially ordered relation on A and 3 is a least element of A with respect to R_3 since (3, x) ∈R, ∀ x∈ A. A has not ***greatest element*** with respect to R_3.

Example: A= {a, b, c} and R={(a, a), (b, b), (c, c), (a, b), (c, b)}, then (A, R) is a partially ordered set. a is not ***least element*** because c∈A but (a, c)∉R. b is not ***least element*** since c∈A but (b, c)∉R. c is not ***least element*** because a∈A but (c, a)∉R.

So that A has not any ***least element*** with respect to a relation R. a is not ***greatest element*** because c∈A but (c, a)∉R. b is ***greatest element*** because for all x∈A, (x, b)∈R. c is not ***greatest element*** because there is at least an element such a∈A but (a, c)∉R.

Example: Let ℤ be a set of integers and $R_1=\{(x, y) \in \mathbb{Z} \times \mathbb{Z}\ ; x \leq y\}$.

R_1 is a partially ordered relation on A but there isn't any least element of A with respect to R.

Theorem:

1. Let R be a partially ordered relation on A. Then if A has a least element, then it is unique.
2. Let R be a partially ordered relation on A. Then if A has a greatest element, then it is unique.

Proof(1): Suppose that there exist at least two least elements let be a and b (a≠b). Then (a, b)∈R and (b, a) ∈R by the definition of least element. But R is partially ordered relation on A(R is anti-symmetric) then a=b. Which is contradiction with our assumption.

Proof(2): H.W

Definition: Let (A, R) be a partially ordered set, an element m∈X is called a ***minimal element*** of A with respect to R, if there is no x∈A such that (x, m) ∈R and x≠m.

Definition: Let (A, R) be a partially ordered set, an element n∈A is called a ***maximal element*** of A with respect to R, if there is no x∈A such that (n, x) ∈R and x≠n.

Example: Let A={a, b, c} and R={(a, a), (b, b), (c, c), (a, b), (c, b)}then (A, R) is a partially ordered set. a is a ***minimal element*** of A with respect to R. c is also a ***minimal element*** of A with respect to R but b is not ***minimal element*** of A with respect to R since (c, b)∈R and

$c\neq b$. b is a ***maximal element*** of A with respect to R but both a and c are not ***maximal element*** of A with respect to R since $(a, b) \in R$, $c\neq b$ and $(c, b) \in R$, $c\neq b$.

Theorem: Let (A, R) is a partial order set. If $a\in \boldsymbol{A}$ is a ***maximal element*** of A with respect to R then a is minimal element of A with respect to R^{-1} .

Proof: Let $a\in A$ be a ***maximal element*** of A with respect to R then there is no $x\in A$ such that $(a, x) \in R$ and $x\neq a$ then there is no $x\in A$ such that $(x, a) \in R^{-1}$ and $x\neq a$. Then is a ***minimal element*** of A with respect to R.

Example: Let A={a, b, c, d} and $\mathbf{I_A}$ ={(a, a), (b, b), (c, c)} then

1. **(A, $\mathbf{I_A}$)** is a partial order set, **(A, $\mathbf{I_A}$)** has not least element but all elements of A are minimal**.**
2. **(A, $\mathbf{I_A}$)** has not greatest element but all elements of A are maximal.

Definition:-Let (A,R) be a partial order set, and $B\subseteq \boldsymbol{A}$ and element $a\in \boldsymbol{A}$ called a ***lower bound*** of B in A if $(a, x)\in R$, $\forall x\in B$. In this case B called ***bounded below*** by **a**.

Definition:-Let (A, R) be a partial order set, and $B\subseteq A$ and element $\mathbf{b}\in A$ called an *upper bound* of B in A if $(x,\mathbf{b}) \in R,\ \forall x \in B$**.** In this case B called ***bounded above*** by b.

Definition:-Let (A, R) be a partial order set, and $B\subseteq A$, then B called ***bounded set*** if B bounded below and bounded above.

Remark: The set of all *upper bound* elements of B in A is called *upper bound set* of B in A and the set of all *lower bound* elements of B in A is called *lower bound set* of B in A.

Example:- Let A={3, 6, 9, 12, 15} , B={6, 12} and $R_1=\{(x, y)\in A\times A,\ x \leq y\}$ then (A, R_1) is a partial order set and {3, 6} is lower bounded set of B in A with respect to R_1 ,(B bounded below by 3, 6).

Definition:-Let (A, R) be a partial order set, and $B\subseteq\mathbf{A}$ and element $\alpha \in A$ is called a ***greatest lower bound(infimum)***for B in A if

1. α is ***lower bounded*** for B in A.
2. $(x, \alpha)\in R$, for all lower bound x for B in A.

Definition:-Let (A, R) be a partial order set, and $B\subseteq A$ and element $\beta \in A$ called ***least upper bound (supremum)*** of B in A if

1. β upper bound for B in A.
2. $(\beta, \mathbf{x})\in \boldsymbol{R}$, for all upper bound x for B in A.

Example: Let A={1,2,3,4}, B={2,3} and R={(x, y)∈A×A; x ≤ y}.

1. 1 is least element of A with respect to R.
2. 4 is greatest element of A with respect to R.
3. 1 is minimal element of A with respect to R.
4. 4 is maximal element of A with respect to R.
5. C={1, 2}is ***lower bounded set*** of B in A with respect to R.
6. 2 is greatest lower bounded of B in A with respect to R since 2 is greatest element of the set C with respect to the relation S=R∩(C×C) ={(1, 1), (1, 2), (2, 2)}.
7. D={3, 4} is upper bounded set of B in A with respect to R.
8. 3 is least upper bounded of B in A with respect to R since 3 is least element of the set D with respect to the relation T=R∩(D×D) ={(3, 3), (3, 4), (4, 4)}.

Exercise: If $A = \{a_1,\ a_2,\ a_3\}, B = \{a_2\}$ and $R = \{(a_m, a_n) \in A \times A: m \leq n\}$then answer the following:

1. Show that (A, R) is a partially ordered set.
2. Find least element (if exist).
3. Find greatest element (if exist).
4. Find minimal element(s).
5. Find maximal element(s).
6. Find lower bounded set of B in A.
7. Find upper bounded set of B in A.
8. Find least upper bounded of B in A (if exist).
9. Find greatest lower bounded of B in A (if exist).

Exercises:- Let (A,R) be a partial order set, and B⊆ ***A***.

1. If B has an infimum in A then it is unique.
2. If B has a supremum in A then it is unique.

Definition:-Let (A, R) be a partial ordered set. Then (A, R) is called ***totally (linearly) ordered set*** if for every x, y ∈A then either (x, y) ∈R or (y, x) ∈R.

Example: Let A={1, 2, 3} and R={(x, y)∈A×A; x ≤ y} then R={(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)}, then (A, R) is a ***totally ordered set.***

Definition:- Let (A, R) be a partial ordered set. Then (A, R) is called *a* ***well-ordered set*** if for every non empty subset B of A has a ***least element.***

Remark: $\boldsymbol{R_B = \{(a, b) \in R; a, b \in B\}}$.

Example: Let A={1, 2, 3} and R={(x, y)∈A×A; x ≤ y} then R={(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)}, then (A, R) is a ***well-ordered set.*** Because the set A has the following nonempty subsets: {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, A and

$R_{\{1\}}$={(x, y)∈ R; x ≤ y}={(1, 1)} and 1is a least element of $R_{\{1\}}$.

$R_{\{2\}}$={(x, y)∈ R; x ≤ y}={(2, 2)} and 2is a least element of $R_{\{2\}}$.

$R_{\{3\}}$={(x, y)∈ R; x ≤ y}={(3, 3)} and 3is a least element of $R_{\{3\}}$.

$R_{\{1,2\}}$={(x, y)∈ R; x ≤ y}={(1, 1), (2, 2), (1, 2)} and 1is a least element of $R_{\{1,2\}}$.

$R_{\{1,3\}}$={(x, y)∈ R; x ≤ y}={(1, 1), (3, 3), (1, 3)} and 1is a least element of $R_{\{1,3\}}$.

$R_{\{2,3\}}$={(x, y)∈ R; x ≤ y}={(2, 2), (3, 3), (2, 3)} and 2 is a least element of $R_{\{2,3\}}$.

R_A={(x, y)∈A× A; x ≤ y}=R={(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)} and 1is a least element of R_A .

Remark:

1. Every totally ordered set is a partially ordered set, but the converse is not true.
2. Every well-ordered set is a totally ordered set, but the converse is not true.
3. (A, R) is called *a* ***well-ordered set*** if for every non empty subset B of A has a ***least element*** with respect to a relation R_B or S where S=R∩(B×B).

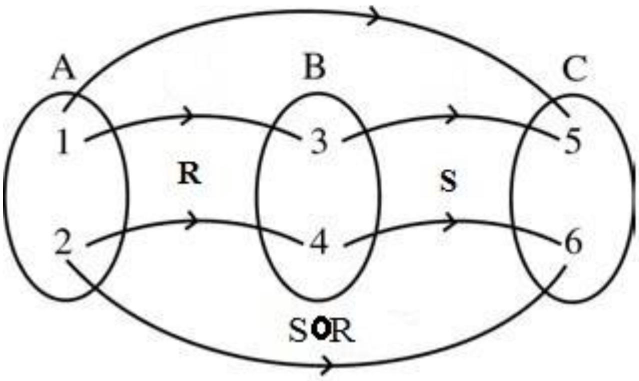
Example:

1. Let A={1, 2, 3} and R={(x, y)∈ ℤ × ℤ; x ≥ y} then (ℤ, R) is a ***totally ordered set*** and ***well-ordered set***.
2. Let B=[0, 1]={x: x∈ ℝ, 0≤x≤1} and R={(x, y)∈B×B, x ≥ y} then (B, R) is a ***totally ordered set*** and but not ***well-ordered set***. Why?

Definition: Let R be a relation from A to B, and S be a relation from B to C. Then the composition of relation R and S is denoted by SoR and defined by

SoR= {(x, z) ∈ $\mathbf{A} \times \mathbf{C}$ | $\exists\, \boldsymbol{y} \in \boldsymbol{B} : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{R} \wedge (\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{S}$}

Example: Let A={1, 2}, B={3,4}, C={5, 6} , R={(1, 3), (2, 4)} and S={(3, 5), (4, 6)} then SoR={(1, 5), (2, 6)}



$$\begin{tikzcd}
A \arrow[r, "R"] \arrow[dr, "SoR"'] & B \arrow[d, "S"] \\
& C
\end{tikzcd}$$

Example: Let A={1, 2, 3} , B={a, b, c, d} and C={w, x, y, z}, R={(1, a), (2, a), (3, d)} and S={(a, x), (b, y), (b, z), (c, z), (d, w)} then SoR={(1, x), (2, x), (3, w)}.

Theorem: Let R be a relation on a set A. Then $RoI_A=I_AoR=R$

Proof: Case 1, To prove $RoI_A=R$. Let $(x, z)\in RoI_A$ then $\exists y\in A$ such that $(x, y)\in I_A$ and $(y, z)\in R$[by the definition of composition of relation]. Then x=y and $(y, z)\in R$ [by the definition of identity relation]. Then $(x, z)\in R$ [since x=y]. This means that if $(x, z)\in RoI_A$ then $(x, z)\in R$, thus, $RoI_A \subseteq R$.

Conversely: Suppose that $(a, c)\in R$ then $(a, a)\in I_A \wedge (a, c) \in R$ [by the definition of identity relation] then $(a, c)\in RoI_A$. This means that if $(a, c)\in R$ then $(a, c)\in RoI_A$. Thus $R\subseteq RoI_A$. Therefore, $RoI_A=R$.

Case 2, To prove $I_AoR=R$ H.W . By case 1 and 2 $RoI_A=I_AoR=R$.

Theorem: Let R, S and T be relations on a set A. Then

1) $(RoS)oT=Ro(SoT)$
2) $(R\cup S)oT=(RoT)\cup(SoT)$
3) $(R\cap S)oT\subseteq(RoT)\cap(SoT)$
4) If $R\subseteq S$, then (i) $RoT\subseteq SoT$ (ii) $ToR\subseteq ToS$

5) (RoS)∩T=∅ $\leftrightarrow (ToR^{-1}) \cap S = \emptyset$

6) $(SoR)^{-1}= R^{-1}o\, S^{-1}$

Proof 2: Let (x, z)∈(R∪S)oT ⇔ ∃ y∈A such that (x, y)∈T∧(y, z)∈(R∪S) [by the definition of composition of relation] .

⇔ (x, y)∈T∧((y, z)∈R∨(y, z)∈S) [By the definition of union]

⇔ ((x, y)∈T∧(y, z)∈R)∨((x, y)∈T∧(y, z)∈S) [by distributive law]

⇔ (x, z) ∈RoT ∨ (x, z) ∈SoT [by the definition of composition of relation]

⇔ (x, z) ∈(RoT)∪(SoT). Therefore (R∪S)oT=(RoT)∪(SoT).

Proof 4 (ii): Suppose that R⊆S, we have to prove that ToR⊆ToS

Let (x, z)∈ToR, then ∃y∈A such that (x, y)∈R∧(y, z)∈T

then (x, y)∈S∧(y, z)∈T [since R⊆S],

then(x, z)∈ ToS [by the definition of composition of relation]

Thus if (x, z)∈ToR then (x, z)∈ToS, Therefore, ToR⊆ToS.

Proof 5: Suppose (ToR^{-1})∩S≠ ∅ iff ∃(x, y)∈(ToR^{-1})∩S iff (x, y)∈(ToR^{-1}) ∧ (x, y)∈S iff ∃ z ∈ A such that ((x, z))∈R^{-1} ∧ (z, y) ∈ T) ∧ (x, y)∈S iff ((z, x)∈R ∧ (z, y) ∈ T) ∧ (x, y)∈S iff (z, x)∈R ∧ ((z, y) ∈ T) ∧ (x, y)∈S) [by associative law]

iff (z, x)∈R ∧ ((x, y)∈S∧ (z, y) ∈T)) [by commutative law]

iff ((z, x)∈R ∧ (x, y)∈S) ∧ (z, y) ∈T[by associative law]

iff (z, y)∈ (SoR) ∧ (z, y) ∈T[by the definition of composition]

iff (z, y)∈ (SoR)∩T[by the definition of intersection].

Thus (RoS)∩T≠ ∅ . Therefore, by **Contrapositive law** if (RoS)∩T=∅ iff (ToR^{-1})∩S=∅.

Theorem: Let R be a relation on a set A. R is a partially ordered relation on A if and only if $\mathrm{R} \cap \mathrm{R}^{-1} = \mathrm{I_A}$ and RoR=R.

Proof: Suppose that R is partially ordered relation on A, to prove $\mathrm{R} \cap \mathrm{R}^{-1} = \mathrm{I_A}$ and RoR=R.

Case 1: To prove $\mathrm{R} \cap \mathrm{R}^{-1} = \mathrm{I_A}$.

Let (x, y) ∈ $\mathrm{R} \cap \mathrm{R}^{-1}$. Iff (x, y) ∈R ∧ (x, y) ∈ R^{-1}[by the definition of intersection]

Iff (x, y) ∈R ∧ (y, x) ∈R [by the definition of R^{-1}], Iff x=y [since R is anti-symmetric relation]. Iff (x, y)∈ $\mathrm{I_A}$, therefore, $\mathrm{R} \cap \mathrm{R}^{-1} \subseteq \mathrm{I_A}$. Let (x, y)∈ R and (x, y)∈ R^{-1}[since R is reflexive relation]. Then (x, x)∈ $\mathrm{R} \cap \mathrm{R}^{-1}$ therefore, $\mathrm{R} \cap \mathrm{R}^{-1} = \mathrm{I_A}$.

Case 2: To prove RoR=R. Let $(x, z) \in RoR$ then $\exists\, y \in A$ such that $(x, y) \in R \wedge (y, z) \in R$[By the definition of composition relation]. Then $(x, z) \in R$ [Since R is transitive relation], thus $RoR \subseteq R$. Let $(x, y) \in R$, then $(x, x) \in R \wedge (x, y) \in R$ [since R is reflexive relation], then $(x, y) \in RoR$[by the definition of composition of relations]. Hence $R \subseteq RoR$. Therefore, R= RoR.

Conversely: Suppose that $R \cap R^{-1} = I_A$ and RoR=R to prove R is partially ordered relation on a set A.

Let $x \in A$ then $(x, x) \in I_A$, then $(x, x) \in R \cap R^{-1}$[since $R \cap R^{-1} = I_A$], then $(x, x) \in R$, So that R is reflexive relation.

Let (x, y), $(y, x) \in R$, then $(x, y) \in R \wedge (y, x) \in R$, then $(x, y) \in R \wedge (x, y) \in R^{-1}$

Then $(x, y) \in R \cap R^{-1}$, then $(x, y) \in I_A$ [Since $R \cap R^{-1} = I_A$], then x=y.

So that R is anti-symmetric relation.

Let (x, y) and $(y, z) \in R$, then $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in RoR$,then $(x, z) \in R$ [R=RoR]. So that R is transitive relation.

Therefore, R is partially ordered relation.

Exercises: Let S and R be two relations on a set A. Then prove or disprove the following:

1) S is transitive iff $SoS \subset S$.
2) If S is reflexive and transitive relation then SoS=S.
3) SoR=RoS
4) $Dom(SoR) \subseteq dom(R)$
5) Ran (SoR) $) \subseteq$ Rang (S)
6) If Ran $(S) \subseteq Dom(R)$ then (RoS)=Dom S.