Question Bank
foundations of Mathematics I and foundations of Mathematics II
2022- 2023 by Dr Hogir

1) First semester

Exercise: Suppose that **p** is a false statement. What is the truth-value of the compound statement **~p**? What is the truth-value of the compound statement **~(~p)**?

Exercise: Suppose that **p** is a false statement, and **q** is a true statement. What is the truth-value of the compound statement **(~p)∧q**? What is the truth-value of the compound statement **p∧(~q)**? What is the truth-value of the compound statement **~(p∧(~q))**?

Exercise: Suppose that the compound statement **p∧(~q)** is true. What are the truth-values of the statements **p** and **q**? You are **deducing** these truth-values.

Exercise: Suppose that p is a false statement, and q is a true statement. What is the truth-value of the compound statement **(~p)**→q? What is the truth-value of the compound statement p→ **(**~ q)? What is the truth-value of the compound statement p→q? What is the truth-value of the compound statement **~(p→ (~q))**?

Exercise: Suppose that p is a false statement, and q is a statement whose truth-value is presently unknown. Suppose that the compound statement **(~p)→q** is true. What is the truth-value of the statement **q**? What is the truth-value of the statement **q**, if you are given that **(~p)→q** is false? What is the truth-value of the compound statement **~(q→ (~q))**?

Exercise: Suppose that the compound statement p→ q is a true statement. In order for p to be true, what *must* the truth-value of q be?

Exercise: Suppose that the compound statement **p→q** is a true statement. Which truth value of **p** assures us that q is true?

Exercise: Suppose that the compound statement **p→q** is false. What are the truth-values of p and of **q**?

Exercise: Show, by constructing its truth table, that **(~(p∨q))↔ (~p)∧(~q)** is a tautology.

Exercise: Show by constructing its truth table that **(~(p∧q))** ↔ **(~p)∨(~q)** is a tautology.

Exercise: Construct the truth table for the compound statement **((~p)∨q)** ↔(**p→q)**. What does the truth table tell you about the two statements **(~p)∨q** and **p→q**?

Exercise: Construct the truth table for the compound statement **(q→p)** ↔(**p→q)**. What does the truth table tell you about the two statements **q→p** and **p→q**?

Exercise: Construct the truth table for the compound statement **(~q→~p)** ↔(**p→q)**. What does the truth table tell you about the two statements **~q→~p** and **p→q**?

Exercise: Construct the truth table for the compound statement **((p∨q)∨r)** ↔ **(p∨(q∨r))**. What does the truth table tell you about the two statements **(p∨q)∨r** and **p∨(q∨r)**?

Exercise: Construct the truth table for the compound statement **((p→q)∧(q→r))** → **(p→r)**.

Exercise: Construct the truth table for the compound statement **((p→q)∨(q→r))** → **(r→p)**.

Exercise: Construct the truth table for the compound statement **((p∧q)∨(p∧r))** ↔ **(p∧(q∨r))**.

Remark:

1. If $p \wedge q$ is true then both p and q are true.

2. If $p \wedge q$ is false then p or q is false.
3. If $p \vee q$ is true then p or q is false.
4. If $p \vee q$ is false then both p and q are false.
5. If $p \rightarrow q$ is true then p is false or both p and q are true.
6. If $p \rightarrow q$ is false then p is true and q are false.
7. If $p \leftrightarrow q$ is true then both p and q are true or false.
8. If $p \leftrightarrow q$ is false then either p or q is true that is the truth value of p is different from q.

Exercise: Let p, q and r be three statements then prove the following:

1. $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$
2. $p \rightarrow (q \rightarrow r) \equiv \sim p \vee (q \rightarrow r)$
3. $p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\sim q \vee r)$.
4. $[\sim(P \vee q)] \longleftrightarrow [(\sim p) \wedge (\sim q)]$is atautology.
5. $\sim[\sim(p \wedge q) \longleftrightarrow ((\sim p) \vee (\sim q))]$ is contradiction.

Theorem: Let $p(x)$ be anopen sentence in x defined on the set$A$, then

1. $\sim (\forall\, \mathrm{x}, \mathrm{p}(\mathrm{x})) \equiv \exists\, \mathrm{x}, \sim p(x)$
2. $\sim (\exists\, \mathrm{x}, \mathrm{p}(\mathrm{x})) \equiv \forall \mathrm{x}, \sim p(x)$
3. $\forall\, x, p(x) \equiv \sim (\exists\, x, \sim p(x))$
4. $\exists\, x, p(x) \equiv \sim (\forall x, \sim p(x))$

Exercise: Let $p(x)$ and$q(x)$ be two open sentence in x defined on the set A. Then prove or disprove the following:

1. $\forall\, x, \big(p(x) \wedge q(x)\big) \equiv \forall\, x, p(x) \wedge \forall\, x, q(x)$. Prove
2. $\forall x, (p(x) \vee q(x)\,) \equiv \forall x, p(x) \vee \forall x, q(x)$.disprove
3. $\forall x, (p(x) \rightarrow q(x)\,) \equiv \forall x, p(x) \rightarrow \forall x, q(x)$. H.W
4. $\forall x, (p(x) \leftrightarrow q(x)\,) \equiv \forall x, p(x) \leftrightarrow \forall x, q(x)$. H.W
5. $\exists x, (p(x) \wedge q(x)\,) \equiv \exists x, p(x) \wedge \exists x, q(x)$. H.W
6. $\exists x, (p(x) \vee q(x)\,) \equiv \exists x, p(x) \vee \exists x, q(x)$. H.W
7. $\exists x, \big(p(x) \rightarrow q(x)\big) \equiv \exists x, p(x) \rightarrow \exists x, q(x)$. H.W
8. $\exists x, \big(p(x) \leftrightarrow q(x)\big) \equiv \exists x, p(x) \leftrightarrow \exists x, q(x)$. H.W

Exercise:

i. If $A = \{1,2\}$, $B = \{2,3\}$, $C = \{1, 3, 4\}$ and $U = \{0, 1, 2,3, 4, 5\}$ then find the following:

1. $A \cup B$	2. $A \cap B$	3. $A - B$	4. $B - A$
5. $A \triangle B$	6. $B \triangle A$	7. A^c	8. $P(A)$

9. $R = \{(a,b) \in A \times B; a - b = 0\}$ 10. R^{-1} 11. Dom R 12. Ran R
13. A∪(B∩C)   14. (A∪B) ∩ (A∪C)   15. A∩(B∩C)   16. (A∩B) ∩(A∩C)
17. A−(B∩C)   18. (A−B) ∩ (A−C)   19. AΔ (B∩C)   20. (AΔB) ∩ (AΔC)
21. A∪(B∪C)   22. (A∪B) ∪ (A∪C)   23. A∩(B∪C)   24. (A∩B) ∪(A∩C)
25. A−(B∪C)   26. (A−B) ∪ (A−C)   27. AΔ (B∪C)   28. (AΔB) ∪ (AΔC).

ii. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, $C = \{x \in \mathbb{R}\text{: } \mathbf{1} \leq \boldsymbol{x} \leq \mathbf{3}\} = [\mathbf{1}, \mathbf{3}]$, $\boldsymbol{D} = \{x \in \mathbb{Q}\text{: } \mathbf{1} \leq \boldsymbol{x} \leq \mathbf{3}\}$, and $U = [0, 5]$ then find the following :

1. $A \cup B$ 2. $C \cap D$ 3. $A - D$ 4. $A \triangle B$
5. A^c 6. $P(A)$ 7. $R = \{(x,y) \in A \times B : y = x + 1\}$
8. $S = \{(X,y) \in P(A) \times B : y \in X\}$ 9. Dom R^{-1} 10. Ran S
11. A∪(B∩C)   12. (A∪B) ∩ (A∪C)   13. A∩(B∩C)   14.(A∩B) ∩(A∩C)
15. A−(B∩C)   16. (A−B) ∩ (A−C)   17. AΔ (B∩C)   18. (AΔB) ∩ (AΔC)
19. A∪(B∪C)   20. (A∪B) ∪ (A∪C)   21. A∩(B∪C)   22.(A∩B) ∪(A∩C)
23. A−(B∪C)   24. (A−B) ∪ (A−C)   25. AΔ (B∪C)   26. (AΔB) ∪ (AΔC);

iii. If $A = \left\{\frac{1}{2}, \frac{1}{4}\right\}$, $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\right\}$ $C = \{x \mid x = 2^{-n},\ n \in \mathbb{N} \cup \{0\}\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\}$ $\boldsymbol{D} = \{x \in \mathbb{Q}\text{: } \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$, and $U = [0, 1]$ then find the following :

1. $A \cup B$ 2. $C \cap D$ 3. $A - C$ 4. $A \triangle B$ 5. D^c 6. $P(\{0\})$ 7. Dom R^{-1}
8. $R = \{(x,y) \in A \times B : y = x^2\}$, 9. $S = \{((a,b),(c,d)) \in R \times R : ad = bc\}$, 10. Ran S.
11. A∪(B∩C)   12. (A∪B) ∩ (A∪C)   13. A∩(B∩C)   14.(A∩B) ∩(A∩C)
15. A−(B∩C)   16. (A−B) ∩ (A−C)   17. AΔ (B∩C)   18. (AΔB) ∩ (AΔC)
19. A∪(B∪C)   20. (A∪B) ∪ (A∪C)   21. A∩(B∪C)   22.(A∩B) ∪(A∩C)
23. A−(B∪C)   24. (A−B) ∪ (A−C)   25. AΔ (B∪C)   26. (AΔB) ∪ (AΔC);

iv. If $A = \{-1,\ 0,\ 1\}$, then write the following relations (by listing all its members) on the set A :

1. $R = \{(a,b) \in A \times A : a \leq b\}$,   2. $S = \{(a,b) \in A \times A : |a| = |b|\}$

3. $T = \{(a,b) \in A \times A : a = b + 1 \text{ or } a = b\}$ 4. R∪S   5. R∪T   6. S∪T

7.R∩S   8. R∩T   9. S∩T   10. R– S   11. R – T   12. S – T

13. RΔS   14. RΔT   15. SΔT   16. R∪(S∩T)   17. (R∪S) ∩ (R∪T)

18. R∩(S∩T)   19. (R∩S) ∩(R∩T)   20.R−(S∩T)   21. (R−S) ∩ (R−T)

22. RΔ (S∩T)   23. (RΔS) ∩ (RΔT)   24. R∪(S∪T)   25. (R∪S) ∪ (R∪T)

26. R∩(S∪T)   27. (R∩S) ∪(R∩T)   28. R−(S∪T)   29.(R−S) ∪ (R−T)

30. $R\Delta(S\cup T)$ 31. $(R\Delta S)\cup(R\Delta T)$

Exercise: Let A, B and C be three sets. If A=B then A-C=B-C. Is the converse is true?
Proof: Let $x \in A-C$ iff $x \in A$ and $x \notin C$ iff $x \in B$ and $x \notin C$ iff $x \in B-C$. The converse is not true for example, if A={1, 2, 3}, B={1, 2} and C= {2,3} then
A-C=B-C={3} but $A\neq B$.

Exercise: Let A, B and C be three sets. Then prove or disprove each of the following:

1. $A\cap(B\cap C)=(A\cap B)\cap(A\cap C)$ 2. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

3. $A-(B\cap C)=(A-B)\cap(A-C)$ 4. $A\Delta(B\cap C)=(A\Delta B)\cap(A\Delta C)$

5. $A\cup(B\cup C)=(A\cup B)\cup(A\cup C)$ 6. $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$;

7. $A-(B\cup C)=(A-B)\cup(A-C)$ 8. $A\Delta(B\cup C)=(A\Delta B)\cup(A\Delta C)$;

9. $A\cup(B-C)=(A\cup B)-(A\cup C)$ 10. $A\cap(B-C)=(A\cap B)-(A\cap C)$;

11. $A-(B-C)=(A-B)-(A-C)$ 12. $A\Delta(B-C)=(A\Delta B)-(A\Delta C)$;

13. $A\cup(B\Delta C)=(A\cup B)\Delta(A\cup C)$ 14. $A\cap(B\Delta C)=(A\cap B)\Delta(A\cap C)$;

15. $A-(B\Delta C)=(A-B)\Delta(A-C)$ 16. $A\Delta(B\Delta C)=(A\Delta B)\Delta(A\Delta C)$;

Q) Let A , B and C are sets and U is a universal set. Prove each of the following:

1. Associative Laws : i. $(A\cap B)\cap C=A\cap(B\cap C)$, ii. $(A\cup B)\cup C=A\cup(B\cup C)$
2. Commutative Laws : i. $A\cap B=B\cap A$, ii. $A\cup B=B\cup A$
3. Distributive Laws: i. $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
ii. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
4. Identity Laws i. $A\cup\emptyset=A$ ii. $A\cap U=A$
5. Complement Laws i. $A\cap A^c=\emptyset$ ii. $A\cup A^c=U$
6. Idempotent Laws i. $A\cap A=A$ ii. $A\cup A=A$
7. Bound Laws i. $A\cap\emptyset=\emptyset$ ii. $A\cup U=U$
8. Absorption Laws: i. $A\cup(A\cap B)=A$ ii. $A\cap(A\cup B)=A$
9. Involution Law: $(A^C)^C=A$
10. 0/1 laws i. $\emptyset^c=U$ ii. $U^c=\emptyset$

11. DeMorgan's Laws i. $(A\cap B)^C = A^C \cup B^C$, ii. $(A\cup B)^C = A^C \cap B^C$

Exercises : Let A and B be two sets and U is a universal set then prove the following:

$A\cap (A^c \cup B) = A \cap B$ and simplify $A \cap (A \cup B^c)^c$.

Theorem: Let A, B, C and D be three sets then:

1.$A\times (B\cap C) = (A\times B)\cap(A\times C)$,

2. $A\times (B\cup C) = (A\times B)\cup(A\times C)$,

3.$(A\times B) \cap (C\times D) = (A\cap C) \times (B\cap D)$

4.$A\times(B - C) = (A\times B) - (A\times C)$, 5. $(A\times B) - (C\times C)=[(A - C)\times B]\cup[A\times(B - C)]$

Exercise: If A ={1, 3, 5, 7} and B={-2, -9, 6}, c={x, y, z}then find the following :

1.$(A\cup B)\times C$ 2.$(A\times C)\cup (B\times C)$ 3.$(A\cup B)\times (B\cup C)$

Q) Let A, B and C be three nonempty sets, then
prove or disprove each the following:

1) $(A\times A) - (B\times C)=[(A - B)\times A]\cup[A\times(A-C)]$
2) $(A\times A) \cap (B\times C)=(A \cap B)\times(A\cap C)]$
3) If $(A \times B) \cup (B \times A) = C \times C$, then A=B=C.
4) $A\cap B = \emptyset$ if and only if $(A \times B) \cap (B \times C) = \emptyset$

Second semester

Exercise: Let R be a relation on a set A. Then prove the following:

1. $(R^{-1})^{-1}$= R
2. R is reflexive relation if and only if $R^{-1}$is reflexive relation.
3. R is symmetric relation if and only if$R^{-1}$is symmetric relation.
4. R is transitive relation if and only if$R^{-1}$is transitive relation.
5. R is equivalent relation if and only if$R^{-1}$is equivalent relation.

Theorem: Let R be a relation on A. Then R is symmetric if and only if $R = R^{-1}$.

Theorem: Let R and S be two relations onset A. Then:

1. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ 2. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Exercise: Let S and T be two reflexive relations on a set A. Are the following relations reflexive, symmetric or transitive?

1. S∩T 2. S∪T 3. S-T 4. SΔT

Theorem: Let R be a relation from A to B. Then

1) Dom R= Ran R^{-1}
2) Ran R= Dom R^{-1}

Definition: Let A be a set. Then the relation I_A on A is called identity relation if $\forall (x, y) \in A \times A, (x, y) \in I_A$ then $x = y$.

Theorem(equivalence theorem)**:** Let R be an equivalence relation on a set A and let a, b ∈A. Then

1) a∈[a]
2) if b∈[a] , then [a]=[b]
3) [a]=[b] if and only if (a, b) ∈R
4) If [a]∩[b]≠ ∅, then [a]=[b]

Exercises: Let ℤ be the set of integers and A_0=3 ℤ ={3x: x∈ ℤ}={...,-6,-3,0,3,6,...}, A_1=1+3 ℤ ={3x+1: x∈ ℤ}={..., -5, -2,1, 4, 7,...} and A_2=2+3 ℤ ={3x+2: x∈ ℤ}={..., -4, -1, 2, 5, 8,...}.

1) Find the following: $A_0 \cup A_1 \cup A_2$, $A_0 \cap A_1$, $A_0 \cap A_2$, $A_1 \cap A_2$
2) If $R_0=\{(x, y) \in A_0 \times A_0\}$, $R_1=\{(x, y) \in A_1 \times A_1\}$ and $R_2=\{(x, y) \in A_2 \times A_2\}$ then show that R_0, R_1 and R_2 are equivalence relations on the sets A_0, A_1 and A_2 respectively.
3) Show that $R_0 \cup R_1 \cup R_2$ is equivalence relation on ℤ.
4) Find [0], [1] and [2].
5) Show that [1]=[7].

Exercises:

1. Let R be a relation on A and $R \subseteq I_A$, then show that R is both symmetric and anti-symmetric.
2. Is the following relations are reflexive, symmetric, anti-symmetric, or transitive on the set A.

a) $R = \{(x, y) \in A \times A, x = y\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

b) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}, x^2 = y^2\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

c) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}, x \leq y\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

d) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}, x + y = 1\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

e) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}, y = x^2 + 3x + 1\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

f) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}, 0 \leq x \leq y \leq 3\}$. If A=$\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\{1, 2, 3\}$.

Q) Let R be a relation on a set A. Then R is anti-symmetric if and only if $R \cap R^{-1} \subseteq I_A$

Q)Let R be a partially ordered relation on A. Then if A has a least element, then it is unique.

Q) Let R be a partially ordered relation on A. Then if A has a greatest element, then it is unique.

Theorem: Let (A, R) is a partial order set. If a$\in$ ***A*** is a ***maximal element*** of A with respect to R then a is minimal element of A with respect to R^{-1} .

Exercise: If $A = \{a_1,\ a_2,\ a_3\}$, $B = \{a_2\}$ and $R = \{(a_m, a_n) \in A \times A: m \leq n\}$then answer the following:

1. Show that (A, R) is a partially ordered set.
2. Find least element (if exist).
3. Find greatest element (if exist).
4. Find minimal element(s).
5. Find maximal element(s).
6. Find lower bounded set of B in A.
7. Find upper bounded set of B in A.
8. Find least upper bounded of B in A (if exist).

9. Find greatest lower bounded of B in A (if exist).

Exercises:- Let (A,R) be a partial order set, and B$\subseteq$ ***A***.

1. If B has an infimum in A then it is unique.
2. If B has a supremum in A then it is unique.

Theorem: Let R be a relation on a set A. Then $RoI_A=I_AoR=R$

Theorem: Let R, S and T be relations on a set A. Then

1) (RoS)oT=Ro(SoT)
2) (R∪S)oT=(RoT)∪(SoT)
3) (R∩S)oT⊆(RoT)∩(SoT)
4) If R⊆S, then (i) RoT⊆SoT (ii) ToR⊆ToS
5) (RoS)∩T=∅ ↔ $(ToR^{-1}) \cap S = \emptyset$
6) $(SoR)^{-1}= R^{-1}o\, S^{-1}$

Theorem: Let R be a relation on a set A. R is a partially ordered relation on A if and only if $R \cap R^{-1} = I_A$ and RoR=R.

Exercises: Let S and R be two relations on a set A. Then prove or disprove the following:

1) S is transitive iff SoS⊂S.
2) If S is reflexive and transitive relation then SoS=S.
3) SoR=RoS
4) Dom(SoR)⊆ dom(R)
5) Ran (SoR))⊆ Rang (S)
6) If Ran (S)⊆ Dom(R) then (RoS)=Dom S.

Chapter four - Functions:

Theorem: A function f: A→B is invertible if and only if f is bijective function.

Theorem: Let $f: A \to B$, $g: B \to C$ and $gof: A \to C$, be functions

1. If $f: A \to B$ and $g: B \to C$ are injective functions, then $gof: A \to C$ is injective function.

2. If $f: A \to B$ and $g: B \to C$ are surjective function, then $gof: A \to C$ is surjective function.
3. If $gof: A \to C$ is injective function, then $f: A \to B$ injective functions
4. If $gof: A \to C$ is surjective function , then $g: B \to C$ is surjective function

EXERSICES:

1. Let g: B → A and h: B → C be two functions if gof=hof for every function f: A → B, then prove that $g = h$.
2. Let $f: [1, \infty] \to \mathbb{R}$ be a function defined by f(x)=$\sqrt{4x-1}$ find range of f.
3. Draw the graph of the following relations and determinate which of them are functions

a) $f = \{(x, y) |\, 2x - 4 = y\}$;

b) $g = \{(x, g(x)) \in \mathbb{R} \times \mathbb{R} \,|\, g(x) = x^2 + 4\}$;

c) $h = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, y = |x| \}$;

d) $J = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x = y^2 + 4\}$;

4. Let S and T be two non- empty sets prove that, there is a one to one correspondence between T×S and S×T.
5. Let f: A → B be a bijective function, then prove that f^{-1}: B → A is a bijective function.

Chapter five - Construction of Numbers

5.1: Cardinality of Sets

Definition: We say that two sets are **equivalent** (denoted by A ~ B) iff there exists a bijection f: A → B. It is not hard to check that ~ is an equivalence relation on the class of all sets:

(1) **A ~ A** for all sets A. (I_A: A → A is a bijection for all sets A)

(2) If **A ~ B** then **B ~ A**. (If f: A → B is a bijection, f^{-1}: B → A is also)

(3) If **A ~ B** and **B ~ C** then **A ~ C**. (If f: A → B is a bijection and g: B → C is a bijection, then g ∘ f: A → C is a bijection).

The equivalence classes under this relation are called **cardinalities.**

Example 1:

1. If $A = \{1,2,3,4,5\}$ and $B = \{4,8,12,16,20\}$ then there exists at least a bijective function $f: A \to B$ where $f(x) = 4x$. Then A ~ B.
2. If $C = \{2,3,4,...\}$ since there exists at least a bijective function $f: \mathbb{N} \to C$. where $f(x) = x$-1. Then $\mathbb{N}$ ~ C.
3. If $D = [0,1] = \{x \in \mathbb{R}; 0 \le x \le 1\}$ and $E = [1,3] = \{x \in \mathbb{R}; 1 \le x \le 3\}$ then there exists at least a bijective function $f: D \to E$ where $f(x) = 2x + 1$. Then **D ~ E**.
4. $\mathbb{R} \sim (0,\infty)$ Since there exists a bijective function $f: \mathbb{R} \to (0,\infty)$ where $f(x) = 2^x$.
5. $(0,1) \sim (1,\infty)$ Since there exists a bijective function $f: (0,1) \to (1,\infty)$ where $f(x) = \frac{1}{x}$.

Example 2: Consider three sets $A_1 = \left\{\frac{1}{n+1}; n \in \mathbb{N}\right\} = \{\frac{1}{2}, \frac{1}{3}, ...\}$, $B_1 = \left\{\frac{1}{n}; n \in \mathbb{N}\right\} = A_1 \cup \{1\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, $C_1 = A_1 \cup \{0\} = \{0, \frac{1}{2}, \frac{1}{3}, ...\}$ and $D_1 = A_1 \cup \{0,1\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$.

1. $\mathbb{N} \sim \mathbf{A_1}$ since there exists at least a bijective function $f: \mathbb{N} \to A_1$ where $f(n) = \frac{1}{n+1}$.
2. $\mathbf{A_1 \sim B_1}$ since there exists at least a bijective function $f: A_1 \to B_1$ where $f(\frac{1}{n}) = \frac{1}{n+1}$.
3. $\mathbf{A_1 \sim C_1}$ since there exists a bijective function $f: A_1 \to C_1$ where

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n+2}, n \in \mathbb{N} \end{cases}.$$

4. $\mathbf{A_1 \sim D_1}$ since there exists a bijective function $f: A_1 \to D_1$ where

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{3} \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n+3}, n \in \mathbb{N} \end{cases}.$$

5. Since $\mathbb{N} \sim \mathbf{A_1}$, $\mathbf{A_1 \sim B_1}$, $\mathbf{A_1 \sim C_1}$ and $\mathbf{A_1 \sim D_1}$ then $\mathbb{N} \sim \mathbf{A_1 \sim B_1 \sim C_1 \sim D_1}$.

Remark: Let $A = A_1 \cup A_2 \cup ... \cup A_n$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $B = B_1 \cup B_2 \cup ... \cup B_n$, $B_i \cap B_j = \emptyset$. If $A_i \sim B_i$ for all $i \in \{1, 2, ... n\}$ then A~B.

Example 3:

1. If A=(0,1) then $A = A_1 \cup A_2$ where $A_1 = \left\{\frac{1}{n+1}; n \in \mathbb{N}\right\}$ and $A_2 = \{x \in A; x \notin A_1\}$.
2. If B=(0,1] then $B = B_1 \cup B_2$ where $B_1 = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ and $B_2 = \{x \in B; x \notin B_1\}$.
3. If C=[0,1) then $C = C_1 \cup C_2$ where $C_1 = \left\{\frac{1}{n+1}; n \in \mathbb{N}\right\} \cup \{0\}$ and $C_2 = \{x \in C; x \notin C_1\}$.
4. If D=[0,1] then $D = D_1 \cup D_2$ where $D_1 = \left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \{0\}$ and $C_2 = \{x \in C; x \notin C_1\}$.

Since $A_1 \sim B_1 \sim C_1 \sim D_1$ and $A_2 = B_2 = C_2 = D_2$ then A~B~C~D. See example 2.

Example 6: Each of the following set is **denumerable:**

1. A_1, B_1, C_1 and D_1 see example 2.
2. $2\mathbb{N} = \{2,4,6,8,\dots\}$ since there exists at least a bijective function $f: \mathbb{N} \to 2\mathbb{N}$ where $f(x) = 2x$.
3. $\mathbb{Z}$ Since there exists at least a bijective function $f: \mathbb{N} \to \mathbb{Z}$ where $f(\mathrm{x}) = \begin{cases} \frac{1-\mathrm{x}}{2} & \text{if x is odd} \\ \frac{\mathrm{x}}{2} & \text{if x is even} \end{cases}$.
4. The set $\mathbb{Q}$.

Example 7: Each of the following sets are not **denumerable**

1. $\mathbb{R}$
2. $[a,b]=\{ x \in \mathbb{R}; a \le x \le b; a < b\}$ for example, [1,2].
3. $(a,b)=\{ \mathrm{x} \in \mathbb{R}; \mathrm{a} < x < b\}$ for example, (0,1).
4. The set of irrational numbers.

Remark:

1. If A is countable and $B \subseteq A$ then B is countable.
2. If A and B are two countable sets then $A \cup B$, $A \cap B$, $A - B$, and $A \triangle B$ are countable sets.
3. If B is uncountable and $B \subseteq A$ then A is uncountable.

Exercises: Show that each of pair of given sets have equal cardinality by describing a bijection from one to the other: ((0,1) and $\mathbb{R}$), $((\sqrt{2}, \infty)$ and $\mathbb{R}$), (A=$\{\dots\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, \dots\}$and $\mathbb{Z}$**),** **and (**The set of even integers and the set of odd integers).

5.2: Construction of Numbers - 1-The Natural Numbers

The Peano Axioms

Thus far we have assumed those properties of the number systems necessary to provide examples and exercises in the earlier chapters. In this chapter we propose to develop the system of numbers assuming only few of its simpler properties. These simple properties known as the **Peano's** Axioms (Postulates) after the Italian mathematician who in 1889 inaugurated the program, may state as follows:

Peano's Axioms: $\mathbb{N}$ is a set with the following properties.

Axiom **I :** $1 \in \mathbb{N}$;

AxiomII : For each $n \in \mathbb{N}$ there exists a unique element $n^+ \in \mathbb{N}$, called Successor of n in $\mathbb{N}$. $(n \in \mathbb{N} \Rightarrow n^+ \in \mathbb{N})$

AxiomIII : For each $n \in \mathbb{N}, n^+ \neq 1$;

Axiom IV(injective**):** For every $m, n \in \mathbb{N}$, if $m^+ = n^+$, then $m = n$;

Axiom V(Principle of Induction): If A is a sub set of $\mathbb{N}$, such that $1 \in A$, and if $k \in A$ implies $k^+ \in A$, then $A = \mathbb{N}$.

ADDITION ON $\mathbb{N}$: Addition(+) on $\mathbb{N}$defined by

I) $n^+ = n + 1$ for every $n \in \mathbb{N}$
II) $m + n^+ = (m + n)^+$ whenever $n + m$ is defined, $\forall m, n \in \mathbb{N}$.

MULTIPLICATION ON $\mathbb{N}$: Multiplication on $\mathbb{N}$ is defined by

I) $n.1 = n$ for every $n \in \mathbb{N}$
II) $m.n^+ = mn + m$, whenever $n.m$ is defined, $\forall m, n \in \mathbb{N}$.

Lemma: If $n \in \mathbb{N}$ and $n \neq 1$, then there exists $m \in \mathbb{N}$ such that $n = m^+$.

Or every natural number different from 1 is a successor that is

Theorem(Closed):- $m + n \in \mathbb{N}$ for every $m, n \in \mathbb{N}$.

Theorem:- For any m, n and p in natural number

1- $(m + n) + p = m + (n + p)$ (Associative law)

2- $n + 1 = 1 + n$

3- $m + n = n + m$ (Commutative law)

4- $If\ \ m + p = n + p$ then $m = n$. (Cancelation law)

5- $m^{+} + n = (m + n)^{+}$

Theorem(Closed):- $m.n \in \mathbb{N}$ for every $m, n \in \mathbb{N}$.

Theorem: - For any m, n and p in $\mathbb{N}$

1) $1.n = n.1$

2) $m^{+}.n = mn + n$

3) $m.n = n.m$ (Commutative law)

4) a- $m.(n + p) = mn + mp$ b- $(m + n).p = mp + np$

5) $(m.n).p = m.(n.p)$ (Associative law)

Theorem:- For any $m \in \mathbb{N}$.

1. If $m + n = m$ then n=0
2. If $m.n = 0\ then\ m = 0 \lor n = 0$
3. If $n.p = m.p \rightarrow n = m\ where\ p \neq 0$.

Lemma: For any $n \in \mathbb{N}$: (1) $n^{1} = n$. (2) $1^{n} = 1$.

Theorem: $\forall\, n, m\ \&\ z \in \mathbb{N}$

1)$n^{m+z} = n^{m}.n^{z}$ 2) $(n^{m})^{z} = n^{mz}$ 3)$(n.m)^{z} = n^{z}.m^{z}$

Theorem:- For any $m, n, p\ and\ q \in \mathbb{N}$

1-If $m < n \land n < p\ then\ m < p$ (< is transitive relation)

1- If $n < m \land m \leq p \rightarrow n < p$.

2- If $n \leq m \land m \leq p \rightarrow n \leq p$

3- If $m \leq n \land n \leq m \rightarrow m = n$.

4- If $n < m \rightarrow n + p < m + p$.

5- If $n \leq m \rightarrow n + p \leq m + p$.

6- If $n < m \land p < q$ then the following: a) $n + p < m + q$ b) $n.p < m.q$

7- $\sim (\exists k \in \mathbb{N} \; such\ that \; n < k < n^{+})$.

8- If $m, n \in \mathbb{N}$ then only one of the following condition is true $m < n$, $m = n$, $m > n$

10. $n < m$ if and only if $n.p < m.p \; where \; p \neq 0$

Theorem :- $\forall \, n, m$ & $z \in \mathbb{N}$,

1) $n < m \; if\ and\ only\ if \; n^z < m^z, z \neq 0$.
2) $(1 < z \land n < m) \; if\ and\ only\ if \; z^n < z^m$.

Chapter Six

Construction of Numbers (Part 2)

Definition: Addition and multiplication on $\mathbb{Z}$ will be defined respectively by

1) $[m, n] + [p, q] = [m + p, n + q]$.
2) $[m, n].[(p, q)] = [mp + nq, mq + np]$.

The Positive, Negative and Zero Integers

Since for every $m, n \in \mathbb{N}$, we have the following cases: $m = n$, $n < m$ or $m < n$

1) If $m = n$ then $[m, n] = [m, m] = [n, n]$ is called zero integer

2) If $m < n$ then $\exists u \in \mathbb{N}$ such that $m + u = n, [m, n] = [m, m + u]$ is called negative integer.
That is $\mathbb{Z}^{-} = \{[m, n]: (m, n) \in \mathbb{N} \times \mathbb{N}, m < n\}$.

3) If $n < m$ then $\exists w \in \mathbb{N}$ such that $n + w = m, [m, n] = [n + w, n]$ is called positive integer.
That is $\mathbb{Z}^{+} = \{[m, n]: (m, n) \in \mathbb{N} \times \mathbb{N}, m > n\}$.

Theorem: Let x, y and $z \in \mathbb{Z}$.

1) $x + y = y + x$

2) $x.y = y.x$

3) $(x + y) + z = x + (y + z)$.

4) $(x.y).z = x.(y.z)$.

5) $x.(y + z) = (x.y) + (x.z)$.

6) $If\ x + y = x + z\ then\ y = z$

7) $If\ x \neq 0\ and\ if\ x.y = x.z\ then\ y = z$.

Theorem:- For every $x, y \in \mathbb{Z}$

1) $x - x = 0$
2) $-(x - y) = y - x$.
3) $x - y = 0\ if\ and\ only\ if\ x = y$.
4) $x.y = 0\ then\ x = 0\ or\ y = 0$.

Remark:- $\forall\ x, y \in \mathbb{Z}$ we use

1) $x \leq y\ iff\ x < y\ or\ x = y$
2) $x \nleq y\ iff\ x \nless y\ and\ x \neq y$.
3) $x > y\ iff\ y < x$.
4) $x \geq y\ iff\ y \leq x$.

Theorem:- $Let\ x, y\ and\ w \in\ \mathbb{Z}$ then

1) $x \nless x$.
2) $If\ x < y\ and\ y < w,\ then\ x < w$.
3) $x < y\ or\ y < x\ or\ x = y$.
4) $If\ x < w\ then\ x + y < w + y$
5) $If\ x < y\ \wedge 0 < w\ then\ x.w < y.w$.

Theorem:- $For\ any\ x, y, w\ and\ u \in \mathbb{Z}$.

1) $[(x < y) \wedge (u < w)] \rightarrow x + u < y + w$
2) $[(x < y) \wedge (u \leq w)] \rightarrow x + u < y + w$

3) $[(x \leq y) \wedge (u < w)] \rightarrow x + u < y + w$

4) $[(x \leq y) \wedge (u \leq w)] \rightarrow x + u \leq y + w$

5) $[(0 < w) \wedge x.w < y.w] \rightarrow x < y$

Definition:- Let $x, y \in \mathbb{Z}$. An integer x is positive if and only if $x > 0$ and An integer y is negative if and only if $y < 0$.

Theorem:- $For\ any\ x, y, and\ w\ \in \mathbb{Z}$

1) $x < y$ if and only if $y - x$ is positive.
2) y is positive if and only if $-y$ is negative.
3) $x < y$ if and only if $-y < -x$
4) The sum and product of two positive integers are positive.
5) The product of two negative integers is positive.
6) The product of positive and negative integer is negative.
7) If $x \neq 0$, then $x^2 > 0$.

4-The Rational Numbers

The system of integers has an obvious defect in that, given integers, $m \neq 0$ and s, the equation mx=s may or may not have a solution. For example, 3x=6 has the solution x=2 but 4x=6 has no solution . This defect is remedied by adjoining to the integers additional numbers to form system $\mathbb{Q}$ of rational numbers.

Definition: Let the binary relation "≈", read " Double wave" be defined on all $\big((m,n),(p,q)\big) \in (\mathbb{Z} \times \mathbb{Z}^*) \times (\mathbb{Z} \times \mathbb{Z}^*)$ by $(m,n) \approx (p,q)$ if and only if $m.q = p.n$. Where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$.

Example:-$\big((2,-3),(-2,3)\big) \in\approx$ since 2.3=-2.-3 and $\big((4,7),(4,7)\big) \in\approx$since 4.7=4.7

Theorem:- The relation ≈ is an equivalence relation on$\mathbb{Z} \times \mathbb{Z}^*$

Definition:- The set of all equivalence classes with respected to the relation ≈ on $\mathbb{Z} \times \mathbb{Z}^*$called the set of all **rational number** and denoted by $\mathbb{Q}$

Theorem: Let x, y, w ∈ $\mathbb{Q}$

1) $x + (y + w) = (x + y) + w$

2) $x + y = y + x$

3) $x.(y + w) = x.y + x.y$

4) $x.(y.w) = (x.y).w$

5) For each $x \in \mathbb{Q}\exists -x \in \mathbb{Q}$ such that $x + (-x) = (-x) + x = 0$

6) $x.1 = 1.x = x$

7) For each $x \in \mathbb{Q}x^{-1} \in \mathbb{Q}$ such that $x.(x^{-1}) = (x^{-1}).x = 1$.

The order relation on rational number

Definition:-Let $[(m,n)],[(p,q)] \in \mathbb{Q}$ then $[(\boldsymbol{m},\boldsymbol{n})] < [(p,q)]$ iff $\boldsymbol{mq}.\boldsymbol{nq} < np.nq$.

Example: [(5,-3)]<[(0,6)] Since (5).(6).(-3).(6)<(-3).(0).(-3).(6) then (-30).(18)<0. Therefore, $[(5,-3)] < [(0,6)]$.

Theorem: For every $x, y, w \in \mathbb{Q}$

1) $x \not< x$

2) If $x < y \land y < w$ *then* $x < w$.

3) For every rational numbers x and y exactly one of the following holds

$x < y$, $x = y$, $y < x$.

4) If $x < y$ then If $x + w < y + w$

5) If $x < y$ *and* $w > 0$ *then* $x.w < y.w$.

5-Real Numbers, Rational Numbers and Complex numbers

Definition: A rational sequence $\{a_n\}$ called convergent sequence , if $\exists a_0$ such that $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ such that $|a_n - a_0| < \varepsilon, \forall n > k$. In this case we say that $\{a_n\}$ convergent to a_0 and we write $\lim_{n\to\infty} a_n = a_0$ *or* $a_n \to a_0$, and a_0 called the limit point of the sequence $\{a_n\}$.

Example 1) Consider the sequence {4}=4, 4, 4, … is converge to 4 since $\forall \varepsilon > 0$ take k=1 then |4-4|< $\varepsilon \forall n > 1$;

2) Consider the sequence $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ is convergent to 0 since $\forall \varepsilon > 0$, $\exists k \in \mathbb{N}$ such that $|\frac{1}{n} - 0| < \varepsilon, \forall n > k$;

$|\frac{1}{n}| < \varepsilon, \forall n > k$ then $\frac{1}{n} < \varepsilon, \forall n > k$ then $n > \frac{1}{\varepsilon}, \forall n > k$, take k $= \left[\!\left[\frac{1}{\varepsilon}\right]\!\right] + 1$ therefor $|\frac{1}{n} - 0| < \varepsilon$, $\forall n > \left[\!\left[\frac{1}{\varepsilon}\right]\!\right] + 1$.

Remark: If a sequence $\{a_n\}$is not convergent then it is called divergent sequence. For example {5n} is a divergent sequence.

Definition:- A sequence $\{a_n\}$ called **Cauchy sequence** if $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon, \forall m, n > k$.

Definition: Let the binary relation $\simeq$ be defined on A=$\{\{x_n\}$; rational Cauchy sequence$\}$ as follows: $(\{x_n\}, \{x_n\}) \in \simeq iff \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. That is the relation $\simeq \subseteq (A \times A)$.

Theorem:- The relation $\simeq$ is an equivalence relation on A × A.

Example:- $\left\{\frac{1}{2^n}\right\} \simeq \{\frac{1}{3^n}\}$, since $\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{1}{3^n} = 0$.

Remark: $[\{x_n\}] = \{\{y_n\}; \{x_n\} \simeq \{y_n\}\} = \{\{y_n\}; \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\}$.

Definition: - Let B be the set of all equivalence classes $[\{x_n\}]$ with respect to the equivalence relation $\simeq$, then the set of real numbers $\mathbb{R} = \{a = \lim_{n\to\infty} x_n ; [\{x_n\}] \in$B$\}$.

The real numbers (axioms)

1) For any $a, b \in \mathbb{R}$,$a + b \in \mathbb{R}$.
2) For any $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.
3) For any $a, b \in \mathbb{R}$, $a + b = b + a$.
4) There exists a unique real number (0) such that $a + 0 = 0 + a = a$, for any a$\in \mathbb{R}$.
5) For every $a \in \mathbb{R}$, there exists a unique $(-a) \in \mathbb{R}$. such that

$$a + (-a) = (-a) + a = 0$$

6) For any $a, b \in \mathbb{R}$, $a.b \in \mathbb{R}$.
7) For any $a, b \in \mathbb{R}$, $a.b = b.a$.
8) There exists a unique real number (1) such that $a.1 = 1.a = a$, for any $a \in \mathbb{R}$.
9) For every $a \in \mathbb{R} - \{0\}$, there exists a unique $(1/a) \in \mathbb{R}$. such that $a.(1/a) = (1/a).a = 1$.

10) For any $a, b, c \in \mathbb{R}$, $(a. b). c = a. (b. c)$.

11) For any $a, b, c \in \mathbb{R}$, $a. (b + c) = a. b + a. c$.

Theorem: For any $a \in \mathbb{R}$, $a. 0 = 0$

Exercise: For any $a, b, c, d \in \mathbb{R}$ and . $b, d \neq 0$ then $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$.

Irrational Numbers: A real number is irrational if it is not rational for example $\sqrt{5}$, $\sqrt[4]{7}$, ... e^2, π, ...are irrational number.

Complex Number: The system of complex number is the number of ordinary algebra . It is the smallest set in which for example, the equation $x^2=a$ can be solved when a is any element of ℝ. In our development of the set complex number, we begin with the product set $\mathbb{R} \times \mathbb{R}$. The binary relation "=" requires $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Now each of the resulting equivalence classes contains but a single element. Hence, we denote a class as (a, b) and so, hereafter, denote $\mathbb{R} \times \mathbb{R}$ by ℂ. That is $\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(x, y) | \, x, y \in \mathbb{R}\}$.

Remark:- (1) $If\ (x, y) \in \mathbb{C}$ then $x + iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ (2) $i = (0,1)$.

The Complex numbers (axioms)

1. For any $a, b \in \mathbb{C}$,$a + b \in \mathbb{C}$.
2. For any $a, b, c \in \mathbb{C}$, $(a + b) + c = a + (b + c)$.
3. For any $a, b \in \mathbb{C}$, $a + b = b + a$.
4. There exists a unique real number (0) such that $a + 0 = 0 + a = a$, for any $a \in \mathbb{C}$.
5. For every $a \in \mathbb{C}$, there exists a unique $(-a) \in \mathbb{C}$. such that $a + (-a) = (-a) + a = 0$
6. For any $a, b \in \mathbb{C}$, $a. b \in \mathbb{C}$.
7. For any $a, b \in \mathbb{C}$, $a. b = b. a$.
8. There exists a unique real number (1) such that $a. 1 = 1. a = a$, for any $a \in \mathbb{C}$.
9. For every $a \in \mathbb{C} - \{0\}$, there exists a unique $(1/a) \in \mathbb{C}$. such that $a. (1/a) = (1/a). a = 1$.
10. For any $a, b, c \in \mathbb{C}$, $(a. b). c = a. (b. c)$.
11. For any $a, b, c \in \mathbb{C}$, $a. (b + c) = a. b + a. c$.