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# The ring of Fermat reals

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**Abstract**

We give the definition of the ring of Fermat reals, a simple extension of the real field containing nilpotent infinitesimals. The construction takes inspiration from smooth infinitesimal analysis, but provides a powerful theory of actual infinitesimals without any need of a background in mathematical logic. In particular it is consistent with classical logic. We face the problem to decide if the product of powers of nilpotent infinitesimals is zero or not, the identity principle for polynomials, the characterization of invertible elements and some applications to Taylor's formulas.


*Keywords:* Actual infinitesimals; Nilpotent infinitesimals; Smooth infinitesimal analysis; Extension of the real field

## 1. Introduction and general problem

In physics one often makes use of informal calculations like

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=1+\frac{v^2}{2c^2}, \qquad \sqrt{1-h_{44}(x)}=1-\frac{1}{2}h_{44}(x) \tag{1}$$

with explicit use of infinitesimals $v/c \ll 1$ or $h_{44}(x) \ll 1$, such that, e.g., $h_{44}(x)^2=0$. For example, the following formula can be found on page 14 of [12] (using the equality sign and not the approximate equality sign)

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$$f(x, t+\tau) = f(x,t) + \tau \cdot \frac{\partial f}{\partial t}(x,t) \tag{2}$$

and was justified using the words "*since* $\tau$ *is very small*". The formulas (1) are a specific instance of the general equation (2). In [9], there is an analogous equality applied to the Newtonian approximation in general relativity.

Using this type of infinitesimals, we can write an *equality*, in some infinitesimal neighborhood, between a smooth function and its tangent straight line; in other words, Taylor's formula without a remainder. Informal methods based on actual infinitesimals are also sometimes used in differential geometry. Some classical examples are the following: (1) tangent vectors are infinitesimal arc of curves traced on a manifold; (2) tangent vectors can be summed using infinitesimal parallelograms; (3) tangent vectors to the tangent bundle are viewed as infinitesimal squares on the manifold; (4) vector fields are sometimes intuitively treated as "infinitesimal transformations" of the space into itself; (5) the Lie brackets of two vector fields are thought as the commutator of the corresponding infinitesimal transformations.

There are obviously many ways to formalize these intuitive reasonings, to obtain a more or less good dialectic between informal and formal thinking. Indeed, there are several theories of actual infinitesimals (for simplicity, we will say "infinitesimals" instead of "actual infinitesimals" as opposed to "potential infinitesimals"; see e.g. [11] and its historical references for an explanation of this terminology). Starting from these theories, we can distinguish between two definitions of infinitesimals: in the first definition there is at least a ring $R$ containing the real field $\mathbb{R}$, and infinitesimals are elements $\varepsilon \in R$ such that $-r < \varepsilon < r$ for every positive standard real $r \in \mathbb{R}_{>0}$. In the second definition, infinitesimals are defined using an algebraic property of nilpotency, i.e. $\varepsilon^n = 0$ for some natural number $n \in \mathbb{N}$. For some types of rings $R$ these definitions can coincide, but in any event they lead only to the trivial infinitesimal $\varepsilon = 0$ if $R = \mathbb{R}$.

However, these two definitions of infinitesimals correspond to separate theories which differ completely in nature and in their underlying ideas. Indeed, these theories can be seen in a more interesting way as belonging to two different classes. In the first one, there are theories that need a certain amount of non-trivial results of mathematical logic; in the second one, there are attempts to define sufficiently strong theories of infinitesimals without the use of non-trivial results of mathematical logic. The first class includes non-standard analysis (NSA) and synthetic differential geometry (SDG, also called smooth infinitesimal analysis), and the second class includes Weil functors, Levi-Civita fields, surreal numbers and geometries over rings containing infinitesimals. More precisely, we can say that to work in NSA and SDG, one needs a formal control that is stronger than the one used in "standard mathematics". In NSA, this control is used to apply the transfer theorem, and in SDG, it must be sufficiently strong to ensure that the proofs belong to intuitionistic logic. Indeed, to use NSA one must be able to formally write the sentences that need to be transferred. On the other hand, since SDG only admits models in intuitionistic logic, we must ensure that our proofs do not use the law of the excluded middle, the classical part of De Morgan's law, some form of the axiom of choice, the implication of double negation toward affirmation, or any other logical principle which is not valid in intuitionistic logic. Physicists, engineers, and even the majority of mathematicians are not used to having this strong formal control in their work; thus, there are attempts to present both NSA and SDG while reducing the necessary formal control as much as possible, even if at some level this is technically impossible. For examples, see [18,4,5] for NSA, and [2,21] for SDG.

In spite of these constraints, NSA is essentially the only theory of infinitesimals with discrete diffusion and a sufficiently large community of working mathematicians publishing results in mathematics and its applications, see [1]. SDG is the only theory of infinitesimals with non-trivial, new and published results in differential geometry concerning infinite dimensional spaces, such as the space of all the diffeomorphisms of a generic (e.g., non-compact) smooth manifold. In NSA we have only a few results concerning differential geometry (see e.g. [26,17], and references therein). Other theories of infinitesimals, at least up to now, do not have the same formal strength of NSA or SDG, nor the same potentiality to be applied in several different areas of mathematics.

One of the aims of the present work is to find a theory of infinitesimals within "standard mathematics" (in the precise sense explained above, with a formal control that is more "standard" and not as strong as the one needed in NSA or SDG), with results comparable with those of SDG. In other words, we do not want to force the reader to learn a stronger formal control of the mathematics he/she is doing. Because it should be included within "standard mathematics", our theory of infinitesimals must be compatible with classical logic. We note that this is not incompatible with results needing a strong formal control (such as a transfer theorem); the theory should be a valid instrument for readers that prefer a strong formal control, but should not concretely force all readers to have such a formal aptitude. For these reasons, we do not wish to frame the present work as in opposition to NSA or SDG. To emphasize this lack of opposition, we note that further development of the present theory of Fermat reals (more precisely, the extension of this method to add new infinitesimal points to diffeological spaces, see [15]) reveals that intuitionistic logic results in the greatest simplification. This further underscores that the aim of the theory of Fermat reals is not simply to develop a classical alternative to SDG, but to develop a theory of nilpotent infinitesimals that, because of its simplicity and intuitive strength, *can* be used in classical logic. This includes the possibility to obtain, for example following ideas similar to those presented in this article, an intuitionistic topos whose simplicity permits to study the model *also* in classical logic, without constraining every reader to use intuitionistic logic. The use of the internal logic remaining a positive feature for a selection of readers.

We can hence frame our construction in the problem proposed by [23]:

> In recent years, several alternative solutions to the problem of generalizing manifolds to include function spaces and spaces with singularities have been proposed in the literature. A particularly appealing one is the theory of convenient vector spaces [...]. These structures are in a way simpler than the sheaves considered in this book, but one should notice that the theory of convenient vector spaces does not include an attempt to develop an appropriate framework for infinitesimal structures, which is one of the main motivations of our approach.

Another point of view about a powerful theory like NSA is that, in spite of the fact that it is often presented using opposed motivations, it lacks the intuitive interpretation of what its powerful formalism permits. For example, what is the intuitive meaning and usefulness of $^\circ\sin(I) \in \mathbb{R}$, i.e., the standard part of the sine of an infinite number $I \in {}^*\mathbb{R}$? This, and the above-mentioned "strong formal control" needed to work in NSA, combined with very strong but scientifically unjustified cultural reasons, may explain the lack of NSA penetration in mathematics, and consequently in its didactics.

Analogously, in SDG from the intuitive and classical point of view, it is odd that we cannot exhibit "examples" of infinitesimals (indeed, in SDG it is only possible to prove that $\neg\neg\exists d \in D$, where $D = \{h \in R \mid h^2 = 0\}$ is the set of first order infinitesimals). Another example of a counter-intuitive property is that any $d \in D$ is simultaneously both positive $d \geqslant 0$ and negative $d \leqslant 0$ (of course one cannot conclude that $d = 0$ because in SDG we only have a partial order and not an order relation). Due to this property, one cannot construct a physical theory containing a fixed infinitesimal parameter. An example is in [30], in which Planck's constant $\hbar$ is taken as an infinitesimal (which must be positive and not negative in the mind of almost every physicist), thereby allowing classic mechanics to be deduced from quantum mechanics. Similar counter-intuitive properties (from the point of view of a physicist, because from a formal point of view they are perfectly acceptable) can be found in other theories of infinitesimals, such as those using ideals of rings of polynomials as a formal scheme to construct a particular type of infinitesimal. Among these theories are "Weil functors" [19,20] and "differential geometry over general base fields and rings", see [6]. The final conclusion after the establishment of this type of counter-intuitive example (even if these theories include several intuitively clear examples and concepts) is that these types of frameworks require one to sometimes follow a formal point of view, losing the dialectic with the corresponding intuitive meaning. For example, in our opinion, it seems difficult to formalize the frequent use of intuitive drawings of infinitesimal quantities used in physics without a total order relation in the ring of scalars, because these quantities are frequently drawn as small segments, and hence they can be considered comparable elements with respect to order.

Another aim of the present work is to construct a new theory of infinitesimals that always preserves a good dialectic between formal properties and intuitive interpretation. As we will see in this and in subsequent articles, this can be done faithfully. To provide an example, the ring ${}^\bullet\mathbb{R}$ can be represented geometrically, using a total order preserving monomorphism $\varphi : ({}^\bullet\mathbb{R}, \leqslant) \to (\mathcal{F}, \preccurlyeq)$, where $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R}^2)$ is a suitable family of lines of the plane $\mathbb{R}^2$ (for a definition of the total order relation on the ring of Fermat reals, see [15]).

Technically, we want to show that it is possible to extend the real field by adding nilpotent infinitesimals, thus arriving at an enlarged real line ${}^\bullet\mathbb{R}$, by means of a very simple construction. Indeed, to define the extension ${}^\bullet\mathbb{R} \supset \mathbb{R}$ we use elementary analysis only.

To avoid misunderstandings we clarify that the purpose of the present work is not to provide an alternative foundation to differential and integral calculus (like NSA), but to develop a theory of nilpotent infinitesimals as a first step in the foundation of a smooth ($\mathcal{C}^\infty$) differential geometry. In particular, our theory should apply to infinite dimensional spaces, like the space of all smooth functions $\mathbf{Man}(M; N)$ between two generic manifolds (e.g., without using a compactness hypothesis on the domain $M$). Our focus on the foundation of differential geometry, excluding the whole calculus, is typical of SDG, Weil functors, and geometries over generic rings. Some preliminary results in this direction are provided in [15]. A more complete comparison between theories of infinitesimals can be found in [3] and in Appendix B of [15].

## 2. Motivations for the name "Fermat reals"

It is well known that historically two possible reductionist constructions of the real field starting from the rationals have been proposed. The first is Dedekind's order completion using sections of rationals, the second is Cauchy's metric space completion. While there is no historical reason to attribute our extension ${}^\bullet\mathbb{R} \supset \mathbb{R}$ of the real field (described below) to Fermat, it is highly likely that he would have liked the underlying spirit and some of the properties of our theory. For example:

1. A formalization of Fermat's infinitesimal method for deriving functions is provable in our theory. We recall that Fermat's idea was to suppose first that $h \neq 0$, thereby constructing the incremental ratio

$$\frac{f(x+h)-f(x)}{h}$$

and, after suitable simplifications (sometimes using infinitesimal properties), to take in the final result $h=0$. Note that this idea is not derived from an accurate historical analysis, which would be beyond the scope of the present work (e.g., see [7,10,13]).

2. Fermat's method of finding the maximum or minimum of a given function $f(x)$ at $x=a$ was to assume that $h$ is extremely small, so that the value of $f(x+h)$ was approximately equal to that of $f(x)$. In modern, algebraic language, it can be said that $f(x+h)=f(x)$ only if $h^2=0$, that is, if $h$ is a first order infinitesimal. Fermat was aware that this is not a "true" equality, but is some kind of approximation [7,10,13]. We will follow a similar idea, in that we define ${}^\bullet\mathbb{R}$ introducing a suitable equivalence relation to represent this equality.

## 3. Definition and algebraic properties of Fermat reals: The basic idea

We start from the idea that a smooth ($\mathcal{C}^\infty$) function $f : {}^\bullet\mathbb{R} \to {}^\bullet\mathbb{R}$ is actually *equal* to its tangent straight line in the first order neighborhood e.g. of the point $x=0$. Formally, we wish to write

$$\forall h \in D\colon \quad f(h) = f(0) + h \cdot f'(0) \tag{3}$$

where $D$ is the subset of ${}^\bullet\mathbb{R}$ which defines the above-mentioned neighborhood of $x=0$. The equality (3) can be seen as a first order Taylor's formula without remainder, because intuitively we think that $h^2=0$ for any $h \in D$ (indeed the property $h^2=0$ defines the first order neighborhood of $x=0$ in ${}^\bullet\mathbb{R}$). These almost trivial considerations lead us to understand many things: ${}^\bullet\mathbb{R}$ must necessarily be a ring and not a field because in a field the equation $h^2=0$ implies $h=0$; moreover we will surely have some limitation in the extension of some function from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. For example in the extension of the square root, because using this function with the usual properties, the equation $h^2=0$ implies $|h|=0$. On the other hand, we are also led to ask whether (3) uniquely determines the derivative $f'(0)$. Indeed, even if it is true that we cannot simplify by $h$, we know that the polynomial coefficients of Taylor's formula are unique in classical analysis. In fact, we will prove that

$$\exists!\, m \in \mathbb{R}\ \forall h \in D\colon \quad f(h) = f(0) + h \cdot m \tag{4}$$

that is the slope of the tangent is uniquely determined in case it is an ordinary real number. We will call formulas like (4) *derivation formula*s.

If we try to construct a model for the formula (4), a natural idea is to think our new numbers in ${}^\bullet\mathbb{R}$ as equivalence classes $[h]$ of usual functions $h : \mathbb{R} \to \mathbb{R}$. In this way, we may hope both to include the real field using classes generated by constant functions, and that the class generated by $h(t)=t$ could be a first order infinitesimal number.

**Remark 1.** Sometimes, but not always, we will use a notation like $h_t := h(t)$ for real functions of the real variable $t$. This permits to decrease the number of parenthesis used in formulas and to leave the classical notation $f(x)$ for functions of the form $f : {}^\bullet\mathbb{R} \to {}^\bullet\mathbb{R}$.

To understand how to define this equivalence relation we have to think at (3) in the following sense:

$$f(h_t) \sim f(0) + h_t \cdot f'(0), \tag{5}$$

where the idea is that we are going to define $\sim$. If we think $h_t$ "sufficiently similar to $t$", we can define $\sim$ so that (5) is equivalent to

$$\lim_{t \to 0^+} \frac{f(h_t) - f(0) - h_t \cdot f'(0)}{t} = 0,$$

that is

$$x \sim y \quad :\Longleftrightarrow \quad \lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0. \tag{6}$$

In this way, (5) is very near to the definition of differentiability for $f$ at 0.

It is important to note that, because of de L'Hôpital's theorem, we have the isomorphism

$$\mathcal{C}^1(\mathbb{R}, \mathbb{R})/\sim \;\simeq\; \mathbb{R}[x]/(x),$$

the left-hand side is (isomorphic to) the usual tangent bundle of $\mathbb{R}$ and thus we obtain nothing new. It is not easy to understand what set of functions we have to choose for $x$, $y$ in (6) so as to obtain a non-trivial structure. The first idea is to take continuous functions at $t = 0$, instead of more regular ones like $\mathcal{C}^1$-functions. In this way, we have that, e.g., $h_k(t) = |t|^{1/k}$ becomes a $k$-th order nilpotent infinitesimal because $h^{k+1} \sim 0$. For almost all the results presented in this article, continuous functions at $t = 0$ work well. However, only in proving the non-trivial property

$$\bigl(\forall x \in {}^\bullet\mathbb{R}\colon\ x \cdot f(x) = 0\bigr) \quad \Longrightarrow \quad \forall x \in {}^\bullet\mathbb{R}\colon\ f(x) = 0 \tag{7}$$

we can see that it does not suffice to take continuous functions at $t = 0$. To prove (7) the functions defined in the following Definition 3 turned out to be very useful.

**Remark 2.** In the following, we will use a slight modification of Landau's little-oh notation: writing $x_t = y_t + o(t)$ as $t \to 0^+$ we will always mean

$$\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0 \quad \text{and} \quad x_0 = y_0 \in \mathbb{R}.$$

In other words, every little-oh function we will consider is continuous as $t \to 0^+$.

**Definition 3.** If $x : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, then we say that $x$ *is nilpotent* iff $|x(t) - x(0)|^k = o(t)$ as $t \to 0^+$, for some $k \in \mathbb{N}$. $\mathcal{N}$ will denote the set of all the nilpotent functions.

For example, any Hölder function $|x(t) - x(s)| \leqslant c \cdot |t - s|^\alpha$ (for some constant $\alpha > 0$) is nilpotent. The choice of nilpotent functions, instead of more regular ones, establishes a great difference of our approach with respect to the classical definition of jets (see e.g. [8,16]), that (6) may recall. Indeed, in our approach all the $\mathcal{C}^1$-functions $x$ with the same value and derivative at $t = 0$ generate the same $\sim$-equivalence relation. Only a non-differentiable function at $t = 0$ like $x(t) = \sqrt{t}$ generates non-trivial nilpotent infinitesimals.

Another problem, necessarily connected with the basic idea (3), is that the use of nilpotent infinitesimals frequently leads to consider terms like $h_1^{i_1} \cdot \dots \cdot h_n^{i_n}$. For this type of products, the first problem is to know whether $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} \neq 0$ and what is the order $k$ of this new infinitesimals, that is for what $k$ we have $(h_1^{i_1} \cdot \dots \cdot h_n^{i_n})^k \neq 0$ but $(h_1^{i_1} \cdot \dots \cdot h_n^{i_n})^{k+1} = 0$. We will have a good frame if we will be able to solve these problems starting from the order of each infinitesimal $h_j$ and from the values of the powers $i_j \in \mathbb{N}$. On the other hand, almost all the examples of nilpotent infinitesimals are sums of terms of the form $h(t) = t^\alpha$, with $0 < \alpha < 1$. These functions have also very good properties in dealing with products of powers. It is for these reasons that we shall focus our attention on the following family of functions $x : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ in the definition (6) of $\sim$:

**Definition 4.** We say that $x$ *is a little-oh polynomial*, and we write $x \in \mathbb{R}_o[t]$ iff

1. $x : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$.
2. We can write

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \text{as } t \to 0^+$$

for suitable

$$k \in \mathbb{N},$$
$$r, \alpha_1, \dots, \alpha_k \in \mathbb{R},$$
$$a_1, \dots, a_k \in \mathbb{R}_{\geqslant 0}.$$

Hence, a little-oh polynomial[1] $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \geqslant 0$, with generic positive powers of $t$, and up to a little-oh function as $t \to 0^+$.

**Example.** Simple examples of little-oh polynomials are the following:

1. $x_t = 1 + t + t^{1/2} + t^{1/3} + o(t)$.
2. $x_t = r \ \forall t$. Note that in this example we can take $k = 0$, and hence $\alpha$ and $a$ are the void sequence of reals, that is the function $\alpha = a : \emptyset \to \mathbb{R}$, if we think of an $n$-tuple $x$ of reals as a function $x : \{1, \dots, n\} \to \mathbb{R}$.
3. $x_t = r + o(t)$.

## 4. First properties of little-oh polynomials

### *4.1. Little-oh polynomials are nilpotent*

First properties of little-oh polynomials are the following: if $x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o_1(t)$ as $t \to 0^+$ and $y_t = s + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + o_2(t)$, then $(x + y) = r + s + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} +$

[1] Actually in the following notation the variable $t$ is mute.

$o_3(t)$ and $(x \cdot y)_t = rs + \sum_{i=1}^k s\alpha_i \cdot t^{a_i} + \sum_{j=1}^N r\beta_j \cdot t^{b_i} + \sum_{i=1}^k \sum_{j=1}^N \alpha_i \beta_j \cdot t^{a_i} t^{b_j} + o_4(t)$, hence the set of little-oh polynomials is closed with respect to pointwise sum and product. Moreover, little-oh polynomials are nilpotent functions (see Definition 3); to prove this, we firstly prove that the set of nilpotent functions $\mathcal{N}$ is a subalgebra of the algebra $\mathbb{R}^{\mathbb{R}}$ of real valued functions. Indeed, let $x$ and $y$ be two nilpotent functions such that $|x - x(0)|^k = o_1(t)$ and $|y - y(0)|^N = o_2(t)$, then we can write $x \cdot y - x(0) \cdot y(0) = x \cdot [y - y(0)] + y(0) \cdot [x - x(0)]$, so that we can consider $|x \cdot [y - y(0)]|^k = |x|^k \cdot |y - y(0)|^k = |x|^k \cdot o_1(t)$ and $\frac{|x|^k \cdot o_1(t)}{t} \to 0$ as $t \to 0^+$ because $|x|^k \to |x(0)|^k$, hence $x \cdot [y - y(0)] \in \mathcal{N}$. Analogously, $y(0) \cdot [x - x(0)] \in \mathcal{N}$ and hence the closure of $\mathcal{N}$ with respect to the product follows from the closure with respect to the sum. The case of the sum follows from the following equalities (where we use $u := x - x_0$, $v := y - y_0$, $|u_t|^k = o_1(t)$ and $|v_t|^N = o_2(t)$ and we have supposed $k \geqslant N$):

$$u^k = o_1(t), \qquad v^k = o_2(t),$$

$$(u + v)^k = \sum_{i=0}^k \binom{k}{i} u^i \cdot v^{k-i},$$

$$\forall i = 0, \dots, k\colon \quad \frac{u_t^i \cdot v_t^{k-i}}{t} = \frac{(u_t^k)^{\frac{i}{k}} \cdot (v_t^k)^{\frac{k-i}{k}}}{t^{\frac{i}{k}} \cdot t^{\frac{k-i}{k}}} = \left(\frac{u_t^k}{t}\right)^{\frac{i}{k}} \cdot \left(\frac{v_t^k}{t}\right)^{\frac{k-i}{k}}.$$

Now we can prove that $\mathbb{R}_o[t]$ is a subalgebra of $\mathcal{N}$. Indeed, every constant $r \in \mathbb{R}$ and every power $t^{a_i}$ are elements of $\mathcal{N}$ and hence $r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} \in \mathcal{N}$, so it remains to prove that if $y \in \mathcal{N}$ and $w = o(t)$, then $y + w \in \mathcal{N}$, but this is a consequence of the fact that every little-oh function is trivially nilpotent, and hence it follows from the closure of $\mathcal{N}$ with respect to the sum.

### 4.2. *Closure of little-oh polynomials with respect to smooth functions*

Now, we want to prove that little-oh polynomials are preserved by smooth functions. That is, if $x \in \mathbb{R}_o[t]$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth, then $f \circ x \in \mathbb{R}_o[t]$. Let us fix some notations:

$$x_t = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + w(t) \quad \text{with } w(t) = o(t),$$

$$h(t) := x(t) - x(0) \quad \forall t \in \mathbb{R}_{\geqslant 0},$$

hence $x_t = x(0) + h_t = r + h_t$. The function $t \mapsto h(t) = \sum_{i=1}^k \alpha_i \cdot t^{a_i} + w(t)$ belongs to $\mathbb{R}_o[t] \subseteq \mathcal{N}$ so we can write $|h|^N = o(t)$ for some $N \in \mathbb{N}$ and as $t \to 0^+$. From Taylor's formula, we have

$$f(x_t) = f(r + h_t) = f(r) + \sum_{i=1}^N \frac{f^{(i)}(r)}{i!} \cdot h_t^i + f(x_t) = f(r + h_t) \tag{8}$$

$$= f(r) + \sum_{i=1}^N \frac{f^{(i)}(r)}{i!} \cdot h_t^i + o\big(h_t^N\big). \tag{9}$$

But

$$\frac{|o(h_t^N)|}{|t|} = \frac{|o(h_t^N)|}{|h_t^N|} \cdot \frac{|h_t^N|}{|t|} \to 0,$$

hence $o(h_t^N) = o(t) \in \mathbb{R}_o[t]$. From this, the formula (8), the fact that $h \in \mathbb{R}_o[t]$ and using the closure of little-oh polynomials with respect to ring operations, the conclusion $f \circ x \in \mathbb{R}_o[t]$ follows.

## 5. Equality and decomposition of Fermat reals

**Definition 5.** Let $x, y \in \mathbb{R}_o[t]$, then we say that $x \sim y$ or that $x = y$ in ${}^\bullet\mathbb{R}$ iff $x(t) = y(t) + o(t)$ as $t \to 0^+$. Because it is easy to prove that $\sim$ is an equivalence relation, we can define ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$, i.e. ${}^\bullet\mathbb{R}$ is the quotient set of $\mathbb{R}_o[t]$ with respect to the equivalence relation $\sim$.

The equivalence relation $\sim$ is a congruence with respect to pointwise operations, hence ${}^\bullet\mathbb{R}$ is a commutative ring. Where it will be useful to simplify notations, we will write "$x = y$ in ${}^\bullet\mathbb{R}$" instead of $x \sim y$, and we will talk directly about the elements of $\mathbb{R}_o[t]$ instead of their equivalence classes; for example we can say that $x = y$ in ${}^\bullet\mathbb{R}$ and $z = w$ in ${}^\bullet\mathbb{R}$ imply $x + z = y + w$ in ${}^\bullet\mathbb{R}$.

The immersion of $\mathbb{R}$ in ${}^\bullet\mathbb{R}$ is $r \mapsto \hat{r}$ defined by $\hat{r}(t) := r$, and in the sequel we will always identify $\hat{\mathbb{R}}$ with $\mathbb{R}$, which is hence a subring of ${}^\bullet\mathbb{R}$. Conversely, if $x \in {}^\bullet\mathbb{R}$ then the map ${}^\circ(-)$ : $x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$, which evaluates each Fermat real in 0, is well defined. We shall call ${}^\circ(-)$ the *standard part map*. Let us also note that, as a vector space over the field $\mathbb{R}$ we have $\dim_{\mathbb{R}} {}^\bullet\mathbb{R} = \infty$, and this underscores even more the difference of our approach with respect to the classical definition of jets (see e.g. [8,16]). Our idea is instead more near to NSA, where standard sets can be extended adding new infinitesimal points, and this is not the point of view of jet theory.

With the following theorem we will introduce the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$, that is a unique notation for its standard part and all its infinitesimal parts.

**Theorem 6.** *If $x \in {}^\bullet\mathbb{R}$, then there exists one and only one sequence*

$$(k, r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k)$$

*such that*

$$k \in \mathbb{N},$$

$$r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k \in \mathbb{R}$$

*and*

1. $x = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i}$ *in* ${}^\bullet\mathbb{R}$.
2. $0 < a_1 < a_2 < \dots < a_k \leqslant 1$.
3. $\alpha_i \neq 0 \ \forall i = 1, \dots, k$.

In this statement we have also to include the void case $k = 0$ and $\alpha = a : \emptyset \to \mathbb{R}$. Obviously, as usual, we use the definition $\sum_{i=1}^0 b_i = 0$ for the sum of an empty set of numbers. As we will see, this is the case where $x$ is a standard real, i.e. $x \in \mathbb{R}$.

In the following we will use the notations $t^a \sim \mathrm{d}t_{1/a} := [t \in \mathbb{R}_{\geqslant 0} \mapsto t^a \in \mathbb{R}]_\sim \in {}^\bullet\mathbb{R}$ so that e.g. $\mathrm{d}t_2 = t^{1/2}$ (in ${}^\bullet\mathbb{R}$) is a second order infinitesimal.[2] In general, as we will see from the definition of order of a generic infinitesimal, $\mathrm{d}t_a$ is an infinitesimal of order $a$. In other words these two notations for the same object permit to emphasize the difference between an actual infinitesimal $\mathrm{d}t_a \in {}^\bullet\mathbb{R}$ and a potential infinitesimal $t^{1/a} \in \mathbb{R}_o[t]$: an actual infinitesimal of order $a \geqslant 1$ corresponds, through the passage to the $\sim$ equivalence class, to a potential infinitesimal of order $\frac{1}{a} \leqslant 1$ (with respect to the classical notion of order of an infinitesimal function from calculus, see e.g. [24,28]).

**Remark 7.** Let us note that $\mathrm{d}t_a \cdot \mathrm{d}t_b = \mathrm{d}t_{\frac{ab}{a+b}}$, moreover $\mathrm{d}t_a^\alpha := (\mathrm{d}t_a)^\alpha = \mathrm{d}t_{\frac{a}{\alpha}}$ for every $\alpha \geqslant 1$, and finally $\mathrm{d}t_a = 0$ for every $a < 1$. For example, $\mathrm{d}t_a^{[a]+1} = 0$ for every $a \in \mathbb{R}_{>0}$, where $[a] \in \mathbb{N}$ is the integer part of $a$, i.e. $[a] \leqslant a < [a] + 1$.

**Existence proof.** Since $x \in \mathbb{R}_o[t]$, we can write $x_t = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o(t)$ as $t \to 0^+$, where $r, \alpha_i \in \mathbb{R}$, $a_i \in \mathbb{R}_{\geqslant 0}$ and $k \in \mathbb{N}$. Hence, $x = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i}$ in ${}^\bullet\mathbb{R}$ and our purpose is to pass from this representation of $x$ to another one that satisfies conditions 1, 2 and 3 of the statement. Since if $a_i > 1$ then $\alpha_i \cdot t^{a_i} = 0$ in ${}^\bullet\mathbb{R}$, we can suppose that $a_i \leqslant 1$ for every $i = 1, \dots, k$. Moreover, we can also suppose $a_i > 0$ for every $i$, because otherwise, if $a_i = 0$, we can replace $r \in \mathbb{R}$ by $r + \sum\{\alpha_i \mid a_i = 0,\ i = 1, \dots, k\}$.

Now, we sum all the terms $t^{a_i}$ having the same $a_i$, that is we can consider

$$\bar{\alpha_i} := \sum\{\alpha_j \mid a_j = a_i,\ j = 1, \dots, k\}$$

so that in ${}^\bullet\mathbb{R}$ we have

$$x = r + \sum_{i \in I} \bar{\alpha_i} \cdot t^{a_i}$$

where $I \subseteq \{1, \dots, k\}$, $\{a_i \mid i \in I\} = \{a, \dots, a_k\}$ and $a_i \neq a_j$ for any $i, j \in I$ with $i \neq j$. Neglecting $\bar{\alpha_i}$ if $\bar{\alpha_i} = 0$ and renaming $a_i$, for $i \in I$, in such a way that $a_i < a_j$ if $i, j \in I$ with $i < j$, we obtain the existence result. Note that if $x = r \in \mathbb{R}$, in the final step of this proof we have $I = \emptyset$.

**Uniqueness proof.** Let us suppose that in ${}^\bullet\mathbb{R}$ we have

$$x = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} = s + \sum_{j=1}^N \beta_j \cdot t^{b_j} \tag{10}$$

where $\alpha_i$, $\beta_j$, $a_i$ and $b_j$ verify the conditions of the statement. First of all ${}^\circ x = x(0) = r = s$ because $a_i, b_j > 0$. Hence, $\alpha_1 t^{a_1} - \beta_1 t^{b_1} + \sum_i \alpha_i \cdot t^{a_i} - \sum_j \beta_j \cdot t^{b_j} = o(t)$. By reduction to the absurd, if we had $a_1 < b_1$, then collecting the term $t^{a_1}$ we would have

[2] Let us point out that we make hereby an innocuous abuse of language using the same notation both for the value of the function, $t^a \in \mathbb{R}$, and for the equivalence class, $t^a \in {}^\bullet\mathbb{R}$.

$$\alpha_1 - \beta_1 t^{b_1-a_1} + \sum_i \alpha_i \cdot t^{a_i-a_1} - \sum_j \beta_j \cdot t^{b_j-a_1} = \frac{o(t)}{t} \cdot t^{1-a_1}. \tag{11}$$

In (11) we have that $\beta_1 t^{b_1-a_1} \to 0$ for $t \to 0^+$ because $a_1 < b_1$ by hypothesis; $\sum_i \alpha_i \cdot t^{a_i-a_1} \to 0$ because $a_1 < a_i$ for $i = 2, \ldots, k$; $\sum_j \beta_j \cdot t^{b_j-a_1} \to 0$ because $a_1 < b_1 < b_j$ for $j = 2, \ldots, N$, and finally $t^{1-a_1}$ is limited because $a_1 \leqslant 1$. Hence, for $t \to 0^+$ we obtain $\alpha_1 = 0$, which conflicts with condition 3 of the statement. We can argue, in a corresponding way, if we had $b_1 < a_1$. In this way, we see that we must have $a_1 = b_1$. From this and from Eq. (11) we obtain

$$\alpha_1 - \beta_1 + \sum_i \alpha_i \cdot t^{a_i-a_1} - \sum_j \beta_j \cdot t^{b_j-a_1} = \frac{o(t)}{t} \cdot t^{1-a_1} \tag{12}$$

and hence for $t \to 0^+$ we obtain $\alpha_1 = \beta_1$. We can now restart from (12) to prove, in the same way, that $a_2 = b_2$, $\alpha_2 = \beta_2$, etc. At the end we must have $k = N$ because, otherwise, if we had e.g. $k < N$, at the end of the previous recursive process, we would have

$$\sum_{j=k+1}^{N} \beta_j \cdot t^{b_j} = o(t).$$

From this, collecting the terms containing $t^{b_{k+1}}$, we obtain

$$t^{b_{k+1}-1} \cdot \left[\beta_{k+1} + \beta_{k+2} \cdot t^{b_{k+2}-b_{k+1}} + \cdots + \beta_N \cdot t^{\beta_N-\beta_{k+1}}\right] \to 0. \tag{13}$$

In this sum $\beta_{k+j} \cdot t^{b_{k+j}-b_{k+1}} \to 0$ as $t \to 0^+$, because $b_{k+1} < b_{k+j}$ for $j > 1$ and hence $\beta_{k+1} + \beta_{k+2} \cdot t^{b_{k+2}-b_{k+1}} + \cdots + \beta_N \cdot t^{\beta_N-\beta_{k+1}} \to \beta_{k+1} \neq 0$, so from (13) we get $t^{b_{k+1}-1} \to 0$, that is $b_{k+1} > 1$, in contradiction with the uniqueness hypothesis $b_{k+1} \leqslant 1$.

Let us note explicitly that the uniqueness proof permits also to affirm that the decomposition is well defined in ${}^\bullet\mathbb{R}$, i.e. that if $x = y$ in ${}^\bullet\mathbb{R}$, then the decomposition of $x$ and the decomposition of $y$ are equal. □

On the basis of this theorem, we introduce two notations: the first one emphasizing the potential nature of an infinitesimal $x \in {}^\bullet\mathbb{R}$, and the second one emphasizing its actual nature.

**Definition 8.** If $x \in {}^\bullet\mathbb{R}$, we say that

$$x = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} \quad \text{is the potential decomposition (of } x\text{)} \tag{14}$$

iff conditions 1, 2, and 3 of Theorem 6 are verified. Of course it is implicit that the symbol of equality in (14) has to be understood in ${}^\bullet\mathbb{R}$.

For example $x = 1 + t^{1/3} + t^{1/2} + t$ is a decomposition because we have increasing powers of $t$. The only decomposition of a standard real $r \in \mathbb{R}$ is the void one, i.e. that with $k = 0$ and $\alpha = a : \emptyset \to \mathbb{R}$; indeed, to see that this is the case, it suffices to go along the existence proof again with this case $x = r \in \mathbb{R}$ (or to prove it directly, e.g. by contradiction).

**Definition 9.** Considering that $t^{a_i} = \mathrm{d}t_{1/a_i}$ (in ${}^\bullet\mathbb{R}$) we can also use the following notation, emphasizing more the fact that $x \in {}^\bullet\mathbb{R}$ is an actual infinitesimal:

$$x = {}^\circ x + \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{b_i} \tag{15}$$

where we have used the notation ${}^\circ x_i := \alpha_i$ and $b_i := 1/a_i$. In this way, the condition that uniquely identifies all $b_i$ is $b_1 > b_2 > \cdots > b_k \geqslant 1$. We call (15) the *actual decomposition* of $x$ or simply the *decomposition* of $x$. We will also use the notation $\mathrm{d}^i x := {}^\circ x_i \cdot \mathrm{d}t_{b_i}$ (and simply $\mathrm{d}x := \mathrm{d}^1 x$) and we will call ${}^\circ x_i$ the *$i$-th standard part of $x$* and $\mathrm{d}^i x$ the *$i$-th infinitesimal part of $x$* or the $i$-th *differential* of $x$. So let us note that we can also write

$$x = {}^\circ x + \sum_i \mathrm{d}^i x$$

and in this notation all the addenda are uniquely determined (the number of them too). Finally, if $k \geqslant 1$ that is if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, we set $\omega(x) := b_1$ and $\omega_i(x) := b_i$. The real number $\omega(x) = b_1$ is the greatest order in the actual decomposition (15), corresponding to the smallest in the potential decomposition (14). It is called the *order* of the Fermat real $x \in {}^\bullet\mathbb{R}$. The number $\omega_i(x) = b_i$ is called the $i$-th order of $x$. If $x \in \mathbb{R}$ we set $\omega(x) := 0$ and $\mathrm{d}^i x := 0$. Observe that in general $\omega(x) = \omega(\mathrm{d}x)$, $\mathrm{d}(\mathrm{d}x) = \mathrm{d}x$ and that, using the notations of the potential decomposition (8), we have $\omega(x) = 1/a_1$.

**Example.** If $x = 1 + t^{1/3} + t^{1/2} + t$, then ${}^\circ x = 1$, $\mathrm{d}x = \mathrm{d}t_3$ and hence $x$ is a third order infinitesimal, i.e. $\omega(x) = 3$, $\mathrm{d}^2 x = \mathrm{d}t_2$ and $\mathrm{d}^3 x = \mathrm{d}t$; finally all the standard parts are ${}^\circ x_i = 1$.

## 6. The ideals $D_k$

In this section, we will introduce the sets of nilpotent infinitesimals corresponding to a $k$-th order neighborhood of 0. Every smooth function restricted to this neighborhood becomes a polynomial of order $k$, obviously given by its $k$-th order Taylor's formula (without remainder). We start with a theorem characterizing infinitesimals of order less than $k$.

**Theorem 10.** *If $x \in {}^\bullet\mathbb{R}$ and $k \in \mathbb{N}_{>1}$, then $x^k = 0$ in ${}^\bullet\mathbb{R}$ if and only if ${}^\circ x = 0$ and $\omega(x) < k$.*

**Proof.** If $x^k = 0$, then taking the standard part map of both sides, we have ${}^\circ(x^k) = ({}^\circ x)^k = 0$ and hence ${}^\circ x = 0$. Moreover, $x^k = 0$ means $x_t^k = o(t)$ and hence $(\frac{x_t}{t^{1/k}})^k \to 0$ and $\frac{x_t}{t^{1/k}} \to 0$. We rewrite this condition using the potential decomposition $x = \sum_{i=1}^k \alpha_i \cdot t^{a_i}$ of $x$ (note that in this way we have $\omega(x) = \frac{1}{a_1}$) obtaining

$$\lim_{t \to 0^+} \sum_i \alpha_i \cdot t^{a_i - \frac{1}{k}} = 0 = \lim_{t \to 0^+} t^{a_1 - \frac{1}{k}} \cdot \left[\alpha_1 + \alpha_2 \cdot t^{a_2 - a_1} + \cdots + \alpha_k \cdot t^{a_k - a_1}\right].$$

But $\alpha_1 + \alpha_2 \cdot t^{a_2 - a_1} + \cdots + \alpha_k \cdot t^{a_k - a_1} \to \alpha_1 \neq 0$, hence we must have that $t^{a_1 - \frac{1}{k}} \to 0$, and so $a_1 > \frac{1}{k}$, that is $\omega(x) < k$.

Vice versa, if ${}^\circ x = 0$ and $\omega(x) < k$, then $x = \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o(t)$, and

$$\lim_{t\to 0^+} \frac{x_t}{t^{1/k}} = \lim_{t\to 0^+} \sum_i \alpha_i \cdot t^{a_i - \frac{1}{k}} + \lim_{t\to 0^+} \frac{o(t)}{t} \cdot t^{1-\frac{1}{k}}.$$

But $t^{1-\frac{1}{k}} \to 0$ because $k > 1$ and $t^{a_i - \frac{1}{k}} \to 0^+$ because $\frac{1}{a_i} \leqslant \frac{1}{a_1} = \omega(x) < k$ and hence $x^k = 0$ in ${}^\bullet\mathbb{R}$. $\square$

If we want that in a $k$-th order infinitesimal neighborhood a smooth function is equal to its $k$-th Taylor's formula, i.e.

$$\forall h \in D_k\colon \quad f(x+h) = \sum_{i=0}^k \frac{h^i}{i!} \cdot f^{(i)}(x), \tag{16}$$

we need to take infinitesimals which are able to delete the remainder, that is, such that $h^{k+1} = 0$. The previous theorem permits to extend the definition of the ideal $D_k$ to real number subscripts instead of natural numbers $k$ only.

**Definition 11.** If $a \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, then

$$D_a := \left\{ x \in {}^\bullet\mathbb{R} \;\middle|\; {}^\circ x = 0,\ \omega(x) < a + 1 \right\}.$$

Moreover, we will simply denote $D_1$ by $D$.

1. If $x = \mathrm{d}t_3$, then $\omega(x) = 3$ and $x \in D_3$. More in general, $\mathrm{d}t_k \in D_a$ if and only if $\omega(\mathrm{d}t_k) = k < a + 1$. For example, $\mathrm{d}t_k \in D$ if and only if $1 \leqslant k < 2$.
2. $D_\infty = \bigcup_a D_a = \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0\}$ is the set of all the infinitesimals of ${}^\bullet\mathbb{R}$.
3. $D_0 = \{0\}$ because the only infinitesimal having order strictly less than 1 is, by definition of order, $x = 0$ (see Definition 9).

As we will see in a subsequent article, defining $x \leqslant y$ *in* ${}^\bullet\mathbb{R}$ iff $x_t \leqslant y_t + z_t$ for $t \geqslant 0$ sufficiently small and for a suitable $z \in {}^\bullet\mathbb{R}$ such that $z = 0$ in ${}^\bullet\mathbb{R}$, we have a totally ordered ring $({}^\bullet\mathbb{R}, \leqslant)$ and the usual relationships between infinitesimals and order relation.

The following theorem gathers several expected properties of the sets $D_a$ and of the order of an infinitesimal $\omega(x)$. In this statement if $r \in \mathbb{R}$, then $\lceil r \rceil$ is the *ceiling* of the real $r$, i.e. the unique integer $\lceil r \rceil \in \mathbb{Z}$ such that $\lceil r \rceil - 1 < r \leqslant \lceil r \rceil$.

**Theorem 12.** *Let $a, b \in \mathbb{R}_{>0}$ and $x, y \in D_\infty$, then*

1. $a \leqslant b \;\Rightarrow\; D_a \subseteq D_b$.
2. $x \in D_{\omega(x)}$.
3. $a \in \mathbb{N} \;\Rightarrow\; D_a = \{x \in {}^\bullet\mathbb{R} \mid x^{a+1} = 0\}$.
4. $x \in D_a \;\Rightarrow\; x^{\lceil a \rceil + 1} = 0$.
5. $x \in D_\infty \setminus \{0\}$ *and* $k = [\omega(x)] \;\Rightarrow\; x \in D_k \setminus D_{k-1}$.
6. $\mathrm{d}(x \cdot y) = \mathrm{d}x \cdot \mathrm{d}y$.
7. $x \cdot y \neq 0 \;\Rightarrow\; \frac{1}{\omega(x\cdot y)} = \frac{1}{\omega(x)} + \frac{1}{\omega(y)}$.

8. $x + y \neq 0 \Rightarrow \omega(x + y) = \max(\omega(x), \omega(y))$.
9. *$D_a$ is an ideal.*

**Proof.** Properties 1 and 2 follow directly from Definition 11 of $D_a$, whereas property 3 follows from Theorem 10. From 1 and 3 property 4 follows: in fact $x \in D_a \subseteq D_{\lceil a \rceil}$ because $a \leqslant \lceil a \rceil$, hence $x^{\lceil a \rceil + 1} = 0$ from property 3. To prove property 5, if $k = [\omega(x)]$, then $k \leqslant \omega(x) < k + 1$, hence directly from Definition 11 the conclusion follows.

To prove property 6 let

$$x = \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{a_i} \quad \text{and} \quad y = \sum_{j=1}^{N} {}^\circ y_j \cdot \mathrm{d}t_{b_j} \tag{17}$$

be the decompositions of $x$ and $y$ (considering that they are infinitesimals, so that ${}^\circ x = {}^\circ y = 0$). Recall that $\mathrm{d}x = {}^\circ x_1 \cdot \mathrm{d}t_{a_1}$ and $\mathrm{d}y = {}^\circ y_1 \cdot \mathrm{d}t_{b_1}$. From (17) we have

$$x \cdot y = \sum_{i=1}^{k} \sum_{j=1}^{N} {}^\circ x_i {}^\circ y_j \, \mathrm{d}t_{a_i} \, \mathrm{d}t_{b_j} = \sum_{i=1}^{k} \sum_{j=1}^{N} {}^\circ x_i {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \tag{18}$$

where we have used Remark 7. But $\omega(x) = a_1 \geqslant a_i$ and $\omega(y) = b_1 \geqslant b_j$ from Definition 9 of decomposition. Hence,

$$\frac{1}{a_1} + \frac{1}{b_1} \leqslant \frac{1}{a_i} + \frac{1}{b_j},$$
$$\frac{a_1 b_1}{a_1 + b_1} \geqslant \frac{a_i b_j}{a_i + b_j}$$

so that the greatest infinitesimal in the product (18) is

$$\mathrm{d}(x \cdot y) = {}^\circ x_1 {}^\circ y_1 \, \mathrm{d}t_{a_1} \, \mathrm{d}t_{b_1} = \mathrm{d}x \cdot \mathrm{d}y.$$

From this proof, property 7 follows, because $x \cdot y \neq 0$ by hypothesis, and hence its order is given by

$$\omega(x \cdot y) = \frac{a_1 b_1}{a_1 + b_1} = \left( \frac{1}{a_1} + \frac{1}{b_1} \right)^{-1} = \left( \frac{1}{\omega(x)} + \frac{1}{\omega(y)} \right)^{-1}.$$

From the decompositions (17) we also have

$$x + y = \sum_{i=1}^{k} {}^\circ x_i \, \mathrm{d}t_{a_i} + \sum_{j=1}^{N} {}^\circ y_j \, \mathrm{d}t_{b_j}$$

and therefore, because by hypothesis $x + y \neq 0$, its order is given by the greatest infinitesimal in this sum, that is

$$\omega(x+y)=\max(a_1,b_1)=\max\bigl(\omega(x),\omega(y)\bigr).$$

It remains to prove property 9. First of all $\omega(0)=0<a+1$, hence $0\in D_a$. If $x, y\in D_a$, then $\omega(x)$ and $\omega(y)$ are strictly less than $a+1$ and hence $x+y\in D_a$ follows from property 8. Finally, if $x\in D_a$ and $y\in{}^\bullet\mathbb{R}$, then $x\cdot y=x\cdot{}^\circ y+x\cdot(y-{}^\circ y)$, so $\omega(x\cdot y)=\max(\omega(x\cdot{}^\circ y),\omega(x\cdot(y-{}^\circ y)))=\max(\omega(x),\omega(x\cdot z))$, where $z:=y-{}^\circ y\in D_\infty$ is an infinitesimal. If $x\cdot z=0$, we have $\omega(x\cdot y)=\omega(x)<a+1$, otherwise from property 7

$$\frac{1}{\omega(x\cdot z)}=\frac{1}{\omega(x)}+\frac{1}{\omega(z)}\geqslant\frac{1}{\omega(x)}$$

and hence $\omega(x\cdot y)\leqslant\omega(x)<a+1$; in any case the conclusion $x\cdot y\in D_a$ follows. □

Property 4 of this theorem cannot be proved substituting the ceiling $\lceil a\rceil$ with the integer part $[a]$. In fact if $a=1.2$ and $x=\mathrm{d}t_{2.1}$, then $\omega(x)=2.1$ and $[a]+1=2$ so that $x^{[a]+1}=x^2=\mathrm{d}t_{\frac{2.1}{2}}\neq 0$ in ${}^\bullet\mathbb{R}$, whereas $\lceil a\rceil+1=3$ and $x^3=\mathrm{d}t_{\frac{2.1}{3}}=0$.

Finally, let us note the increasing sequence of ideals/neighborhoods of zero:

$$\{0\}=D_0\subset D=D_1\subset D_2\subset\cdots\subset D_k\subset\cdots\subset D_\infty. \tag{19}$$

Because of (19) and of the property $\mathrm{d}t_a=0$ if $a<1$, we can say that $\mathrm{d}t$ is the smallest infinitesimals and $\mathrm{d}t_2$, $\mathrm{d}t_3$, etc. are greater infinitesimals.

Because of properties 7 and 8 of the previous theorem, we have that $v(x):=\frac{1}{\omega(x)}$ if $x\in{}^\bullet\mathbb{R}_{\neq 0}$ and $v(0):=+\infty$ is a valuation on the ring ${}^\bullet\mathbb{R}$, i.e. it is a function $v:{}^\bullet\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ such that $v(0)=+\infty$, $v(x)\in\mathbb{R}$ for $x\neq 0$, and such that $v(x\cdot y)=v(x)+v(y)$ and $v(x+y)\geqslant\min(v(x),v(y))$ (in our case the equality holds). This permits to mention here some analogies between the A. Robinson's valuation field ${}^\rho\mathbb{R}$ (also called field of asymptotic numbers, see [25, 22]) and our ring of Fermat reals. Fixing an invertible infinitesimal $\rho\in{}^*\mathbb{R}$, the field of asymptotic numbers can be easily defined as the quotient field ${}^\rho\mathbb{R}:=\mathcal{M}_\rho({}^*\mathbb{R})/\mathcal{N}_\rho({}^*\mathbb{R})$, where

$$\mathcal{M}_\rho({}^*\mathbb{R}):=\bigl\{\zeta\in{}^*\mathbb{R}\bigm|\exists m\in\mathbb{N}_{>0}\colon\ |\zeta|\leqslant\rho^{-m}\bigr\},$$
$$\mathcal{N}_\rho({}^*\mathbb{R}):=\bigl\{\zeta\in{}^*\mathbb{R}\bigm|\forall n\in\mathbb{N}_{>0}\colon\ |\zeta|\leqslant\rho^{n}\bigr\}$$

are the sets of $\rho$-moderate and $\rho$-negligible non-standard reals. Even if ${}^\rho\mathbb{R}$ is a field, and hence we cannot have non-zero nilpotent elements, we can use a nilpotent-like language of suitable equivalence relations to deal with formulas like (16). For example, the valuation of ${}^\rho\mathbb{R}$ is defined by $v(x):={}^\circ(\log_\rho|x|)$, and if we define $x\simeq y$ iff $v(x-y)>1$, then this relation preserves the ring operations and we have that $h^{a+1}\simeq 0$ if $a=\frac{1}{v(h)}$. The relation $\simeq$ is the analogous of our equality in ${}^\bullet\mathbb{R}$, because it is not hard to prove that $x\simeq y$ iff $\frac{x-y}{\rho}$ is an infinitesimal of ${}^\rho\mathbb{R}$. The dependence by the fixed infinitesimal $\rho\in{}^*\mathbb{R}_{\neq 0}$ is tied with the index set used in the construction of ${}^*\mathbb{R}$, and hence is not different from the choice of the infinitesimal function $t\in\mathbb{R}_{\geqslant 0}\mapsto t\in\mathbb{R}$ that we used in our Definition 5 of ${}^\bullet\mathbb{R}$. In some situation we may have a natural, problem-related, choice for $\rho$, like in the case of algebras of generalized functions, where $\rho$ is the infinitesimal generated by $\varphi\mapsto\mathrm{diam}(\mathrm{supp}(\varphi))$, in case $\varphi$ is a non-zero test function (see [29] for more details).

## 7. Products of powers of nilpotent infinitesimals

In this section we introduce several useful instruments to determine whether a product of the form $h_1^{i_1} \cdot \dots \cdot h_n^{i_n}$ , with $h_k \in D_\infty \setminus \{0\}$, is zero or whether it belongs to some $D_k$. Generally speaking, this problem is non-trivial in a ring (e.g., in SDG there is not an effective procedure to address this problem, see e.g. [21]) and its solutions will be very useful in proofs of infinitesimal Taylor's formulas.

**Theorem 13.** *Let $h_1, \dots, h_n \in D_\infty \setminus \{0\}$ and $i_1, \dots, i_n \in \mathbb{N}$, then*

1. $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0 \Leftrightarrow \sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$.
2. $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} \neq 0 \Rightarrow \frac{1}{\omega(h_1^{i_1} \cdot \dots \cdot h_n^{i_n})} = \sum_{k=1}^n \frac{i_k}{\omega(h_k)}$.

**Proof.** Let

$$h_k = \sum_{r=1}^{N_k} \alpha_{kr} t^{a_{kr}} \tag{20}$$

be the potential decomposition of $h_k$ for $k = 1, \dots, n$. Then by Definition 8 of potential decomposition and Definition 9 of order, we have $0 < a_{k1} < a_{k2} < \dots < a_{kN_k} \leqslant 1$ and $j_k := \omega(h_k) = \frac{1}{a_{k1}}$, hence $\frac{1}{j_k} \leqslant a_{kr}$ for every $r = 1, \dots, N_k$. Therefore from (20), collecting the terms containing $t^{1/j_k}$ we have

$$h_k = t^{1/j_k} \cdot \left(\alpha_{k1} + \alpha_{k2} t^{a_{k2}-1/j_k} + \dots + \alpha_{kN_k} t^{a_{kN_k}-1/j_k}\right)$$

and hence

$$\begin{aligned} h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}} \cdot \left(\alpha_{11} + \alpha_{12} t^{a_{12}-\frac{1}{j_1}} + \dots + \alpha_{1N_1} t^{a_{1N_1}-\frac{1}{j_1}}\right)^{i_1} \cdot \dots \\ \cdot \left(\alpha_{n1} + \alpha_{n2} t^{a_{n2}-\frac{1}{j_n}} + \dots + \alpha_{nN_n} t^{a_{nN_n}-\frac{1}{j_n}}\right)^{i_n}. \end{aligned} \tag{21}$$

Hence, if $\sum_k \frac{i_k}{j_k} > 1$ we have that $t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}} = 0$ in ${}^\bullet\mathbb{R}$, so also $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0$. Vice versa, if $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0$, then the right-hand side of (21) is a $o(t)$ as $t \to 0^+$, that is

$$\begin{aligned} t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}-1} \cdot \left(\alpha_{11} + \alpha_{12} t^{a_{12}-\frac{1}{j_1}} + \dots + \alpha_{1N_1} t^{a_{1N_1}-\frac{1}{j_1}}\right)^{i_1} \cdot \dots \\ \cdot \left(\alpha_{n1} + \alpha_{n2} t^{a_{n2}-\frac{1}{j_n}} + \dots + \alpha_{nN_n} t^{a_{nN_n}-\frac{1}{j_n}}\right)^{i_n} \to 0. \end{aligned}$$

But each term $(\alpha_{k1} + \alpha_{k2} t^{a_{k2}-\frac{1}{j_k}} + \dots + \alpha_{kN_k} t^{a_{kN_k}-\frac{1}{j_k}})^{i_k} \to \alpha_k^{i_k} \neq 0$ so, necessarily, we must have $\frac{i_1}{j_1} + \dots + \frac{i_n}{j_n} - 1 > 0$, and this concludes the proof of 1.

To prove property 2 it suffices to apply recursively property 7 of Theorem 12, in fact

$$\begin{aligned}\frac{1}{\omega(h_1^{i_1}\cdot\dots\cdot h_n^{i_n})} &= \frac{1}{\omega(h_1^{i_1})}+\frac{1}{\omega(h_2^{i_2}\cdot\dots\cdot h_n^{i_n})}\\ &= \frac{1}{\omega(h_1\cdot\overset{i_1}{\dots}\cdot h_1)}+\frac{1}{\omega(h_2^{i_2}\cdot\dots\cdot h_n^{i_n})}\\ &= \dots = \frac{i_1}{\omega(h_1)}+\frac{1}{\omega(h_2^{i_2}\cdot\dots\cdot h_n^{i_n})} = \frac{i_1}{\omega(h_1)}+\dots+\frac{i_n}{\omega(h_n)}\end{aligned}$$

and this concludes the proof. □

**Example 14.** $\omega(\mathrm{d}t_{a_1}^{i_1}\cdot\dots\cdot\mathrm{d}t_{a_n}^{i_n})^{-1}=\sum_k\frac{i_k}{\omega(\mathrm{d}t_{a_k})}=\sum_k\frac{i_k}{a_k}$ and $\mathrm{d}t_{a_1}^{i_1}\cdot\dots\cdot\mathrm{d}t_{a_n}^{i_n}=0$ if and only if $\sum_k\frac{i_k}{a_k}>1$. For example, $\mathrm{d}t\cdot h=0$ for every $h\in D_\infty$.

From this theorem we can derive four simple corollaries that will be useful in the course of the present work. Some of these corollaries are useful because they give properties of powers like $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}$ in cases where exact values of the orders $\omega(h_k)$ are unknown. The first corollary gives a necessary and sufficient condition to have $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\in D_p\setminus\{0\}$.

**Corollary 15.** *In the hypotheses of the previous Theorem* 13 *let* $p\in\mathbb{R}_{>0}$, *then we have*

$$h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\in D_p\setminus\{0\}\quad\iff\quad\frac{1}{p+1}<\sum_{k=1}^n\frac{i_k}{\omega(h_k)}\leqslant 1.$$

**Proof.** This follows almost directly from Theorem 13. In fact if $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\in D_p\setminus\{0\}$, then its order is given by $\omega(h_1^{i_1}\cdot\dots\cdot h_n^{i_n})=[\sum_k\frac{i_k}{\omega(h_k)}]^{-1}=:a$ and moreover, $a\geqslant 1$ because $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\neq 0$. Furthermore, $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}$ being an element of $D_p$, we also have $a<p+1$, from which the conclusion $\frac{1}{p+1}<\frac{1}{a}\leqslant 1$ follows.

Vice versa, if $\frac{1}{p+1}<\frac{1}{a}:=\sum_k\frac{i_k}{\omega(h_k)}\leqslant 1$, then from Theorem 13 we have $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\neq 0$ and $\omega(h_1^{i_1}\cdot\dots\cdot h_n^{i_n})=a$; but $a<p+1$ by hypothesis, hence $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}\in D_p$. □

Now, we will prove a sufficient condition to have $h_1^{i_1}\cdot\dots\cdot h_n^{i_n}=0$, starting from the hypotheses $h_k\in D_{j_k}$ only, that is $\omega(h_k)<j_k+1$. The typical situation where this applies is for $j_k=[\omega(h_k)]\in\mathbb{N}$.

**Corollary 16.** *Let* $h_k\in D_{j_k}$ *for* $k=1,\dots,n$ *and* $i_1,\dots,i_n\in\mathbb{N}$, *then*

$$\sum_{k=1}^n\frac{i_k}{j_k+1}\geqslant 1\quad\Longrightarrow\quad h_1^{i_1}\cdot\dots\cdot h_n^{i_n}=0.$$

In fact $\sum_{k=1}^n\frac{i_k}{\omega(h_k)}>\sum_{k=1}^n\frac{i_k}{j_k+1}\geqslant 1$ because $\omega(h_k)<j_k+1$, hence the conclusion follows from Theorem 13.

Let $h,k\in D$; we want to see if $h\cdot k=0$. Because in this case $\sum_k\frac{i_k}{j_k+1}=\frac{1}{2}+\frac{1}{2}=1$ we always have

$$h \cdot k = 0. \tag{22}$$

This is an important conceptual difference between Fermat reals and the ring of SDG, where the product of two first order infinitesimal is not necessarily zero. The consequences of this property have a deep effect on the development of the theory of Fermat reals, and force us to develop several new concepts that enable us to generalize the derivation formula (4) to functions defined on infinitesimal domains, such as $f : D \to {}^\bullet\mathbb{R}$ (see [15]). We note that, within the simple Definition 5, the equality (22) has an intuitively clear meaning, and to preserve this intuition we keep the equality instead of completely changing the theory to something less intuitive. As we will see in a subsequent article, for a generic ring $R$, a total order relation on the subset $\{h \in R \mid h^2 = 0\}$ necessarily implies equalities of the form (22) (see [15]).

The next corollary solves the same problem of the previous one, but starting from the hypotheses $h_k^{j_k} = 0$:

**Corollary 17.** *If $h_1, \dots, h_n \in D_\infty$ and $h_k^{j_k} = 0$ for $j_1, \dots, j_n \in \mathbb{N}$, then if $i_1, \dots, i_n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \frac{i_k}{j_k} \geqslant 1 \quad \Longrightarrow \quad h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0.$$

In fact if $h_k^{j_k} = 0$, then $j_k > 0$ and $h_k \in D_{j_k - 1}$ by Theorem 12, so the conclusion follows from the previous corollary.

Finally, the latter corollary permits e.g. to pass from a formula like

$$\forall h \in D_p^n\colon \quad f(h) = \sum_{\substack{i \in \mathbb{N}^n \\ |i| \leqslant p}} h^i \cdot a_i$$

to a formula like

$$\forall h \in D_q^n\colon \quad f(h) = \sum_{\substack{i \in \mathbb{N}^n \\ |i| \leqslant q}} h^i \cdot a_i$$

where $q < p$. In the previous formulas $D_a^n = D_a \times \overset{n}{\cdots} \times D_a$ and we have used the classical multi-indexes notations, e.g. $h^i = h_1^{i_1} \cdot \dots \cdot h_n^{i_n}$ and $|i| = \sum_{k=1}^n i_k$.

**Corollary 18.** *Let $p \in \mathbb{N}_{>0}$ and $h_k \in D_p$ for each $k = 1, \dots, n$; $i \in \mathbb{N}^n$ and $h \in D_\infty^n$. Then*

$$|i| > p \quad \Longrightarrow \quad h^i = 0.$$

To prove it, we only have to apply Corollary 16:

$$\sum_{k=1}^{n} \frac{i_k}{p+1} = \frac{\sum_k i_k}{p+1} = \frac{|i|}{p+1} \geqslant \frac{p+1}{p+1} = 1.$$

Let us note explicitly that the possibility to prove all these results about products of powers of nilpotent infinitesimals is essentially tied with the choice of little-oh polynomials in the definition of the equivalence relation $\sim$ in Definition 4. Equally effective and useful results are not provable for the more general family of nilpotent functions (see e.g. [14]).

## 8. Identity principle for polynomials

In this section we want to prove that if a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ of ${}^\bullet\mathbb{R}$ is identically zero, then $a_k = 0$ for all $k = 0, \ldots, n$. To prove this conclusion, it suffices to mean "identically zero" as "equal to zero for every $x$ belonging to the extension of an open subset of $\mathbb{R}$". Therefore, we firstly define what this extension is.

**Definition 19.** If $U$ is an open subset of $\mathbb{R}^n$, then ${}^\bullet U := \{x \in {}^\bullet\mathbb{R}^n \mid {}^\circ x \in U\}$. Here, with the symbol ${}^\bullet\mathbb{R}^n$ we mean ${}^\bullet\mathbb{R}^n := {}^\bullet\mathbb{R} \times \overset{n}{\cdots\cdots} \times {}^\bullet\mathbb{R}$.

The identity principle for polynomials can now be stated in the following way:

**Theorem 20.** *Let $a_0, \ldots, a_n \in {}^\bullet\mathbb{R}$ and $U$ be an open neighborhood of $0$ in $\mathbb{R}$ such that*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \quad \text{in } {}^\bullet\mathbb{R}\ \forall x \in {}^\bullet U. \tag{23}$$

*Then*

$$a_0 = a_1 = \cdots = a_n = 0 \quad \text{in } {}^\bullet\mathbb{R}.$$

**Proof.** Because $U$ is an open neighborhood of $0$ in $\mathbb{R}$, we can always find $x_1, \ldots, x_{n+1} \in U$ such that $x_i \neq x_j$ for $i, j = 1, \ldots, n+1$ with $i \neq j$. Hence, from hypothesis (23) we have

$$a_n x_k^n + \cdots + a_1 x_k + a_0 = 0 \quad \text{in } {}^\bullet\mathbb{R}\ \forall k = 1, \ldots, n+1.$$

That is, in vectorial form

$$(a_n, \ldots, a_0) \cdot \begin{bmatrix} x_1^n & x_2^n & \ldots & x_{n+1}^n \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_{n+1}^{n-1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \ldots & x_{n+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix} = 0 \quad \text{in } {}^\bullet\mathbb{R}.$$

This matrix $V$ is a Vandermonde matrix, hence it is invertible

$$(a_n, \ldots, a_0) \cdot V = \underline{0} \quad \text{in } {}^\bullet\mathbb{R}^{n+1},$$
$$(a_n, \ldots, a_0) \cdot V \cdot V^{-1} = \underline{0} \quad \text{in } {}^\bullet\mathbb{R}^{n+1},$$

hence $a_k = 0$ in ${}^\bullet\mathbb{R}$ for every $k = 0, \ldots, n$. $\square$

This theorem can be extended to polynomials with more than one variable using recursively the previous theorem, one variable per time.

## 9. Invertible Fermat reals

We can see more formally that, to prove (3), we cannot embed the reals $\mathbb{R}$ into a field but only into a ring, necessarily containing nilpotent element. In fact, applying (3) to the function $f(h) = h^2$ for $h \in D$, where $D \subseteq {}^\bullet\mathbb{R}$ is a given subset of ${}^\bullet\mathbb{R}$, we have

$$f(h) = h^2 = f(0) + h \cdot f'(0) = 0 \quad \forall h \in D,$$

where we have supposed the preservation of the equality $f'(0) = 0$ from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. In other words, if $D$ and $f(h) = h^2$ verify (3), then necessarily each element $h \in D$ must be a new type of number whose square is zero. Of course, in a field the only subset $D$ verifying this property is $D = \{0\}$.

Because we cannot have property (3) and a field at the same time, we need a sufficiently good family of cancellation laws as substitutes. The simplest one of them is also useful to prove the uniqueness of (4):

**Theorem 21.** *If $x \in {}^\bullet\mathbb{R}$ is a Fermat real and $r, s \in \mathbb{R}$ are standard real numbers, then*

$$(x \cdot r = x \cdot s \text{ in } {}^\bullet\mathbb{R} \text{ and } x \neq 0) \quad \Longrightarrow \quad r = s.$$

**Remark.** As a consequence of this result, we can always cancel a non-zero Fermat real in an equality of the form $x \cdot r = x \cdot s$ where $r$, $s$ are *standard* reals. This is obviously tied with the uniqueness part of (4) and implies that formula (4) uniquely identifies the first derivative in case it is a standard real number.

**Proof.** From Definition 5 of equality in ${}^\bullet\mathbb{R}$ and from $x \cdot r = x \cdot s$ we have

$$\lim_{t \to 0^+} \frac{x_t \cdot (r - s)}{t} = 0.$$

But if we had $r \neq s$ this would implies $\lim_{t \to 0^+} \frac{x_t}{t} = 0$, that is $x = 0$ in ${}^\bullet\mathbb{R}$ and this contradicts the hypothesis $x \neq 0$. □

The last result of this section takes its ideas from similar situations of formal power series and gives also a formula to compute the inverse of an invertible Fermat real.

**Theorem 22.** *Let $x = {}^\circ x + \sum_{i=1}^n {}^\circ x_i \cdot \mathrm{d}t_{a_i}$ be the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$. Then*

*$x$ is invertible*

*if and only if ${}^\circ x \neq 0$, and in this case*

$$\frac{1}{x} = \frac{1}{{}^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left( \sum_{i=1}^n \frac{{}^\circ x_i}{{}^\circ x} \cdot \mathrm{d}t_{a_i} \right)^j. \tag{24}$$

In the formula (24) we have to note that the series is actually a finite sum because any $\mathrm{d}t_{a_i}$ is nilpotent.

1. $(1+\mathrm{d}t_2)^{-1} = 1 - \mathrm{d}t_2 + \mathrm{d}t_2^2 - \mathrm{d}t_2^3 + \cdots = 1 - \mathrm{d}t_2 + \mathrm{d}t$ because $\mathrm{d}t_2^3 = 0$.
2. $(1+\mathrm{d}t_3)^{-1} = 1 - \mathrm{d}t_3 + \mathrm{d}t_3^2 - \mathrm{d}t_3^3 + \mathrm{d}t_3^4 - \cdots = 1 - \mathrm{d}t_3 + \mathrm{d}t_3^2 - \mathrm{d}t$.

**Proof.** If $x \cdot y = 1$ for some $y \in {}^\bullet\mathbb{R}$, then, taking the standard parts of each side we have $^\circ x \cdot {}^\circ y = 1$ and hence $^\circ x \neq 0$. Vice versa, the idea is to start from the series

$$\frac{1}{1+r} = \sum_{j=0}^{+\infty} (-1)^j \cdot r^j \quad \forall r \in \mathbb{R}\colon\ |r| < 1$$

and, intuitively, to define

$$\left({}^\circ x + \sum_i {}^\circ x_i \,\mathrm{d}t_{a_i}\right)^{-1} = {}^\circ x^{-1} \cdot \left(1 + \sum_i \frac{{}^\circ x_i}{{}^\circ x}\,\mathrm{d}t_{a_i}\right)^{-1} = {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_i \frac{{}^\circ x_i}{{}^\circ x}\,\mathrm{d}t_{a_i}\right)^j.$$

So, let $y := {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot (\sum_i \frac{{}^\circ x_i}{{}^\circ x}\,\mathrm{d}t_{a_i})^j$ and $h := x - {}^\circ x = \sum_i {}^\circ x_i \,\mathrm{d}t_{a_i} \in D_\infty$ so that we can also write

$$y = {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \frac{h^j}{{}^\circ x^j}.$$

But $h \in {}^\bullet\mathbb{R}$ is a little-oh polynomial with $h(0) = 0$, so it is also continuous, hence for a sufficiently small $\delta > 0$ we have

$$\forall t \in (-\delta, \delta)\colon \quad \left|\frac{h_t}{{}^\circ x}\right| < 1.$$

Therefore,

$$\forall t \in (-\delta, \delta)\colon \quad y_t = \frac{1}{{}^\circ x} \cdot \left(1 + \frac{h_t}{{}^\circ x}\right)^{-1} = \frac{1}{{}^\circ x + h_t} = \frac{1}{x_t}.$$

From this equality and from Definition 5 it follows $x \cdot y = 1$ in ${}^\bullet\mathbb{R}$. $\square$

## 10. The derivation formula

In this section we want to give a proof of (4), because it has been the principal motivation for the construction of the ring of Fermat reals ${}^\bullet\mathbb{R}$. Anyhow, before considering the proof of the derivation formula, we have to extend a given smooth function $f : \mathbb{R} \to \mathbb{R}$ to a certain function ${}^\bullet f : {}^\bullet\mathbb{R} \to {}^\bullet\mathbb{R}$.

**Definition 23.** Let $A$ be an open subset of $\mathbb{R}^n$, $f : A \to \mathbb{R}$ a smooth function and $x \in {}^\bullet A$, then we define

$$^\bullet f(x) := f \circ x \quad \text{in } {}^\bullet\mathbb{R}.$$

In other words, using the notation $[x]_\sim \in {}^\bullet\mathbb{R}$ for the equivalence class generated by $x \in \mathbb{R}_o[t]$ modulo the relation $\sim$ defined in Definition 5, we can write the previous definition as ${}^\bullet f([x]_\sim) := [f \circ x]_\sim$.

This definition is correct because we have seen that little-oh polynomials are preserved by smooth functions, and because the function $f$ is locally Lipschitz, so

$$\left|\frac{f(x_t) - f(y_t)}{t}\right| \leqslant K \cdot \left|\frac{x_t - y_t}{t}\right| \quad \forall t \in (-\delta, \delta)$$

for a sufficiently small $\delta$ and some constant $K$, and hence if $x = y$ in ${}^\bullet\mathbb{R}$, then also ${}^\bullet f(x) = {}^\bullet f(y)$ in ${}^\bullet\mathbb{R}$.

The function ${}^\bullet f$ is an extension of $f$, that is

$$^\bullet f(r) = f(r) \quad \text{in } {}^\bullet\mathbb{R}\ \forall r \in \mathbb{R},$$

as it follows directly from the definition of equality in ${}^\bullet\mathbb{R}$ (i.e. Definition 5), thus we can still use the symbol $f(x)$ both for $x \in {}^\bullet\mathbb{R}$ and $x \in \mathbb{R}$ without confusion. After the introduction of the extension of smooth functions, we can also state the following useful *elementary transfer theorem* for equalities, whose proof follows directly from the previous definitions:

**Theorem 24.** *Let $A$ be an open subset of $\mathbb{R}^n$, and $\tau, \sigma : A \to \mathbb{R}$ be smooth functions. Then*

$$\forall x \in {}^\bullet A\colon \quad {}^\bullet\tau(x) = {}^\bullet\sigma(x)$$

*iff*

$$\forall r \in A\colon \quad \tau(r) = \sigma(r).$$

Now, we will prove the derivation formula (4).

**Theorem 25.** *Let $A$ be an open set in $\mathbb{R}$, $x \in A$ and $f : A \to \mathbb{R}$ a smooth function, then*

$$\exists!\, m \in \mathbb{R}\ \forall h \in D\colon \quad f(x+h) = f(x) + h \cdot m. \tag{25}$$

*In this case we have $m = f'(x)$, where $f'(x)$ is the usual derivative of $f$ at $x$.*

**Proof.** Uniqueness follows from the previous cancellation law (Theorem 21). Indeed if $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$ both verify (25), then $h \cdot m_1 = h \cdot m_2$ for every $h \in D$. But there exists a non-zero first order infinitesimal, e.g. $\mathrm{d}t \in D$, so from Theorem 21 it follows $m_1 = m_2$.

To prove the existence part, take $h \in D$, so that $h^2 = 0$ in ${}^\bullet\mathbb{R}$, i.e. $h_t^2 = o(t)$ for $t \to 0^+$. But $f$ is smooth, hence from its second order Taylor's formula we have

$$f(x+h_t)=f(x)+h_t\cdot f'(x)+\frac{h_t^2}{2}\cdot f''(x)+o\big(h_t^2\big).$$

But

$$\frac{o(h_t^2)}{t}=\frac{o(h_t^2)}{h_t^2}\cdot\frac{h_t^2}{t}\to 0\quad\text{for } t\to 0^+$$

so

$$\frac{h_t^2}{2}\cdot f''(x)+o\big(h_t^2\big)=o_1(t)\quad\text{for } t\to 0^+$$

and we can write

$$f(x+h_t)=f(x)+h_t\cdot f'(x)+o_1(t)\quad\text{for } t\to 0^+$$

that is

$$f(x+h)=f(x)+h\cdot f'(x)\quad\text{in } {}^\bullet\mathbb{R}$$

and this proves the existence part because $f'(x)\in\mathbb{R}$. $\square$

For example $e^h=1+h$, $\sin(h)=h$ and $\cos(h)=1$ for every $h\in D$.

Analogously, we can prove the following infinitesimal Taylor's formula.

**Lemma 26.** *Let $A$ be an open set in $\mathbb{R}^d$, $x\in A$, $n\in\mathbb{N}_{>0}$ and $f:A\to\mathbb{R}$ a smooth function, then*

$$\forall h\in D_n^d:\quad f(x+h)=\sum_{\substack{j\in\mathbb{N}^d\\|j|\leqslant n}}\frac{h^j}{j!}\cdot\frac{\partial^{|j|}f}{\partial x^j}(x).$$

For example $\sin(h)=h-\frac{h^3}{6}$ if $h\in D_3$ so that $h^4=0$.

It is possible to generalize several results of the present work to functions of class $\mathcal{C}^n$ only, instead of smooth ones. However, it is an explicit purpose of this work to simplify statements of results, definitions and notations, even if, as a result of this searching for simplicity, its applicability will only hold for a more restricted class of functions. Some more general results, stated for $\mathcal{C}^n$ functions, but less simple, can be found in [14].

Note that $m=f'(x)\in\mathbb{R}$, i.e. the slope is a standard real number, and that we can use the previous formula with standard real numbers $x$ only, and not with a generic $x\in{}^\bullet\mathbb{R}$, but we shall remove this limitation in subsequent works (see e.g. [15]).

In other words we can say that the derivation formula (4) allows us to differentiate the usual differentiable functions using a language with infinitesimal numbers and to obtain from this an ordinary function.

If we apply this theorem to the smooth function $p(r):=\int_x^{x+r}f(t)\,\mathrm{d}t$, for $f$ smooth, then we immediately obtain the following

**Corollary 27.** *Let $A$ be open in $\mathbb{R}$, $x \in A$ and $f : A \to \mathbb{R}$ smooth. Then*

$$\forall h \in D\colon \quad \int_x^{x+h} f(t)\ \mathrm{d}t = h \cdot f(x).$$

*Moreover, $f(x) \in \mathbb{R}$ is uniquely determined by this equality.*

We close this section by introducing a very simple notation useful to emphasize some equalities: if $h, k \in {}^\bullet\mathbb{R}$ then we say that $\exists h/k$ iff $\exists! r \in \mathbb{R}\colon\ h = r \cdot k$, and obviously we denote this $r \in \mathbb{R}$ with $h/k$. Therefore, we can say, e.g., that

$$f'(x) = \frac{f(x+h) - f(x)}{h} \quad \forall h \in D_{\neq 0},$$
$$f(x) = \frac{1}{h} \cdot \int_x^{x+h} f(t)\ \mathrm{d}t \quad \forall h \in D_{\neq 0}.$$

Moreover, we can prove some natural properties of this "ratio", like the following one

$$\exists \frac{u}{v}, \frac{x}{y} \quad \text{and} \quad vy \neq 0 \quad \Longrightarrow \quad \frac{u}{v} + \frac{x}{y} = \frac{uy + vx}{vy}.$$

**Example 28.** Consider e.g. $x = 1 + 2\,\mathrm{d}t_3 + \mathrm{d}t_2 + 5\,\mathrm{d}t_{4/3}$, then using the previous ratio we can find a formula to calculate all the coefficients of this decomposition. Indeed, let us consider first the term $2\,\mathrm{d}t_3$: if we multiply both sides by $\mathrm{d}t_{3/2}$, where

$$\frac{3}{2} = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_3)}}$$

we obtain

$$(x - {}^\circ x) \cdot \mathrm{d}t_{3/2} = 2\,\mathrm{d}t_3\,\mathrm{d}t_{3/2} + \mathrm{d}t_2\,\mathrm{d}t_{3/2} + 5\,\mathrm{d}t_{4/3}\,\mathrm{d}t_{3/2}$$

but $\mathrm{d}t_3\,\mathrm{d}t_{3/2} = \mathrm{d}t$ whereas $\mathrm{d}t_a\,\mathrm{d}t_{3/2} = 0$ if $a < 3$, so

$$\frac{(x - {}^\circ x)\,\mathrm{d}t_{3/2}}{\mathrm{d}t} = 2.$$

Analogously, we have

$$\frac{(x - {}^\circ x - 2\,\mathrm{d}t_3)\,\mathrm{d}t_2}{\mathrm{d}t} = 1 \quad \text{and} \quad \frac{(x - {}^\circ x - 2\,\mathrm{d}t_3 - \mathrm{d}t_2)\,\mathrm{d}t_4}{\mathrm{d}t} = 5$$

where

$$2 = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_2)}} \quad \text{and} \quad 4 = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_{4/3})}}.$$

Using the same idea, we can prove the recursive formula

$$\alpha_{i+1} = \frac{1}{1-\frac{1}{\omega_{i+1}(x)}} \quad \Longrightarrow \quad \frac{(x - {}^\circ x - \sum_{k=1}^{i} x_k \, \mathrm{d}t_{\omega_k(x)}) \cdot \mathrm{d}t_{\alpha_{i+1}}}{\mathrm{d}t} = x_{i+1}.$$

Finally, directly from the definition of decomposition it follows

$$\alpha \neq \frac{1}{1-\frac{1}{\omega_{i+1}(x)}} \ \forall i \quad \Longrightarrow \quad \frac{(x - {}^\circ x - \sum_{k=1}^{i} x_k \, \mathrm{d}t_{\omega_k(x)}) \cdot \mathrm{d}t_{\alpha}}{\mathrm{d}t} = 0,$$

$$\frac{(x - {}^\circ x - \sum_{k=1}^{i} x_k \, \mathrm{d}t_{\omega_k(x)}) \cdot \mathrm{d}t_{\alpha}}{\mathrm{d}t} \neq 0 \quad \Longrightarrow \quad \alpha = \frac{1}{1-\frac{1}{\omega_{i+1}(x)}}.$$

In this way, all the terms of the decomposition of a Fermat real are uniquely determined by these recursive formulas.

## 11. Conclusions

The problem of transforming informal infinitesimal methods into a rigorous theory has been faced by several authors. The most commonly used theories (NSA and SDG) require a good knowledge of mathematical logic and a strong formal control. Others, like Weil functors (see e.g. [20]) or the Levi-Civita field (see e.g. [27]), are mainly based on formal/algebraic methods and may lack intuitive meaning. In this initial work, we have shown that it is possible to bypass the inconsistency of SIA using classical logic and modifying the Kock–Lawvere axiom, corresponding to our derivation formula (see [21]). This has been performed always maintaining a very good intuitive meaning. In subsequent articles we will present the order properties of the ring of Fermat reals, their geometric representability, the differential and integral calculus for smooth functions defined on open sets of the form ${}^\bullet U \subseteq {}^\bullet\mathbb{R}$ or in infinitesimal sets like $D_{\alpha_1} \times \cdots \times D_{\alpha_n}$. Also, we will extend this method, that has taken us from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$, to a generic diffeological space (see [15]). The present paper is therefore the first step of a program aimed at the development of a modified version of SDG that *can* be studied using classical logic.

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