Robotic Systems

Chapter IV
Forward Kinematics

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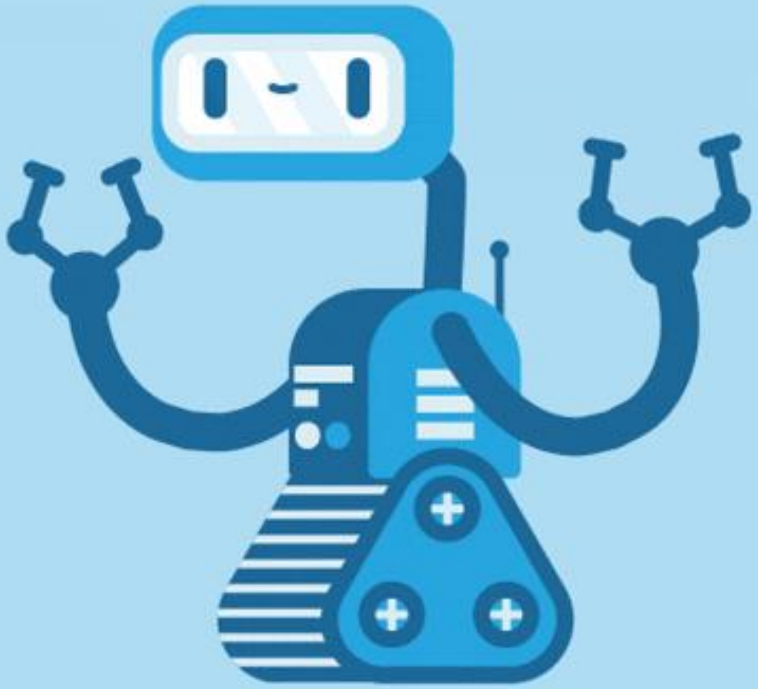
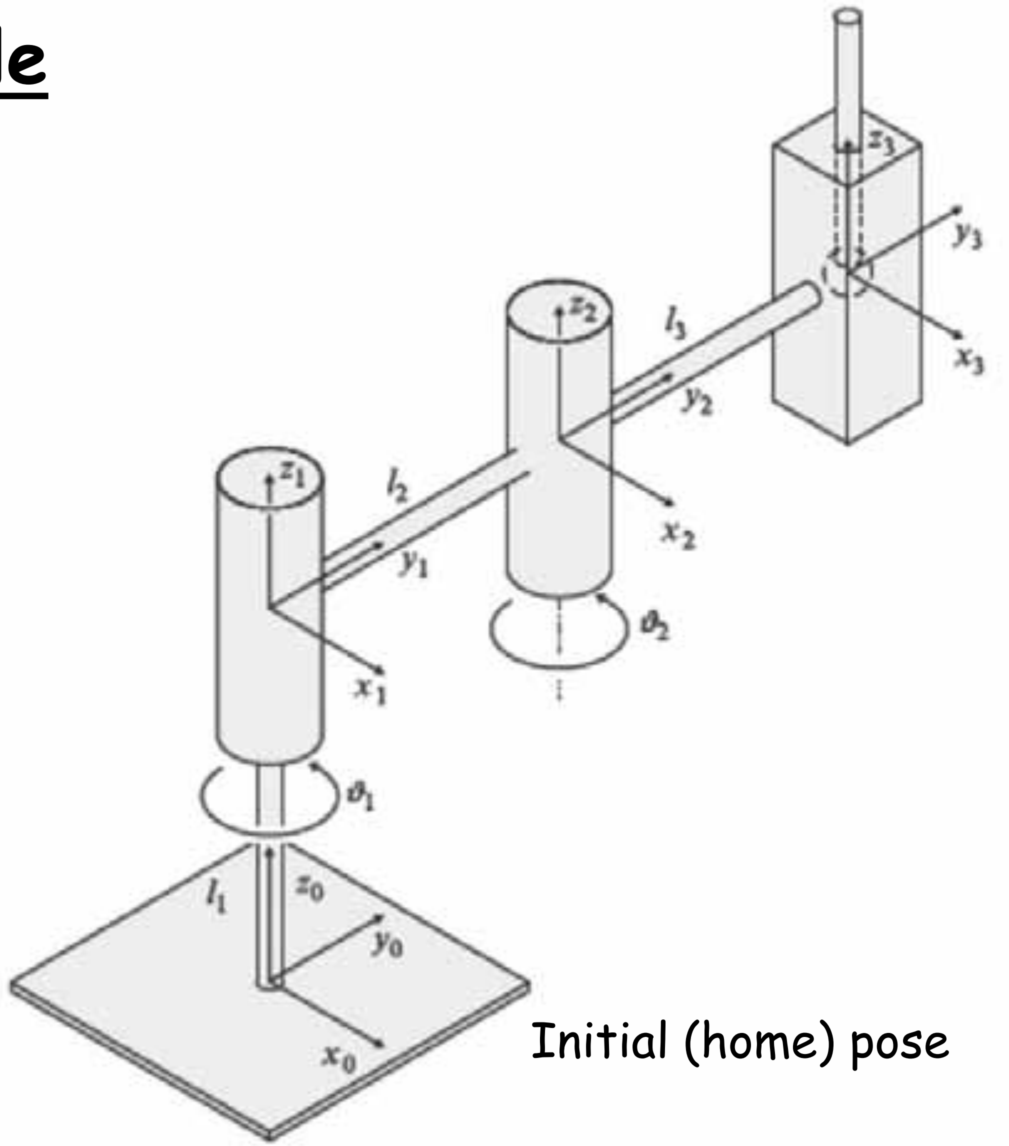


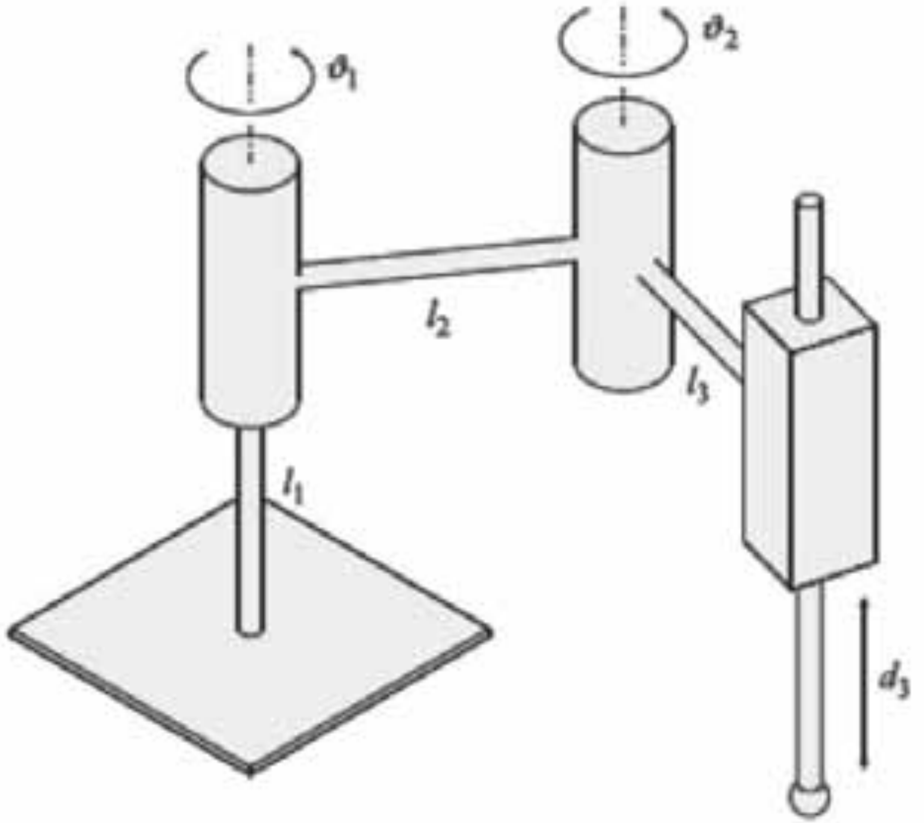
Forward Kinematics

The calculation of the position and orientation of the end-effector frame from joint coordinates.

Example



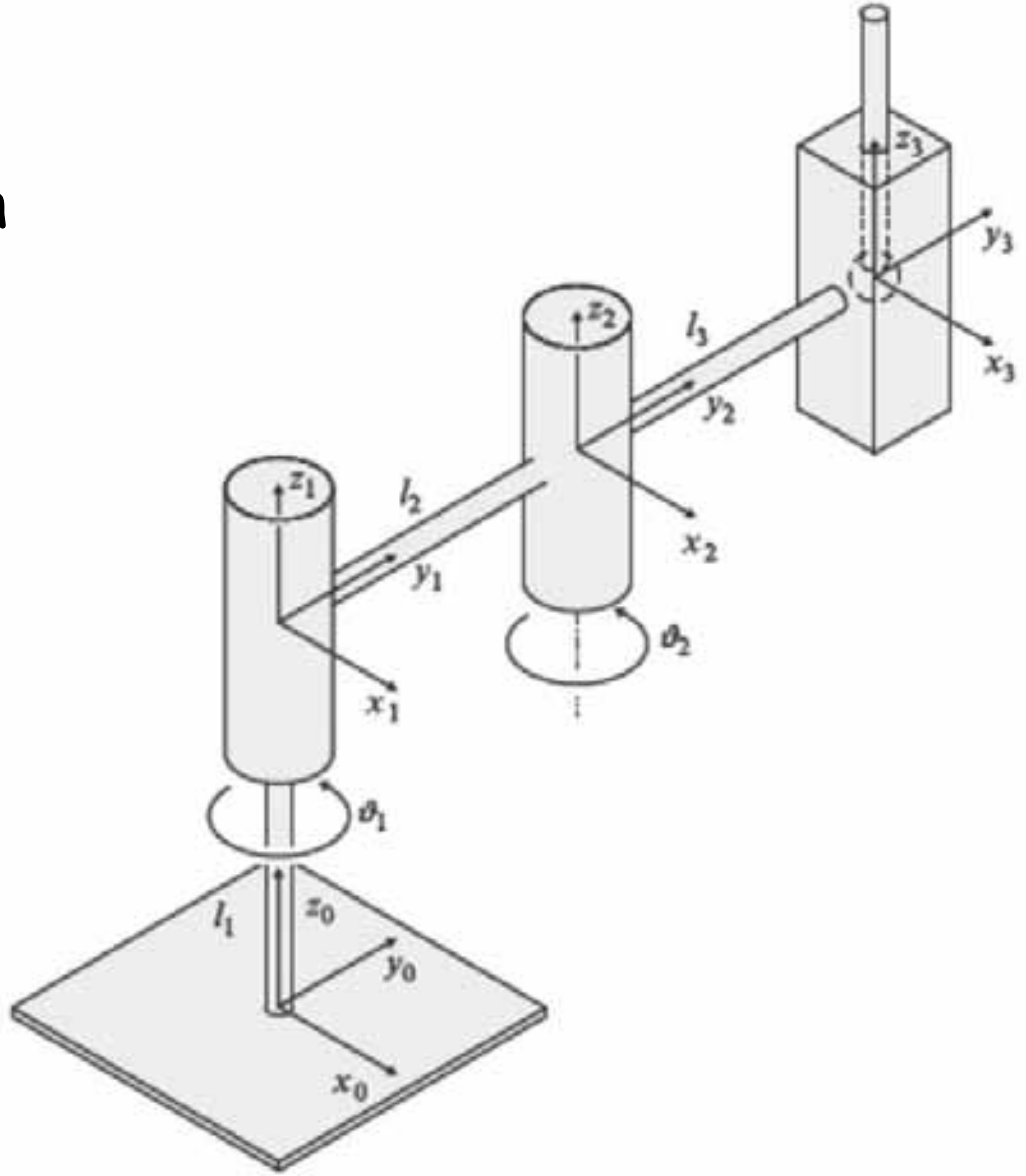
Initial (home) pose



Arbitrary pose

The pose of the end frame x3,y3,z3 with respect to the reference frame x0,y0,z0 is expressed by the following post multiplication of three pairs of homogenous transformation matrices:

$$ {}^{0}\mathbf{H}_3 = ({}^{0}\mathbf{H}_1\mathbf{D}_1)\cdot({}^{1}\mathbf{H}_2\mathbf{D}_2)\cdot({}^{2}\mathbf{H}_3\mathbf{D}_3). $$

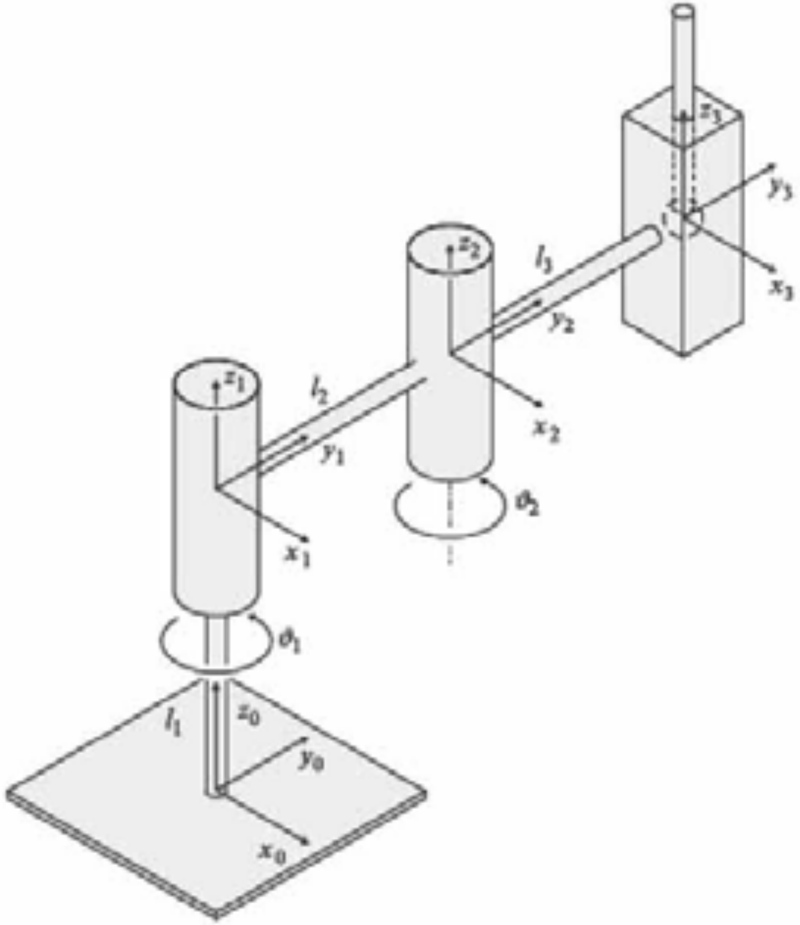


$$ {}^0\mathbf{H}_1\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1\mathbf{H}_2\mathbf{D}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c2 & -s2 & 0 & 0 \\ s2 & c2 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . $$

$$ {}^2\mathbf{H}_3\mathbf{D}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

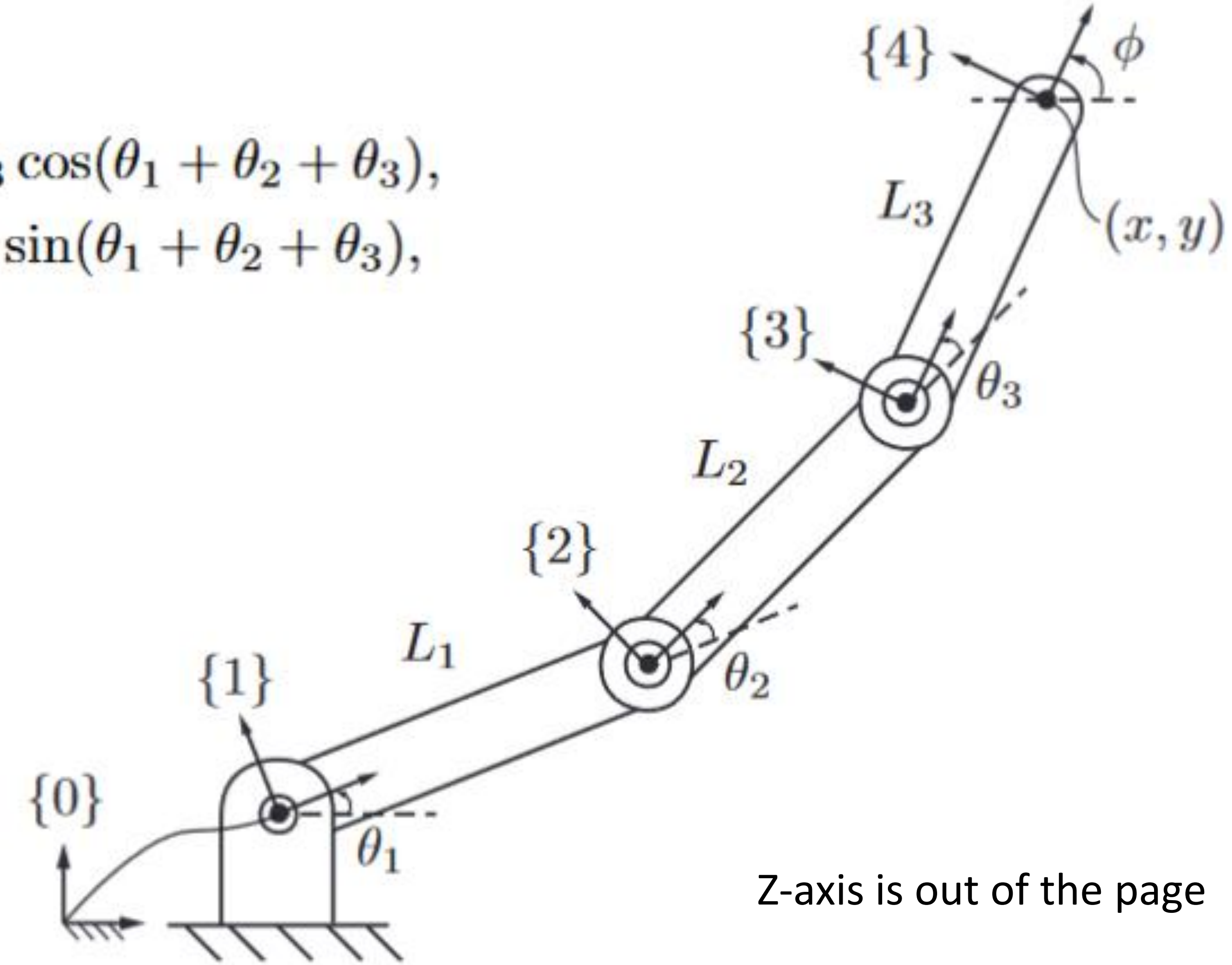
$$ {}^0\mathbf{H}_3 = \begin{bmatrix} c12 & -s12 & 0 & -l_3 s12 - l_2 s1 \\ s12 & c12 & 0 & l_3 c12 + l_2 c1 \\ 0 & 0 & 1 & l_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$ c12 = \cos(\vartheta_1 + \vartheta_2) = c1c2 - s1s2 \text{ and } s12 = \sin(\vartheta_1 + \vartheta_2) = s1c2 + c1s2. $$

Example: 3R planar open chain

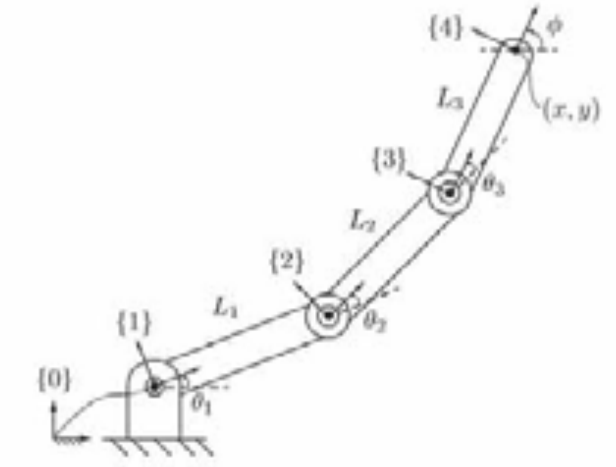
$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3),\\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3),\\
\phi &= \theta_1+\theta_2+\theta_3.
\end{aligned}
$$



Z-axis is out of the page

Example

The forward kinematics can be written as a product of four homogeneous transformation matrices:



$$T_{04} = T_{01}T_{12}T_{23}T_{34},$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

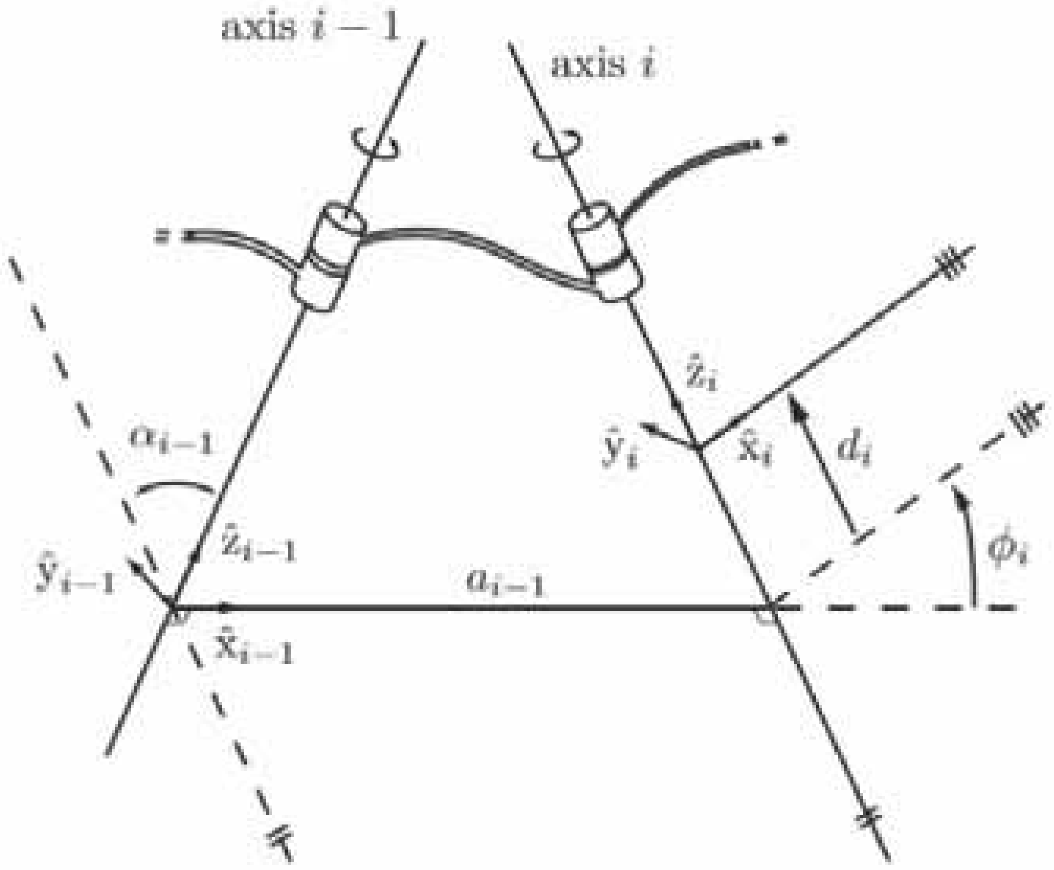
$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denavit–Hartenberg Convention

The forward kinematic is given by:

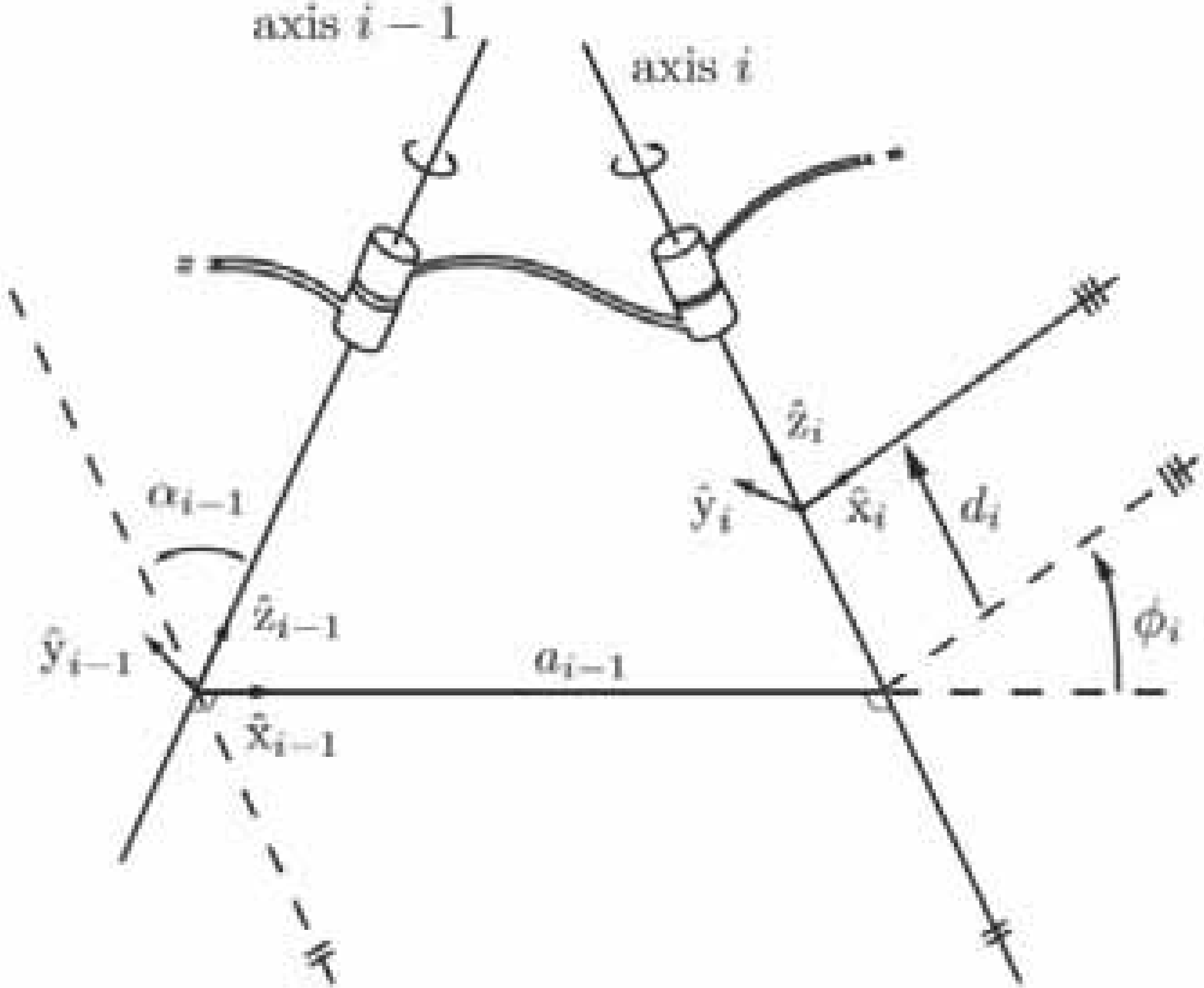
$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2)\cdots T_{n-1,n}(\theta_n)$$

With a set of rules for assigning link frames.



Rule 1

Assigning $\hat{z}_i$ axis coincide with joint i and z_{i-1} axis coincide with joint axis i-1. the direction of positive rotation is determined by the right hand rule.

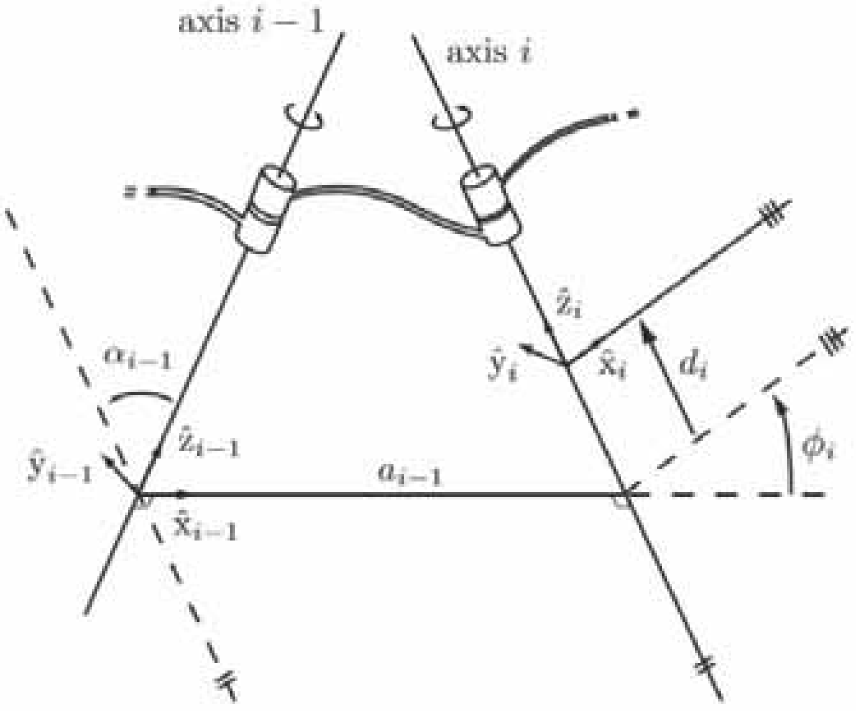


Rule 2: Origin of the link reference frame

Find the line segment that orthogonally intersects both the joint axes $\hat{z}_{i-1}$ and $\hat{z}_I$.

The $\hat{x}$-axis is chosen to be in the direction of the mutually perpendicular line pointing from the (i–1)-axis to the i-axis.

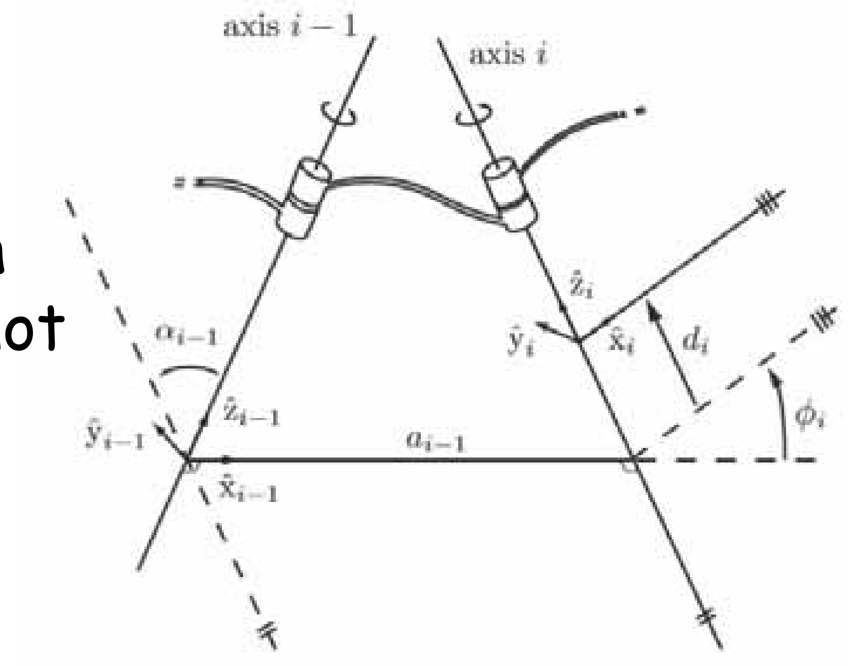
The $\hat{y}$-axis is then uniquely determined from the cross product $\hat{x}\ X\ \hat{y} = \hat{z}$.



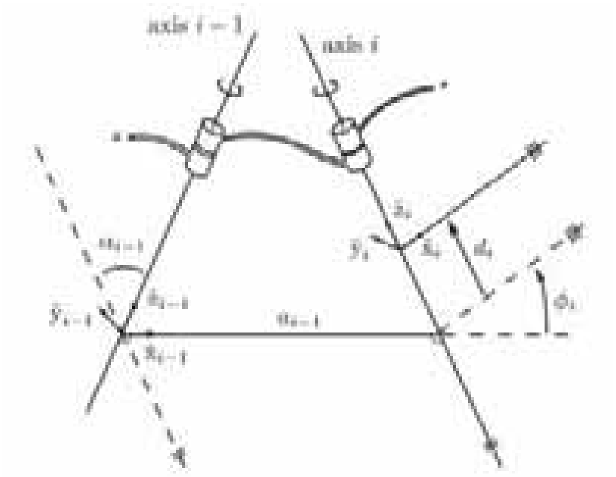
Rule 3: The D-H parameters

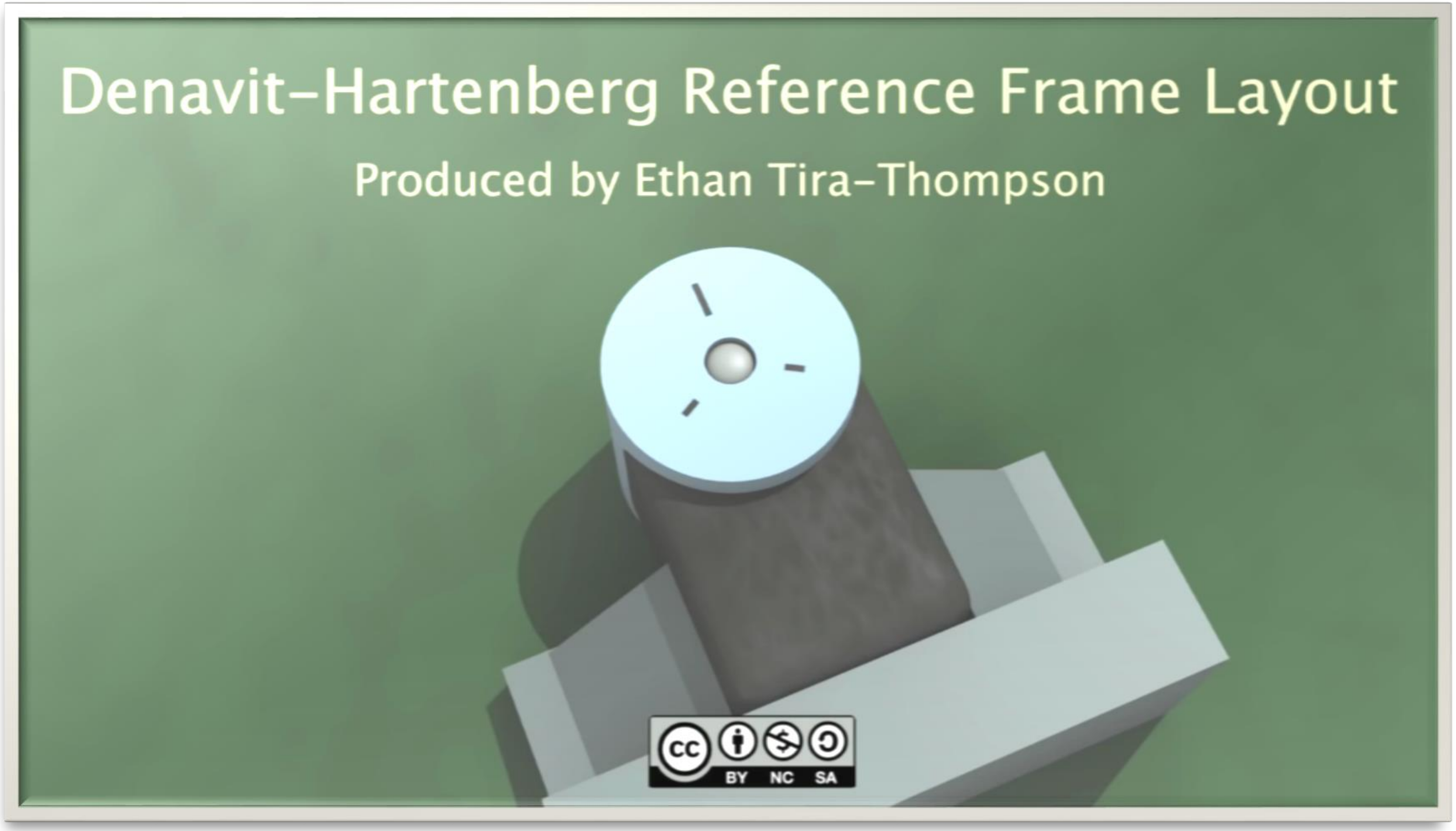
Define four parameters that exactly specify $T_{i-1,i}$.

1. The length of the mutually perpendicular line, denoted by the scalar a_{i-1} is called the **link length** of link i-1. Despite its name, this link length does not necessarily correspond to the actual length of the physical link.
2. The **link twist** α_{i-1} between $\hat{z}_{i-1}$ and $\hat{z}_i$ measured about $\hat{x}_{i-1}$.
3. The link offset d_i is the distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of the link-i frame.
4. The joint angle Φ_i is the angle from $\hat{x}_{i-1}$ to $\hat{x}i$ about Z_{i-1}.



Video Segment



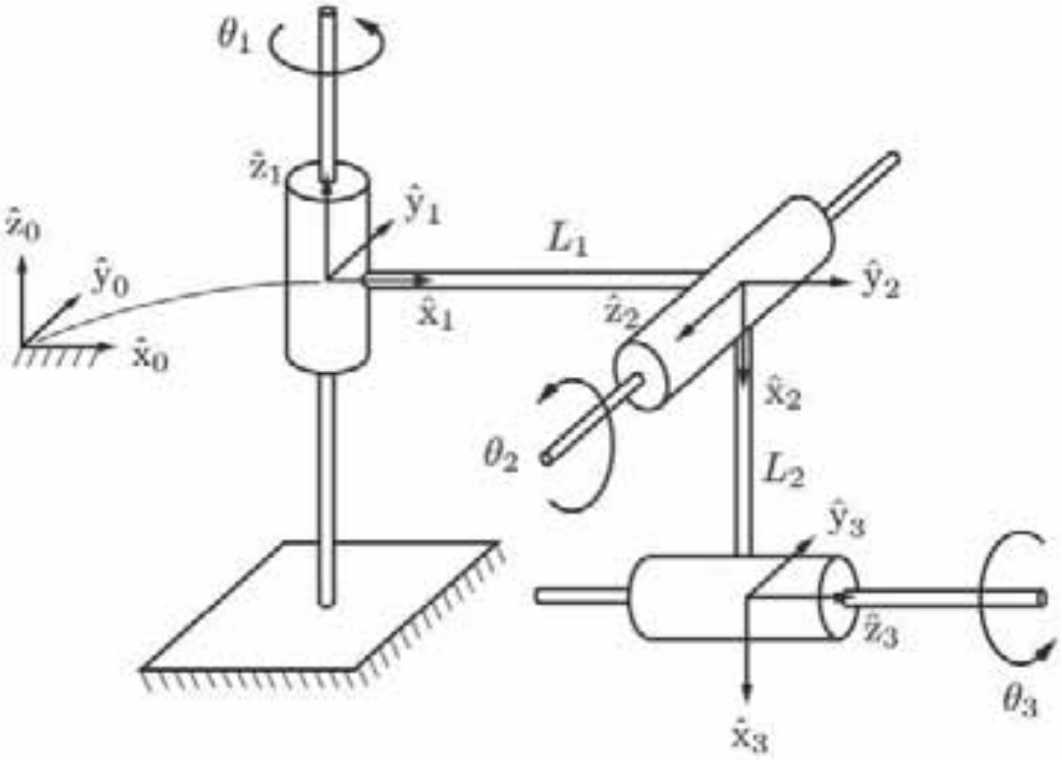


Example: 3R Spatial open chain

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	L_1	0	$\theta_2 - 90°$
3	−90°	L_2	0	θ_3

D-H parameters

1. The length of the mutually perpendicular line, denoted by the scalar a_{i-1} is called the **link length** of link i-1. Despite its name, this link length does not necessarily correspond to the actual length of the physical link.
2. 2. The **link twist** α_{i-1} between $\hat{z}_{i-1}$ and $\hat{z}_i$ measured about $\hat{x}_{i-1}$.
3. The link offset d_i is the distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of the link-i frame.
4. The joint angle Φ_i is the angle from $\hat{x}_{i-1}$ to $\hat{x}_I$.



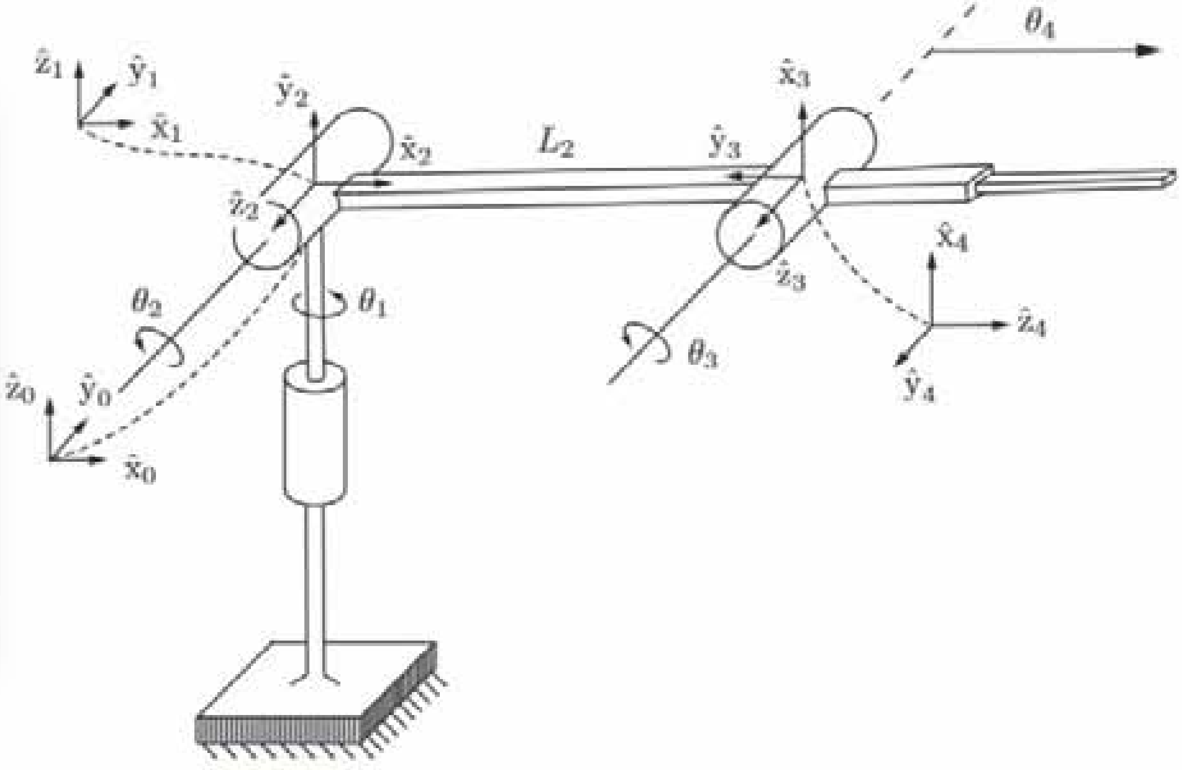
Manipulator Forward Kinematic

$$
\begin{aligned}
T_{i-1,i} &= \text{Rot}(\hat{x}, \alpha_{i-1})\text{Trans}(\hat{x}, a_{i-1})\text{Trans}(\hat{z}, d_i)\text{Rot}(\hat{z}, \phi_i) \\
&= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\text{Rot}(\hat{x}, \alpha_{i-1}) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\text{Trans}(\hat{x}, a_{i-1}) &= \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\text{Trans}(\hat{z}, d_i) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\text{Rot}(\hat{z}, \phi_i) &= \begin{bmatrix} \cos\phi_{i-1} & -\sin\phi_{i-1} & 0 & 0 \\ \sin\phi_{i-1} & \cos\phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Example: RRRP spatial open chain

1. The length of the mutually perpendicular line, denoted by the scalar a_{i-1} is called the **link length** of link i-1. Despite its name, this link length does not necessarily correspond to the actual length of the physical link.
2. 2. The **link twist** α_{i-1} between $\hat{z}_{i-1}$ and $\hat{z}_i$ measured about $\hat{x}_{i-1}$.
3. The link offset d_i is the distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of the link-i frame.
4. The joint angle Φ_i is the angle from $\hat{x}_{i-1}$ to $\hat{x}_I$.



i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90°$
4	90°	0	θ_4	0

Manipulator Forward Kinematic

$$T_{i-1,i} = \text{Rot}(\hat{x},\alpha_{i-1})\text{Trans}(\hat{x},a_{i-1})\text{Trans}(\hat{z},d_i)\text{Rot}(\hat{z},\phi_i)$$

$$= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

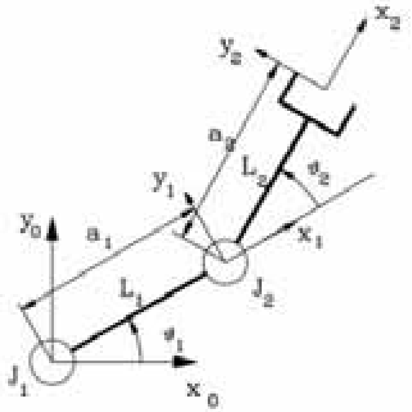
$$\text{Rot}(\hat{x},\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{Trans}(\hat{x},a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{Trans}(\hat{z},d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{Rot}(\hat{z},\phi_i) = \begin{bmatrix} \cos\phi_{i-1} & -\sin\phi_{i-1} & 0 & 0 \\ \sin\phi_{i-1} & \cos\phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example: 2dof planar manipulator



Denavit-Hartenberg parameters

	d	θ	a	α
L1	0	θ_1	a_1	0^o
L2	0	θ_2	a_2	0^o

The ${}^{i-1}\mathbf{H}_i$ matrices result:

$$
{}^0\mathbf{H}_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1 C_1 \\ S_1 & C_1 & 0 & a_1 S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^1\mathbf{H}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0\mathbf{T}_2 = {}^0\mathbf{H}_1\, {}^1\mathbf{H}_2 =
$$

$$
\begin{bmatrix} C_{12} & -S_{12} & 0 & a_1 C_1 + a_2 C_{12} \\ S_{12} & C_{12} & 0 & a_1 S_1 + a_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Reading List:

Kevin M. Lynch and Frank C. Park, 2017, Modern Robotics, 1st Edition, Cambridge University Press, chapter 4.

Watching list

https://www.youtube.com/watch?v=rA9tm0gTln8

Questions?