Scattering Theory

Lec.3

Case Study1:
Low energy Scattering

Low energy Scattering:

Example

Consider scattering by finite potential well, charactrized by $V = V_0$ for $r < a$, and $V = 0$ for $r \geq a$. Here, V_0 is constant. The potential is repulsive for $V_0 > 0$, and attractive for $V_0 < 0$.

The outside wavefunction is given by:

$$R_0(r) = e^{i\delta_0}\left[\cos\delta_0\, j_0(kr) - \sin\delta_0\, y_0(kr)\right]$$

$$= \frac{e^{i\delta_0} \sin(kr + \delta_0)}{kr} \quad \text{---- (45)}$$

The inside wavefunction

$$R_0 = B \frac{\sin(k' r)}{r}, \qquad B \text{ is constant}$$

$$E - V_0 = \frac{\hbar^2 k'^2}{2m}$$

For $E > V_0$, equation (45) is applies.
However, for $E < V_0$, we have

$$R_0(r) = B\,\frac{\sinh(\bar{k}r)}{r}$$

$$V_0 - E = \frac{\hbar^2 \bar{k}^2}{2m}$$

and its radial derivative

$$\tan(ka + \delta_0) = \frac{k}{\bar{k}}\tan(\bar{k}a)$$

for $E > V_0$

for $E > V_0$

$$\tan(ka + \delta_0) = \frac{k}{\bar{k}}\tanh(\bar{k}a)$$

for $E < V_0$

Now: Consider attractive potential for which $E > V_0$, suppose $|V_0| >> E$

$$ka + \delta_0 \simeq \frac{k}{\bar{k}}\tan(\bar{k}a)$$

$$\delta_0 \simeq ka\left[\frac{\tan(\bar{k}a)}{\bar{k}a} - 1\right]$$

The scattering cross-section will be:

$$\sigma_{tot} = \frac{4\pi}{k^2} \sin^2 \delta_0$$

$$= 4\pi a^2 \left[\frac{\tan(k'a)}{k'a} - 1 \right]^2 \quad \text{---} (46)$$

$$k'a = \sqrt{k^2 a^2 + \frac{2m|V_0|a^2}{\hbar^2}}$$

for small value of ka

$$\boxed{k'a \simeq \sqrt{\frac{2m|V_0|a^2}{\hbar^2}}}$$

~~[illegible]~~

The total cross-section (for s-wave) is independent of the energy of incident particles.

Note For ($k'a \simeq 4.49$) at which $\delta \longrightarrow \pi$, and the scattering cross-section (46) vanishes. In fact, the the σ is not exactly zero, ~~[illegible]~~ becaus contributeis from $\ell > 0$ partial waves It follows that for certain values of (V_0 and k) which gives almost perfect transmition of incident wave. This effect is called "Ramsauer-Townsend" effect.

Case study2: Resonance

Resonance:

let as assume that the quantity $\sqrt{\frac{2m}{\hbar^2}|V_0|a^2}$ is slightly less than $\pi/2$. Then by increasing incident energy $\bar{k}a$ which is mentioned in the previous lecture

$$\bar{k}a = \sqrt{k^2a^2 + \frac{2m|V_0|a^2}{\hbar^2}}$$

can approach the value $\pi/2$.

which leads to

$$\tan(\bar{k}a) \longrightarrow \infty$$

so we cannot able to assume the the right side of equation:

$$\tan(ka + \delta_0) = \frac{k}{\bar{k}} \tan(\bar{k}a)$$

is small.

In fact, it follow from that at the value of the incident energy when $\bar{k}a = \frac{\pi}{2}$, then we also have

$$ka + \delta_0 = \pi/2 \quad \text{or} \quad \delta_0 \simeq \frac{\pi}{2}$$

(As we assume $ka \ll 1$).

So, in this case

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0 = 4\pi a^2\left(\frac{1}{k^2a^2}\right) \qquad \text{----(47)}$$

The magnitude of cross-section is much larger than that given in Eq(46) for $k'a \neq \frac{\pi}{2}$

Note

The origin of this is quit simple, the condition:

$$\sqrt{\frac{2m}{\hbar^2}|V_0|a^2} = \pi/2$$

is equivalent to the condition of sperical well of depth V_0 posses abound state at (zero energy). Therefor, for potential well that verify above equation, the energy of scattering system is same as the energy of the bound state. So, the incident particle would like to form a bound state in the potential well. But, the bound state is not stable because the systeme has small (positive energy).

This sort of scattering resonance is best understood as the capture of an incident particle to form a metastabe bound state.

* The cross-section for resonance scattering ~~___~~ is much larger than that for non-scattering.

** We can see that there is a resonance effect when the phase-shift of the S-wave packet takes the value $\frac{\pi}{2}$. suppose that δ_ℓ reaches the value $\frac{\pi}{2}$ at the incident energy E_0. So that:

$$\delta_0(E_0) = \frac{\pi}{2} \qquad \text{-----} \quad (48)$$

let us expand $\cot \delta_\ell$ in the vicinity of resonance energy, so

$$\cot\delta_\ell(E) = \cot\delta_\ell(E_0) + \left(\frac{d\cot\delta_\ell}{dE}\right)_{E=E_0}(E-E_0) + \cdots$$

$$= -\left(\frac{1}{\sin^2\delta_\ell}\frac{d\delta_\ell}{dE}\right)_{E=E_0}(E-E_0) + \cdots\cdots$$

---- (49)

Defining

$$\left(\frac{d\delta_\ell}{dE}\right)_{E=E_0} = \frac{2}{\Gamma}$$

we obtain:

$$\cot\delta_\ell(E) = -\frac{2}{\Gamma}(E-E_0)+\cdots$$

where Γ is the width of the resonance (comes later).

Jus remember that from

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

that the contribution of the ℓ th partial waves to the scattering cross-section is:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$$

$$= \frac{4\pi}{k^2}(2\ell+1)\frac{1}{1+\cot^2\delta_\ell}$$

Thus

$$\boxed{\sigma_\ell \simeq \frac{4\pi}{k^2}(2\ell+1)\frac{\Gamma^2/4}{(E-E_0)^2+\Gamma^2/4}}$$

---- (50)

Eq (50) is the famous (Breit - Wigner) formulla.

The variation of the partial - cross section σ_ℓ with the incident energy has the form of classical resonance curve. The quantity Γ is width of the resonance (Energy). We can explain equation (50) as it describing the absorption of an incident particle to form a metastable state, of energy E_0 and lifetime $\tau = \hbar/\Gamma$.