زانكۆى سەلاحەدین - هەولێر
Salahaddin University-Erbil

Exact and numerical solution of integral equation

Research project

Submitted to The Department of Mathematics In Partial Fulfillment Of The Requirements For The Degree Of BSc. In MATHEMATICS

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CERTIFICATION OF THE SUPERVISOR

I certify that this work was prepared under my supervision at the Department of Mathematics /College of Education/ Salahaddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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ABSTRACT

In this work we find the Exact and numerical solution of integral equation which $\int_a^x K(x,y)\,\varphi(y)dy = f(x)$,used to find the Approximate numerical solutions, this method is tested on Linear Fredholm Integral Equation (LFIE) and Volterra Integral Equation (LVIE) of second kind, was highly accurate.

The examples are given to explain the solution procedures. The Comparison of numerical solution was compatible with exact solutions.

Contents

CHAPTER ONE

INTRODUCTION

1.1 Introduction

The theory and applications of integral equations, or, as it is often called, of the inversion of definite integrals, have come suddenly into prominence and have held during the last half dozen years a central place in the attention of mathematicians. By an integral equation, is understood an equation in which the unknown function occurs under one or more signs of definite integration. Mathematicians have so far devoted their attention mainly to two peculiarly simple types of integral equations, - the linear equations of the first and second kinds, - and we shall not in this tract attempt to go beyond these cases. We shall also restrict ourselves to equations in which only simple (as distinguished from multiple) integrals occur. This restriction, however, is quite an unessential one made solely to avoid unprofitable complications at the start, since the results we shall obtain usually admit of an obvious extension to the case of multiple integrals without the introduction of any new difficulties. In this respect integral equations are in striking contrast to the closely related differential equations, where the passage from ordinary to partial differential equation. (M.Bochar 2015)

Integral equation formulations are a competitive strategy in computational electromagnetics but, lamentably, are often plagued by ill-conditioning and by related numerical instabilities that can jeopardize their effectiveness in several real case scenarios. Luckily, however, it is possible to leverage effective preconditioning and regularization strategies that can cure a large majority of these problems. Not surprisingly, integral equation preconditioning is currently a quite active field of research. To give the reader a prepositive overview of the state of the art, this paper will review and discuss the main advancements in the field of integral equation preconditioning in electromagnetics summarizing the strengths and weaknesses of each technique. The contribution will guide the reader through the choices of the right preconditioner for a given application scenario. This will be complemented by new analyses and discussions which will provide a further. (Adrien, et al. 2021)

An integral equation is defined as an equation in which the unknown function u(x) to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. An integral equation can be classified as a linear or nonlinear integral equation as we have seen in the ordinary and partial differential equations. we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations. (Rahman 2013)

CHAPTER TWO

Background

there are a number of classifications of linear integral equations that distinguish different kinds of equations. The following are the most frequently studied. (Hochstadt 2011)

$$\int_a^b K(x,y)\,\varphi(y)dy = f(x), \qquad (1)$$

$$\varphi(x) - \lambda\int_a^b K(x,y)\,\varphi(y)dy = f(x), \qquad (2)$$

$$\text{a(x)}\,\varphi(x) - \lambda\int_a^b K(x,y)\,\varphi(y)dy = f(x), \qquad (3)$$

The above equations (1) -(3) are generally known as Fred Holm equations of the first, second, and third kind, respectively. The interval (a,b)may in general be a finite interval or

(-$\infty, b]$, $[a,\infty)$, or $(-\infty,\infty)$, where a and b are finite. If a(x) does not vanish one can divide (3) by a(x) to reduce it to (2). The functions $f(x)$, a(x), and $K(x,y)$ are presumably known functions and the function $\varphi(x)$ is unknown . The parameter λ could be absorbed in the function $K(x,y)$, but it is convenient to retain it in the equation. Its role will become clearer when the operators in question will be studied. The function $K(x,y)$ is generally known as the kernel of the equation.

A second class of equations are the Volterra equations of the first, second, and third kind, namely

$$\int_a^x K(x,y)\,\varphi(y)dy = f(x), \qquad (4)$$

$$\varphi(x) - \lambda\int_a^x K(x,y)\,\varphi(y)dy = f(x), \qquad (5)$$

$$a(x)\,\varphi(x) - \lambda\int_a^x K(x,y)\,\varphi(y)dy = f(x), \qquad (6)$$

One can view these as special cases of Fred Holm equations. The letter reduce to the corresponding Volterra equations if $K(x,y) = 0$ for y > x . Nevertheless, Volterra equations have many interesting properties that do not emerge from the general theory of Fred Holm equations so that a separate study is definitely warranted.

Equations (1)-(6) have one thing in common; they are all linear equation

That is, the functions φ enters the equations in a linear manner so that

$$\int_a^b K(x,y)\,[c1\varphi1(y) + c2\varphi2(y)]dy$$

$$=c1\int_a^b K(x,y)\,\varphi1(y)dy + c2\int_a^b K(x,y)\,\varphi2(y)dy.$$

Definition of a function 2.1: A function f from a set A to a set B (f: A → B) is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is called the domain of the function f. The range or codomain of the function is the set of elements in B that are in correspondence with elements in A. In the case of functions described as equations in two variables, the variable x is the independent variable and the variable y is the dependent variable. In general, a function is denoted as f(x) (read f of x), where f is the name of the function, x is the domain value and f(x) is the range value y for a given x. The process of finding the value of f(x) for a given value of x is called evaluating a function. (B.George and weir 2014)

Definition of Integral Equations 2.2: Integral equation is an equation in which the unknown, say a function of numerical variable, occurs under an integral. That means a functional equation involving the unknown function under one or more integrals. (Hochstadt 2011)

The main types of integral equations are the following:

Definition of Volterra integral equations 2.3: The most standard form of Volterra linear integral equations is of the form

$$\varphi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt \qquad (2.1)$$

where the limits of integration are function of x and the unknown function u(x) appears linearly under the integral sign. If the function φ(x) = 1, then equation (2.1) simply become

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt \qquad (2.2)$$

and this equation is known as the Volterra integral equation of the second kind; whereas if φ(x)= 0, then equation (2.1)
becomes

$$f(x)+\lambda\int_a^x K(x,t)\, u(t)dt=0 \qquad (2.3)$$

which is known as the Volterra equation of the first kind. (Rahman 2013)

Definition of Fredholm integral equations 2.4:

The most standard form of Fred Holm linear integral equations is given by the form

$$\varphi(x)u(x)=f(x)+\lambda\int_a^x K(x,t)\,u(t)dt \qquad (2.4)$$

where the limits of integration a and b are constants and the unknown function u(x) appears linearly under the integral sign. If the function φ(x) = 1, then (2.4) becomes simply

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\,u(t)dt \qquad (2.5)$$

and this equation is called Fred Holm integral equation of second kind; whereas if φ(x) = 0, then (2.4) yields

$$f(x) + \lambda \int_a^x K(x,t)\,u(t)dt \qquad (2.6)$$

which is called Fredholm integral equation of the first kind. (Rahman 2013)

2.5 The series solution method

We shall introduce a practical method to handle the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt$$

In the series solution method we shall follow a parallel approach known as the Frobenius series solution usually applied in solving the ordinary differential equation around an ordinary point (see Ref. [9]). The method is applicable provided that $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylor's expansion around $x = 0$. Accordingly, $u(x)$ can be expressed by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the coefficients a and x are constants that are required to be determined. Substitution of equation (2.38) into the above Volterra equation yields

$$\sum_{n=0}^{\infty} a_n x^n = f(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{\infty} a_n t^n dt$$

so that using a few terms of the expansion in both sides, we find

$$\begin{aligned} a_0 \quad & + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ = \quad & f(x) + \lambda \int_0^x K(x,t) a_0 dt + \lambda \int_0^x K(x,t) a_1 t dt \\ & + \lambda \int_0^x K(x,t) a_2 t^2 dt + \cdots + \lambda \int_0^x K(x,t) a_n t^n dt + \cdots \end{aligned}$$

In view of equation (2.40), the integral equation will be reduced to several traditional integrals, with defined integrals having terms of the form $t^n, n \geq 0$ only. We then write the Taylor's expansions for f(x) and evaluate the first few integrals in equation (2.40). Having performed the integration, we equate the coefficients of like powers of x in both sides of equation (2.40). This will lead to a complete determination of the unknown coefficients $a_0, a_1, a_2, \ldots a_n$ .... Consequently, substituting these coefficients $a_n, n \geq 0$, which are determined in equation (2.40), produces the solution in a series form. We will illustrate the series solution method by a simple example. (Rahman 2013)

2.6Integro-differential equations

In the early 1990, Vito Volterra studied the phenomenon of population growth, and new types of equations have been developed and termed as the Integro-differential equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign. In the electrical engineering problem, the current $I(t)$ flowing in a closed circuit can be obtained in the form of the following Integro-differential equations,

$$L\frac{dI}{dt} + RI + \frac{1}{c}\int_0^t I(\tau)d\tau = f(t), \qquad I(0) = I_0 \tag{2.7}$$

Where L is the inductance, R the resistance, C the capacitance, and $f(t)$ the applied voltage. Similar examples can be cited as follows:

$$u''(x) = f(x) + \lambda \int_0^x (x-t)u(t)dt, \quad u(0) = 0, u'(0) = 1, \tag{2.8}$$

$$u'(x) = f(x) + \lambda \int_0^1 (xt)u(t)dt, \quad u(0) = 1 \tag{2.9}$$

Equation (2.7) and (2.8) are of Volterra type Integro-differential equations**,** whereas equation (2.9) Fredholm type Integro-differential equations**.** These terminologies were concluded because of the presence of indefinite and definite integrals. (Rahman 2013)

2.7 The method of successive approximations

In this method, we replace the unknown function $u(x)$ under the integral sign of the Volterra equation any selective real-valued continuous function $u_0(x)$,

called the zeroth approximation. This substitution will give the first approximation $u_1(x)$ by

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)u_0(t)dt \tag{2.10}$$

It is obvious that $u_1(x)$ is continuous if $f(x), K(x,t)$, and $u_0(x)$ are continuous. The second approximation $u_2(x)$ can be obtained similarly by replacing $u_0(x)$ in equation (2.3) by $u_1(x)$ obtained above. And we find

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t)u_1(t)dt \tag{2.11}$$

Continuing in this manner, we obtain an infinite sequence of functions

$$u_0(x), u_1(x), u_2(x), \dots, u_n(x), \dots$$

that satisfies the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt \tag{2.12}$$

for $n = 1,2,3, \dots$ and $u_0(x)$ is equivalent to any selected real-valued function. The most commonly selected function for $u_0(x)$ are 0,1 , and x. Thus, at the limit, the solution $u(x)$ of the obtained as

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{2.13}$$

so that the resulting solution $u(x)$ is independent of the choice of the zeroth approximation $u_0(x)$. This process of approximation is extremely simple. However, if we follow the Picard's successive approximation method, we need to set $u_0(x) = f(x)$, and determine $u_1(x)$ and other successive approximation as follows:

$$
\begin{aligned}
u_1(x) = &\quad f(x) + \lambda \int_0^x K(x,t) f(t)dt \\
u_2(x) = &\quad f(x) + \lambda \int_0^x K(x,t) u_1(t)dt \\
&\quad \ldots \ldots \\
u_{n-1}(x) = &\quad f(x) + \lambda \int_0^x K(x,t) u_{n-2}(t)dt \\
u_n(x) = &\quad f(x) + \lambda \int_0^x K(x,t) u_{n-1}(t)dt
\end{aligned}
\tag{2.14}
$$

The last equation is the recurrence relation. Consider

$$
\begin{aligned}
u_2(x) - u_1(x) = &\quad \lambda \int_0^x K(x,t) \left[f(t) + \lambda \int_0^t K(t,\tau) f(\tau)d\tau \right] dt \\
&\quad -\lambda \int_0^x K(x,t) f(t)dt \\
= &\quad \lambda^2 \int_0^x K(x,t) \int_0^t K(t,\tau) f(\tau) d\tau dt \\
= &\quad \lambda^2 \psi_2(x)
\end{aligned}
\tag{2.15}
$$

where

$$
\psi_2(x) = \int_0^x K(x,t)dt \int_0^t K(t,\tau) f(\tau)d\tau \tag{2.16}
$$

Thus, it can be easily observed from equation (2.15) that

$$
u_n(x) = \sum_{m=0}^{n} \lambda^m \psi_m(x) \tag{2.17}
$$

if $\psi_0(x) = f(x)$, and further that

$$
\psi_m(x) = \int_0^x K(x,t) \psi_{m-1}(t)dt \tag{2.18}
$$

where $m = 1,2,3, \ldots$ and hence $\psi_1(x) = \int_0^x K(x,t) f(t)dt$.

The repeated integrals in equation (2.16) may be considered as a double integral over the triangular region indicated in Figure 2.1; thus interchanging the order of

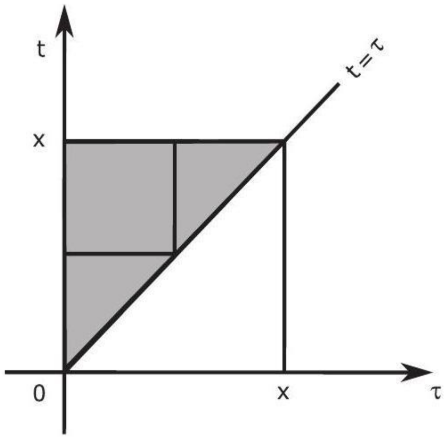

Figure 2.1: Double integration over the triangular region (shaded area).

integration, we obtain

$$\begin{aligned}\psi_2(x) &= \int_0^x f(\tau)d\tau \int_\tau^x K(x,t)K(t,\tau)dt \\ &= \int_0^x K_2(x,\tau)f(\tau)d\tau\end{aligned}$$

where $K_2(x,\tau) = \int_\tau^x K(x,t)K(t,\tau)dt$. Similarly, we find in general

$$\psi_m(x) = \int_0^x K_m(x,\tau)f(\tau)d\tau,\ m = 1,2,3,\ldots \tag{2.19}$$

where the iterative kernels $K_1(x,t) \equiv K(x,t), K_2(x,t), K_3(x,t), \ldots$ are defined by the recurrence formula

$$K_{m+1}(x,t) = \int_t^x K(x,\tau)K_m(\tau,t)d\tau,\ m = 1,2,3,\ldots \tag{2.20}$$

Thus, the solution for $u_n(x)$ can be written as

$$u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \psi_m(x) \tag{2.21}$$

It is also plausible that we should be led to the solution means of the sum if it exists, of the infinite series defined by equation (2.17). Thus, we have using equation (2.19)

$$
\begin{aligned}
u_n(x) &= f(x) + \sum_{m=1}^{n} \lambda^m \int_0^x K_m(x,\tau) f(\tau) d\tau \\
&= f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \right\} f(\tau) d\tau, \qquad (2.22)
\end{aligned}
$$

hence it is also plausible that the solution of equation (2.1) will be given by as $n \to \infty$

$$
\begin{aligned}
\lim_{n\to\infty} u_n(x) &= u(x) \\
&= f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \right\} f(\tau) d\tau \\
&= f(x) + \lambda \int_0^x H(x,\tau;\lambda) f(\tau) d\tau
\end{aligned}
$$

where

$$
H(x,\tau;\lambda) = \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \qquad (2.23)
$$

is known as the resolvent kernel. (Rahman 2013)

CHAPTER THREE

Example 3.1: Obtain the solution of the Volterra equation using the series method.

$u(x) = x + \int_0^x (t-x)u(t)dt$ ....(3.1)

using the series method.

Solution:

$u(x) = \sum_{k=0}^{\infty} a^k x_k$

$$u(x) = x + \int_0^x (t-x)u(t)dt$$

$$\sum_{k=0}^{\infty} a_k x^k = x + \int_0^x (t-x)\sum_{k=0}^{\infty} a_x x^k$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$

$= x + \int_0^x (t-x)(a_0 + a_2t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + a_8t^8 + a_9t^9 + \cdots)dt$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$

$= x + \int_0^x [\, a_0t + a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + a_5t^6 + a_6t^7 + a_7t^8 + a_8t^9 + a_9t^{10} + \cdots] -$

$$a_0x + a_1tx + a_2t^2 + a_3t^3x + a_4t^4x + a_5t^5x + a_6t^6x + a_7t^7x + a_8t^8x + a_9t^9x + \cdots]$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$

$= x + (a_0\frac{t^2}{2} + a_1\frac{t^3}{3} + a_2\frac{t^4}{4} + a_3\frac{t^5}{5} + a_4\frac{t^6}{6} + a_5\frac{t^7}{7} + a_6\frac{t^8}{8} + a_7\frac{t^9}{9} + a_8\frac{t^{10}}{10} + a_9\frac{t^{11}}{11})]_0^x -$

$$(a_0tx + a_1\frac{t^2}{2}x + a_2\frac{t^3}{3}x + a_3\frac{t^4}{4}x + a_4\frac{t^5}{5}x + a_5\frac{t^6}{6}x + a_6\frac{t^7}{7}x + a_7\frac{t^8}{82}x + a_8\frac{t^9}{9}x + a_9\frac{t^{10}}{10}x)\,]_0^x$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$

$= x + (a_0\frac{x^2}{2} + a_1\frac{x^3}{3} + a_2\frac{x^4}{4} + a_3\frac{x^5}{5} + a_4\frac{x^6}{6} + a_5\frac{x^7}{7} + a_6\frac{x^8}{8} + a_7\frac{x^9}{9} + a_8\frac{x^{10}}{10} +$
$a_9\frac{x^{11}}{11})$

$-(a_0x^2 + a_1\frac{x^3}{2} + a_2\frac{x^4}{3} + a_3\frac{x^5}{4} + a_4\frac{x^6}{5} + a_5\frac{x^7}{6} + a_6\frac{x^8}{7} + a_7\frac{x^9}{8} + a_8\frac{x^{10}}{9} +$
$a_9\frac{x^{11}}{10})$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$
$= x - a_0\left(\frac{x^2}{2} - x^2\right) - a_1\left(\frac{x^3}{3} - \frac{x^3}{2}\right) - a_2\left(\frac{x^4}{4} - \frac{x^4}{3}\right) - a_3\left(\frac{x^5}{5} - \frac{x^5}{4}\right) -$
$a_4\left(\frac{x^6}{6} - \frac{x^6}{5}\right) - a_5\left(\frac{x^7}{7} - \frac{x^7}{6}\right) - a_6\left(\frac{x^8}{8} - \frac{x^8}{7}\right) - a_7\left(\frac{x^9}{9} - \frac{x^9}{8}\right) - a_8\left(\frac{x^{10}}{10} - \frac{x^{10}}{9}\right) -$
$a_9\left(\frac{x^{11}}{11} - \frac{x^{11}}{10}\right)$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots$$
$$= x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 - \frac{a_2}{12}x^4 - \frac{a_3}{20}x^5 - \frac{a_4}{30}x^6 - \frac{a_5}{42}x^7 - \frac{a_6}{56}x^8 - \frac{a_7}{72}x^9 - \frac{a_8}{90}x^{10} - \frac{a_9}{110}x^{11}$$

$a_0 = 0, a_1 = 1$

$a_2 = -\frac{a_0}{2}$

$a_3 = -\frac{a_1}{6}$

$a_4 = -\frac{a_2}{12}$

$a_5 = -\frac{a_3}{20}$

$a_6 = -\frac{a_4}{30}$

$a_7 = -\frac{a_5}{42}$

$a_8 = -\frac{a_6}{56}$

$$a_9 = -\frac{a_7}{72}$$

$$a_0 = a_2 = a_4 = a_6 = a_8 = 0$$

$a_{2m} = 0$ for m ≥ 0 ....(3.2)

$a_{2m+1} = \frac{(-1)^m}{2m+1}$ for m ≥ 0 ....(3.3)

Substituting equ (3.2) and (3.3) in equ (3.1) we get

$$u(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \; x^{2n+1}$$

$$u(x) = \sin x$$

Example 3.2:

Solve the following Volterra integral equation of the second kind of successive approximation method

$$u(x) = x + \int_0^x (t - x) u_0(t) dt \qquad \text{....... (3.4)}$$

Solution: we first select any real valued function for the zeroth approximation, hence we set

$$u_0(x) = 0 \qquad \text{....... (3.5)}$$

Substituting in to the right hand side of (eq.3.4) we find

$$u(x) = x + \int_0^x (t - x) u_0(t) dt \qquad \text{....... (3.6)}$$

And this gives the first approximation of $u(x)$ by

$$u(x) = x \qquad \text{....... (3.7)}$$

Inserting (eq.3.7) in to (eq.3.6) to replace $u_0(x)$ we obtain

$$u_2(x) = x + \int_0^x (t - x)(t)dt$$

Where by integration we determine the second approximation of $u(x)$ by

$$u_2(x) = x - \frac{1}{3!}x^3$$

Continuing in the same manner we find the third approximation of $u(x)$ given by

$$u_3(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

Accordingly, the nth approximation is given by

$$u_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \ \frac{x^{2k-1}}{(2k-1)!} \ , n \geq 1$$

Consequently, the solution $u(x)$ of (eq.5) is given by

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$u(x) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} (-1)^{k-1} \ \frac{x^{2k-1}}{(2k-1)!}\right) \quad \dots\dots(3.8)$$

$$u(x) = \ \sin x$$

To show that $u(x)$ obtained in (eq.3.8) does not depend on the selection of $u_0(x)$, we will solve the equation (eq.3.4) by selecting

$$u_0(x) = x$$

using The new selection of $u_0(x)$ in the right hand side of (eq.3.4) we obtain

$$u_1(x) = x + \int_0^x (t - x)tdt$$

Which gives the first approximation by

$$u_1(x) = x - \frac{1}{3!}x^3$$

$$u_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

In a parallel manner we find that

$$u_n(x) = \sum_{k=0}^{n} (-1)^k \ \frac{x^{2k+1}}{(2k+1)!} \ , n \geq 0$$

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$u(x) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} (-1)^k \ \frac{x^{2k+1}}{(2k+1)!} \right)$$

$$u(x) = \ \sin x$$

Example 3.3: Solve the following Volterra integer differential equation by using the series solution method.

$u^{''} = x \cosh x - \int_0^x tu(t)dt \,, \quad u(0) = 0 \,, \ u^{''}(0) = 1$ ……. (3.9)

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n \, x^n \qquad \ldots \ldots (3.10)$$

Into both sides of the equation (3.9) and using the Taylor expansion of $\cosh x$ we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n \, x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \qquad \ldots \ldots (3.11))$$

Using the initial Conditions Yields

$a_0 = 0,$ …….(3.12)

$a_1 = 1,$ …….(3.13)

Evaluating the traditional integrals that involve terms of the form t^n , $n \geq 0$,and using few terms from both sides yields

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$

$$= x\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(\frac{1}{3}x^3 + \frac{1}{4}a_nx^4 + \cdots\right) \qquad \ldots\ldots. (3.14)$$

Equating the coefficients of like powers of x in both sides we find

$$a_2 = 0,$$

$$a_3 = \frac{1}{3!}, \qquad \ldots\ldots.(3.15)$$

$$a_4 = 0,$$

And generally

$$a_{2n} = 0, \text{ for } n \geq 0, \qquad \ldots\ldots.(3.16)$$

$$a_{2n+1} = \frac{1}{(2n+1)!}, \text{ for } n \geq 0 \qquad \ldots\ldots.(3.17)$$

Using (3.10) we find the solution $u(x)$ in a series from

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \qquad \ldots\ldots. (3.18)$$

And in a closed from, the exact solution is given by

$$u(x) = \sinh x\,, \qquad \ldots\ldots.(3.19)$$

Obtained upon using the Taylor Series of $\sinh x$.

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پوختە:

ئەو لێکۆڵینەوەیە کە بۆ(Exact and numerical solution of integral equation

)یە کەهاوکێشەکەی بریتیە لە$\int_a^x K(x,y)\,\varphi(y)dy = f(x)$باکاردێ بۆ

دۆزینەوەی شیکاری هاوکێشە بە نزیکەیی لە نۆمیریکاڵ دا ئەو میسۆدە یان

دەتوانین بڵێین ئەو ڕیگایە یە بەکاردێ لەگەڵ (Linear fredhold integral

equation) وە لەگەڵ (Volterrn integral equation)کە لە پلە دوودا بوو.وە ئەو

نمونەکانیشی کە دراون بۆ ڕونکردنەوەی کرداری ئەنجامەکەیە ،وە بەراوردی

شیکارە نزیکەکەی نیومێریکاڵ و شیکارە دەقیق (ئەسڵەکە)دەکات .