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Salahaddin University-Erbil

The numerical treatment of Boundary value problem (B.V.P)

Research Project

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of BSc. in MATHEMATIC

Prepared by:

Muhemmed Majeed Qadir

Supervised by:

Assist. Prof. Dr. Ivan Subhi Latif

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CERTIFICATION OF THE SUPERVISOR

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:

Supervisor**: Dr. Ivan Subhi Latif**

Scientific grade: **Assistant Professor**

Signature:

Supervisor: **Dr. Pakhshan M Hassan**

Scientific grade: **Doctor**

Date: 07 / 04 / 2023

In view of available recommendations, I forward this word for debate by the examining committee.

Signature:

Name: **Dr. Rashad Rashid Haji**

Scientific grade**: Assistant Professor**

Chairman of the Mathematics Department

Date: 07 / 04 / 2023

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Abstract

In this report we studied numerical methods for solving boundary value problem. First, we declared some methods for solving it like Taylor method Euler method Runge-kutta method, finite difference method and shooting method. Finally, some examples given to illustrate these methods.

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CHAPTER ONE

INTRODUCTION

Numerical methods are essential for solving differential equations that cannot be solved analytically. The finite element methods are used for solving equilibrium configurations, whereas finite difference methods are used for approximating derivatives and solving a wide range of differential equations in various fields.

The calculus of finite differences involves constructing approximation formulae for derivatives by sampling function values at nearby points. These approximation formulae can then be used to develop finite difference methods, which can be applied to solve differential equations numerically. Finite difference methods are versatile and can be designed to solve many types of differential equations, making them a powerful tool in various scientific and engineering fields. (Oliver, 2013) (Todd, 1977)

Differential equations are fundamental to modeling and understanding many physical phenomena in science and engineering. In many cases, analytical solutions to these equations may not be possible or practical, and numerical methods are used to approximate the solution.

As you mentioned, the general solution of an n^{th}order differential equation contains n arbitrary constants. These constants can be determined by specifying n conditions, which are typically either initial conditions or boundary conditions.

An initial value problem involves specifying all n conditions at a single initial point, typically the starting time for a time-dependent problem. This type of problem is often encountered in physics and engineering when modeling systems that evolve over time. (AMES, 1997)

A boundary value problem involves specifying conditions at two or more points, often at the boundaries of a region or domain. This type of problem is typically encountered in problems involving steady-state or equilibrium solutions, such as in heat transfer or fluid mechanics. (Sivaiah, 2013)

Numerical methods, such as finite difference, finite element, and spectral methods, are commonly used to solve both initial value and boundary value problems numerically. These methods involve discretizing the problem domain and solving a system of algebraic equations that approximate the differential equation at the discrete points. The accuracy of the numerical solution depends on the choice of discretization method, step size, and other parameters, and can be assessed through error analysis. (Sivaiah, 2013)

CHAPTER TWO

BACKGROUND

Definition 2.1: (Holmes, 2000)

A Boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

$$y'' = f(x, y, y'), \qquad a \le x \le b \quad y(a) = \alpha \ \text{ and } \ y(b) = \beta$$

Definition 2.2: (Holmes, 2000)

A boundary condition which specifies the value of the function itself is a Dirichlet boundary condition, or first-type boundary condition. For example, if one end of an iron rod is held at absolute zero, then the value of the problem would be known at that point in space.

$y(a) = \alpha$ and $y(b) = \beta$

A boundary condition which specifies the value of the normal derivative of the function is a Neumann boundary condition, or second-type boundary condition.

Taylor's Series method 2.3: (Iserles, 2009)

Consider the one-dimensional initial value problem

$y' = f(x, y), \ y(x_0) = y_0$

where f is a function of two variables x and y and (x_o, y_0) is a known point on the solution curve.

If the existence of all higher order partial derivatives is assumed for y at $x = x_0$, then by Taylor series the value of y at any neighboring point $x + y$ can be written as

$y(x_0+h)=y(x_0)+hy'(x_0)+\frac{h^2}{2!y''(x_0)}+\frac{h^3}{3!y'''(x_0)}+\cdots$ where ' represents the derivative with respect to- x. Since at x_o, y_0 is known, y' at x_0 can be found by computing $f(x_0+y_0)$. Similarly higher derivatives of y at x_0 also can be computed by making use of the relation

$$y'=f(x,y)$$

$$y''=f_x+f_y y'$$

$$y'''=f_{xx}+2f_{xy}y'+f_{yy}{y'}^2+f_y y''$$

and so on. Then

$$y(x_0+h)=y(x_0)+hf+\frac{h^2(f_x+f_y y')}{2!}+\frac{h^3\left(f_{xx}+2f_{xy}y'+f_{yy}{y'}^2+f_y y''\right)}{3!}+O(h^4)$$

Hence the value of y at any neighboring point x_0+h can be obtained by summing the above infinite series. However, in any practical computation, the summation has to be terminated after some finite number of terms. If the series has been terminated after the p^{th}derivative term, then the approximated formula is called the Taylor series approximation to y of order p and the error is of order $p+1$.The same can be repeated to obtain y at other points of x in the interval $[x_o, x_n]$ in a marching process.

Euler method 2.4: (He.Ji-Huan, 2019)

Here, suppose that h is taken to be a small than the Taylor series expansion be truncated after the first derivative term then:

$$y(x_o+h)=y(x_0)+hy'(x_0)+\frac{h^2y''(\varepsilon)}{2}, \quad x_0<\varepsilon<x_0+h$$

After we find $y(x_o+h)$ we can find solution at $x_2=x_0+2h$ and then to $x_3=x_0+3h$ using the following iterative formulas:

$$y_{n+1}=y_n+hy'_n+O(h^2)$$

But, from the IVP, we have that

$y' = f(x, y)$ so $y'_n = y'(x_n) = f(x_n, y_n)$

Thus

$\therefore y_{n+1} = y_n + hf(x_n, y_n), \quad n = 1,2,3, \cdots$

This formula is called Euler formula.

The linear shooting method 2.5: -

The following theorem gives general condition that ensure that the solution to a second order boundary value problem exists and is unique. (Burden & Faires, 2010)

Theorem 2.6: - (Burden & Faires, 2010)

Suppose the function f in the boundary value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta$$

Is continuous on the set

$D = \{(x, y, y') |\ a \le x \le b,\ -\infty < y < \infty,\ -\infty < y' < \infty\},$

And that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on D.If

i. $\frac{\partial f}{\partial y}(x, y, y') > 0 \quad \forall (x, y, y') \in$ D,and

ii. a constant M exists, with

$$\left|\frac{\partial f}{\partial y'}(x, y, y')\right| \le M \qquad \forall (x, y, y') \in D,$$

Then the boundary value problem has a unique solution.

Corollary 2.7: - (Burden & Faires, 2010)

If the linear boundary value problem

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta$$

Satisfy

i. $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$,

ii.$q(x) > 0$ on $[a, b]$.

Then the boundary value problem has a unique solution.

A linear boundary value problem can be solved by formatting linear combination of the solution to two initial value problems.

Finite Difference Approximations Method 2.8: - (Oliver, 2013)

The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form, and the solutions are related to grid points. Thus, a finite difference solution basically involves three steps:

1. Dividing the solution into grids of nodes.

2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.

3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

In this method the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite difference approximation and the resulting linear system of equations are solved by any standard procedure. These roots are valued of required solution at the pivotal points.

The finite-difference approximations to the different derivatives are derived as given below:

By Taylor series:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!y''(x)} + \cdots \qquad \cdots (1)$$

$$\frac{y(x+h) - y(x)}{h} = y'(x) + \frac{h}{2!\,y''(x)} + \cdots$$

Hence $y'(x) = \frac{y(x+h)-y(x)}{h} - \frac{h}{2!y''(x)} + \cdots$

i.e. $y'(x) = \frac{y(x+h)-y(x)}{h} + O(h) \qquad \cdots (2)$

Equation (2) is forward difference approximation for $y'(x)$.

Also $y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!y''(x)} + \cdots \qquad \cdots (3)$

$$y'(x) = \frac{y(x)-y(x-h)}{h} + O(h) \qquad \cdots (4)$$

Equation (3) is backward difference approximation for $y'(x)$.

A central difference approximation for $y'(x)$ can be got as follows.

Subtracting (3) from (1), and dividing by $2h$,we have

$$y'(x) + O(h^2) = \frac{y(x+h)-y(x-h)}{2h} \qquad \cdots (5)$$

Equation (5) gives a better approximation for $y'(x)$ than what is given in eq. (2) or (4) adding (3) and (1), we get

$$y(x+h) + y(x-h) = 2y(x) + h^2y''(x) + \frac{h^4}{24}y^4(x) + \cdots$$

$$y''(x) = \frac{y(x+h)+y(x-h)}{h^2} + O(h^2) \qquad \cdots (6)$$

Equation (6) is taken as a difference approximation for $y''(x)$

Therefore, at $x = x_i$,from Eq. (5) and (6), we get

$$y'_i = \frac{y_{i+1}-y_{i-1}}{2h} \qquad \cdots (7)$$

And $y''_i = \frac{y_{i-1}+y_{i+1}-2y_i}{h^2} \qquad \cdots (8)$

Neglecting $0(h^2)$ if h is small.

Hence, the expressions for the central difference approximations to the first four derivatives of y are given below:

$y'_i = \frac{y_{i+1}-y_{i-1}}{2h}$

$y''_i = \frac{y_{i-1}+y_{i+1}-2y_i}{h^2}$

$y'''_i = \frac{y_{i+2}-2y_{i+1}+2y_{i-1}-2y_{i-2}}{2h^3}$

$y^4{}_i = \frac{y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}}{h^4}$

The given interval (a, b) is divided into n subintervals, each of width h.Then

$$x_i = x_0 + ih = a + ih$$

$$y_i = y(x_i) = y(a + ih)$$

First type:

Suppose a boundary value problem.

$$y'' + a(x)y' + b(x)y = c(x)$$

Together with the boundary condition , $y(x_0) = \alpha,\ y(x_n) = \beta$ is given, when $x \in (x_0, x_n)$.

We replace $y'(x)$ and $y''(x)$ by the difference formulas (7) and (8) reduce to

$$\frac{y_{i-1} + y_{i+1} - 2y_i}{2h} + a(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + b(x_i)y_i = c(x_i)$$

Simplifying, we get

$$y_{i+1}\left(1 + \frac{h}{2}a_i\right) + y_i(h^2b_i - 2) + y_{i-1}(1 - \frac{h}{2}a_i) = c_ih^2 \qquad \cdots (9)$$

Where $i = 1,2,3, \cdots n - 1$

$$y_0 = \alpha$$

$$y_n = \beta$$

$$a_i = a(x_i)$$

$$b_i = b(x_i)$$

$$c_i = c(x_i)$$

Equation (9) will give's $(n-1)$ equations for $i = 1,2,3,\cdots n-1$ which is tridiagonal system and together with $y_0 = \alpha, y_n = \beta$ we get $(n+1)$ equations in the $(n+1)$ unknowns.

$$y_0, y_1, y_2, \cdots, y_n$$

Solving from there $(n+1)$ equation, we get $y_0, y_1, y_2, \cdots, y_n$ values, i.e., the values of y at $x = x_0, x_1, x_2, \cdots, x_n$.

Runge-kutta method 2.9: (Oliver, 2013)

Consider the equation:

$$y' = \mathrm{f}(\mathrm{x}, \mathrm{y}), \quad y(x_o) = y_0$$

To solve this equation, we use the following way:

$$y_{n+1} = y_n + ak_1 + bk_2$$

Where $k_1 = h(x_n, y_n)$, $k_2 = hf(x_n + \alpha h, y_n + \beta k_1)$

Such that: $a + b = 1$ $\alpha b = \frac{1}{2}$ $\beta b = \frac{1}{2}$

Here we have three equations with four unknowns. To find this unknown we can choose one value arbitrarily and we find the others.

CHAPTER THREE

Example 3.1: Solve $y' = x + y,\ y(0) = 1$ by Taylor's series method (Definition 2.3). Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution: Differentiating successively, we get

$y' = x + y \qquad y'(0)= 1 \qquad [\therefore\ y(0)= 1]$

$y'' = 1 + y' \qquad y''(0)= 2$

$y''' = y'' \qquad y'''(0)= 2$

$y''' = y''' \qquad y'''(0) = 2$

Taylor's series is

$$y = y_0 + (x - x_0)y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \cdots$$

Here $x_0 = 0,\ y_0 = 1$

$$\therefore\ \ y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(4)\cdots$$

Thus $y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\ \cdots$

=1.1103

And $y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} +\ \cdots$

$$= 1.2427$$

Example 3.2: Find by Taylor's series method (Definition 2.3), the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $\frac{dy}{dx} = x^2\, y - 1, y(0) = 1$.

Solution: Differentiating successively, we get

$y' = x^2 y - 1 \rightarrow \qquad y'(0) = -1 \qquad [\therefore\ y(0) = 1]$

$y'' = 2xy + x^2 y', \qquad y''(0) = 0$

$y''' = 2y + 4xy' + x2y'', \qquad y'''(0) = 2$

$y^{iv} = 6y' + 6xy'' + x2y''', \qquad y^4(0) = -6.$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence $y(0.1) = 0.90033$ and $y(0.21) = 0.8022$

Example 3.3: Employ Taylor's method (Definition 2.3), to obtain approximate value of y at $x = 0.2$ for the differential equation $\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution: (a) We have $y' = 2y + 3e^x$; $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$

$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$

$y^4 = 2y''' + 3e^x, \qquad y^4(0) = 2y'''(0) + 3 = 45$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^4(0) + \cdots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \cdots$$

$$= 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{15}{8}x^4 + \cdots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.2)^4 + \cdots 0.8110. \quad \cdots (1)$

(b)Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F being e^{-2x}, the solution is

$ye^{-2x} = \int 3e^x e^{-2x} dx + c = -3e^x + c$ or $y = -3e^x + ce^{2x}$

Since $y = 0$ when $x = 0$,

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112. \quad \cdots (2)$

Comparing (1) and (2), it is clear that (1) approximates to the exact value up to three decimal places.

Example3.4: using Euler method (Definition 2.4) to solve $\frac{dy}{dx} = 3x + 2y$, $y(0) = 1$ with take $h = 0.02$.

Solution: since $f(x, y) = 3x + 2y$ and $y_0 = y(x_0 = 0) = 1$ and $x_i = (0.02)$, so

$n = 0 \rightarrow y_1 = y_0 + hy'_0 = y_0 + hf(x_0, y_0) =$

$$1 + 0.02(3 * 0 + 2 * 1) = 1.04$$

$n = 1 \rightarrow y_2 = y_1 + hf(x_1, y_1) =$

$$1.04 + 0.02(3 * 0.02 + 2 * 1.04) = 1.0828$$

$n = 2 \rightarrow y_3 = y_2 + hf(x_2, y_2) =$

$$1.0828 + 0.02(3 * 0.02 + 2 * 1.0828) = 1.1285$$

(Table 3.1) Solution of the Euler method.

$\mathbf{x_i}$	$\mathbf{y_i}$
0	1
0.02	1.04
0.04	1.0828
0.06	1.1285
0.08	1.1772
$\vdots$	$\vdots$

To have a good accuracy we reduce the step size, here take $h = 0.004$.

Example 3.5: Using Euler's method to solve $y' = 1 - y \quad y(0) = 0$. Find y at $x = 0.1$ and $x = 0.2$ Compare the results of exact equation.

Solution:

Given $y' = 1 - y, y(0) = 0 \quad \cdots (1)$

We know that, the Euler's formula for numerical solution of a differential equation

$y' = f(x, y)$ is $y_{n+1} = y_n + hf(x_n, y_n) \qquad \cdots (2)$

The given differential equation is,

$y' = 1 - y$

$\therefore \ f(x, y) = 1 - y$

Also we have,

Put $n = 0$ in (2), we have

$$y_1 = y(0.1) = y_0 + hf(x_0, y_0)$$

$$= 0 + (0.1)(1) = 0.1.$$

$$x_1 = x_0 + h = 0 + (0.1) = 0.1.$$

Now, put $n = 1$ in (2), we have

$$y_2 = y(0.2) = y_1 + hf(x_1, y_1)$$
$$= 0.1 + (0.1)(1 - 0.1) = 0.19.$$

Hence, $y(0.1) = 0.1$and $y(0.2) = 0.19$.

The exact solution of the given equation

$y' = 1 - y$ is given by

$$\frac{dy}{1-y} = dx \rightarrow log(1-y) = x + c$$

Put $x = 0$ and $y = 0$ we get $c = 0$

$$\therefore 1 - y = e^x \rightarrow y = 1 - e^x$$

$\therefore$ $y(0.1) = 1 - e^{0.1} = 0.1052$ and

$$y(0.2) = 1 - e^{0.2} = 0.2214$$

Example 3.6: Using Euler's method to solve by $y' = \frac{y-x}{y+x}$ with initial condition $y(0) = 1$ and find y when $x = 0.1$ correct to four decimal places.

Solution: Given $y' = \frac{y-x}{y+x}$, $y(0) = 1$

$i.e.$, $y_0 = 1$ when $x_0 = 0$

Let $h = 0.02$ so that

$$x_1 = 0.02\,, x_2 = 0.04\,, x_3 = 0.06\,, x_4 = 0.08\,, x_5 = 0.1.$$

Using Euler's formula, we get

$$y_1 = y(0.02) = y`_0 + hf(x_{0,}y_0)$$
$$= y_0 + 0.02\left[\frac{y_0 - x_0}{y_0 + x_0}\right]$$
$$1 + (0.02)\left[\frac{1-0}{1+0}\right] = 1.0200.$$
$$y_2 = y(0.04) = y_1 + hf(x_{1,}y_1)$$
$$= y_1 + 0.02\left[\frac{y_1 - x_1}{y_1 + x_1}\right]$$

$$= 1.0200 + (0.02)\left[\frac{1.0200 - 0.02}{1.0200 + 0.02}\right] = 1.0392$$

$$y_3 = y(0.06) = y_2 + hf(x_{2,}y_2)$$

$$= y_2 + 0.02\left[\frac{y_2 - x_2}{y_2 + x_2}\right]$$

$$= 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right] = 1.0577$$

Similarly, $$y_4 = y(0.08) = y_3 + hf(x_{3,}y_3)$$

$$= y_3 + 0.02\left[\frac{y_3 - x_3}{y_3 + x_3}\right]$$

$$= 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right] = 1.0756$$

$$y_5 = y(0.08) = y_4 + hf(x_{4,}y_4)$$

$$= y_4 + 0.02\left[\frac{y_4 - x_4}{y_4 + x_4}\right]$$

$$= 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right] = 1.0928$$

$$\therefore y(0.1) = 1.0928$$

Example 3.7: Use (Theorem 2.6) to show that the boundary-value problem

$y'' + e^{-xy} + siny' = 0$, for $1 \le x \le 2$, with $y(1) = y(2) = 0$, has a unique solution.

Solution: We have $f(x, y, y') = -e^{-xy} - siny'$.and for all x in $[1, 2]$, $f_y(x, y, y') = xe^{-xy} > 0$ and $|\ f_{y'}(x, y, y')\ | = |-cosy'| \le 1$.

So, the problem has a unique solution.

Example 3.8: Use the linear shooting method to solve the following boundary value problem.

$y'' = -xy' + y + 2x + \frac{2}{x}$, $y(1) = 0$, $y(2) = 4\ln 2$, $h = 0.2$

Solution: To find the solution of the given linear boundary value problem, firstly, we convert the given problem into two initial values problem as.

$$y''_1 = -xy'_1 + y_1 + 2x + \frac{2}{x}, \qquad y_1(1) = 0, \quad y'_1(1) = 0 \qquad \cdots (1)$$

And

$$y''_2 = -xy'_2 + y_2, \qquad y_2(1) = 0, \quad y'_2(1) = 1 \qquad \cdots (2)$$

To covert the above two second order differential equation into the system of first order equations, we substitute

$y'_1 = f_x, y_1, z_1 = z_1$

let $y'_1 = f(x, y_1, z_1) = z_1$, Then $\quad y''_1 = z'_1 \qquad \cdots (3)$

by substituting equation (3) in (1) we get

$$z'_1 = g(x, y_1, z_1) = -xz_1 + y_1 + 2x + \frac{2}{x},$$

$$y_1(1) = 0, \qquad z_1(1) = 0.$$

Similarly, the second differential equation can be converted as

Let $y'_2 = f(x, y_2, z_2) = z_2$, Then $\quad y''_2 = z'_2 \qquad \cdots (4)$

by substituting equation (4) in (2) we get

$$z'_2 = g(x, y_2, z_2) = -xz_2 + y_2$$

$$y_2(1) = 0, \qquad z_2(1) = 1.$$

Solve the above systems using Euler's method

$$y_{i+1} = y_i + hf(x_i, y_{1i}, z_{1i})$$

$$z_{i+1} = z_i + hg(x_i, y_{2i}, z_{2i})$$

For each $i = 0,1,2,3, . . n - 1$.

To get solutions y_1 and $y_{2.}$ We can have the solution of the given linear boundary value problem

First system:

$$y'_1 = f(x, y_1, z_1) = z_1$$

$$z'_1 = g(x, y_1, z_1) = -xz_1 + y_1 + 2x + \frac{2}{x}$$

$$y_1(1) = 0, \qquad z_1(1) = 0.$$

Since $[a, b] = [1,2]$

$$x_0 = 1,\ x_1 = 1.2,\ \ x_3 = 1.4, x_4 = 1.6,\ x_5 = 1.8, x_6 = 2$$

With $h = 0.2$,

$$y_{i+1} = y_i + hf(x_i, y_{1i}, z_{1i})$$

$$y_{11} = y_{10} + (0.2)f(x_0, y_{10}, z_{10}) = 0$$

$$z_{11} = z_{10} + (0.2)g(x_0, y_{20}, z_{20}) = 0.2 * 4 = 0.8$$

$$y_{12} = y_{11} + (0.2)f(x_1, y_{11}, z_{11}) = 0 + 0.2(0.8) = 0.16$$

$$z_{12} = z_{11} + (0.5)g(x_1, y_{11}, z_{11}) = 2 + 0.5 * (-2 + 2.4 + 1.66) = 3.03$$

Also, for second system:

$$y'_2 = f(x, y_2, z_2) = z_2$$

$$z'_2 = g(x, y_2, z_2) = -xz_2 + y_2$$

$$y_2(1) = 0 \qquad , \qquad z_2(1) = 1$$

$$y_{21} = y_{20} + hf(x_0, y_{20}, z_{20})$$

$$= 0 + (0.2)(-1) = \ -0.2$$

$$z_{21} = z_{20} + hg(x_0, y_{20}, z_{20})$$

$$= 1 + 0.2(-1 + 0) = 1 + (-0.2) = 0.8$$

Example (3.9): - Linear Shooting To approximate the solution of the boundary-value problem $-y'' + p(x)y' + q(x)y + r(x) = 0$, for $a \leq x \leq b$, with $y(a) = \alpha$ and $y(b) = \beta$,

INPUT endpoints a, b; boundary conditions α, β; number of subintervals N.

OUTPUT approximations $w_{1,i}$ to $y(x_i)$; $w_{2,i}$ to $y'(x_i)$ for each $i = 0, 1, \ldots, N$.

Step 1 Set $h = (b - a)/N$;

$u_{1,0} = \alpha$;

$u_{2,0} = 0;$

$u_{3,0} = 0;$

$u_{4,0} = 1.$

Step 2 For $i = 0, \ldots, N - 1$ do Steps 3 and 4.

(The Runge-Kutta method for systems is used in Steps 3 and 4.)

Step 3 Set $x = a + ih$**.**

Step 4 Set $k_{1,1} = hu_{2,i}$;

$k_{1,2} = h[\ p(x)u_{2,i} + q(x)u_{1,i} + r(x)]$;

$k_{2,1} = h[\ u_{2,i} + \frac{1}{2} k_{1,2}];$

$k_{2,2} = h[p\left(x + \frac{h}{2}\right)(u_{2,i} + \frac{1}{2} k_{1,2}) + q\left(x + \frac{h}{2}\right)\left(u_{1,i} + \frac{1}{2} k_{1,1}\right) + r\left(x + \frac{h}{2}\right)];$

$k_{3,1} = h[u_{2,i} + \frac{1}{2} k_{2,2}]$;

$k_{3,2} = h[p\left(x + \frac{h}{2}\right)\left(u_{2,i} + \frac{1}{2} k_{2,2}\right) + q\left(x + \frac{h}{2}\right)\left(u_{1,i} + \frac{1}{2} k_{2,1}\right) + r\left(x + \frac{h}{2}\right)];$

$k_{4,1} = h[u_{2,i} + k_{3,2}];$

$k_{4,2} = h[p(x + h)\left(u_{2,i} + k_{3,2}\right) + q(x + h)\left(u_{1,i} + k_{3,1}\right) + r(x + h)]$;

$u_{1,i+1} = u_{1,i} + \frac{1}{6}[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}];$

$u_{2,i+1} = u_{2,i} + \frac{1}{6}[k_{1,2} + 2k_{2,2} + 2kk_{3,2} + k_{4,2}];$

$k'_{1,1} = hv_{2,i};$

$k'_{1,2} = h[p(x)v_{2,i} + q(x)v_{1,i}]$;

$k'_{2,1} = h[v_{2,i} + \frac{1}{2} k'_{1,2}];$

$k'_{2,2} = h[p\left(x + \frac{h}{2}\right) (v_{2,i} + \frac{1}{2} k'_{1,2}) + q\left(x + \frac{h}{2}\right) (v_{1,i} + \frac{1}{2}k'_{1,1})];$

$k'_{3,1} = h[v_{2,i} + \frac{1}{2} k'_{2,2}];$

$k'_{3,2} = h[p(x + \frac{h}{2}) (v_{2,i} + \frac{1}{2}k'_{2,2} + q(x + \frac{h}{2}) (v_{1,i} + \frac{1}{2} k'_{2,1}];$

$k'_{4,1} = h[v_{2,i} + k'_{3,2}];$

$k'_{4,2} = h[p(x + h)\left(v_{2,i} + k'_{3,2}\right) + q(x + h)\left(v_{1,i} + k'_{3,1}\right)] ;$

$v_{1,i+1} = v_{1,i} + \frac{1}{6}[k'_{1,1} + 2k'_{2,1} + 2k'_{3,1} + k'_{4,1}];$

$v_{2,i+1} = v_{2,i} + \frac{1}{6}[k'_{1,2} + 2k'_{2,2} + 2\, k'_{3,2} + k'_{4,2}]$.

Step 5 Set $w_{1,0} = \alpha;$

$w_{2,0} = \frac{\beta - u_{1,n}}{v_{1,n}};$

OUTPUT $(a, w_{1,0}, w_{2,0})$.

Step 6 For $i = 1, \dots, N$

set $w1 = u_{1,i} + w_{2,0}v_{1,i};$

$w2 = u_{2,i} + w_{2,0}v_{2,i};$

$x = a + ih;$

OUTPUT $(x_i, w1, w2)$. (Output is $x_i, w_{1,i}, w_{2,i}$).

Step 7 STOP. (The process is complete.)

Example (3.10): - The boundary-value problem

$$y'' = \frac{2}{x}y' + \frac{2}{x}y + \frac{\sin(lnx)}{x^2}$$

Has the exact solution

$$y = c_1x + \frac{c_2}{x^2} - \frac{3}{10}\sin(lnx) - \frac{1}{10}\cos(lnx),$$

Where $c_2 = \frac{1}{70}[8 - 12\ \sin(ln2) - 4\cos(ln2)] \approx -0.03920701320$

And $c_1 = \frac{11}{10} - c_2 \approx 1.1392070132.$

This problem requires approximating the solutions to the initial-value problems

$y''_1 = -\frac{2}{x}y'_1 + \frac{1}{x^2}y_1 + \frac{\sin(lnx)}{x^2},\ 1 \le x \le 2 \quad y_1(1) = 0, \qquad y'_1(1) = 0.$

And $y''_2 = -\frac{2}{x}y'_2 + \frac{2}{x^2}y_2,\ 1 \le x \le 2 \quad y_2(1) = 0,\ y'_2(1) = 1.$

The results of the calculations, with $N = 10$ and $h = 0.1$, The value listed as u_1, approximates$y_1(x_i), v_i$, approximates $y_2(x_i), w_i$and w; approximates $y(x_i)$.

(Table3.2) The results of the calculation.

x_i	$u_{1,i}$	$v_{1,i}$	w_1	$y(x_i)$	$\lvert y(x_i) - w_1\rvert$
1	1.00000000	0.00000000	1.00000000	1.00000000	–
1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
1.2	1.03245472	0.16851175	1.18708471	1.18708484	1.34×10^{-7}
1.3	1.06674375	0.23608704	1.28338227	1.28338236	9.78×10^{-8}
1.4	1.10928795	0.29659067	1.38144589	1.38144595	6.02×10^{-8}
1.5	1.15830000	0.35184379	1.48115939	1.48115942	3.06×10^{-8}
1.6	1.21248372	0.40311695	1.58239245	1.58239246	1.08×10^{-8}
1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}
1.9	1.39750618	0.54098928	1.89392951	1.89392951	4.41×10^{-9}
2	1.46472815	0.58332538	2.00000000	2.00000000	

The accuracy is expected because the fourth-order Runge-Kutta method gives $O(h^4)$ accuracy to the solutions of the initial-value problems. Unfortunately, there can be problems hidden in this technique, because of round-off errors. If $y_1(x)$ rapidly increases as x goes from a to b, then $u_{1,N} \approx y_1(b)$ will be large. Should β be small in magnitude compared to $u_{1,N}$ the term $w_{2,0} = \frac{(\beta - u_{1,N})}{v_{1,N}}$ will be approximately

$u_{1,N}/ v_{1,N}$. The computations in Step 6 then become

$$w1 = u_{1,i} + w_{2,0}\, v_{1,i} \approx u_{1,i} - \left(\frac{u_{1,N}}{v_{1,N}}\right) v_{1,i},$$

$$w2 = u_{2,i} + w_{2,0}\, v_{2,i} \approx u_{2,i} - \left(\frac{u_{1,N}}{v_{1,N}}\right) v_{2,i},$$

which allows a possibility of a loss of significant digits due to cancellation. However, since $u_{1,i}$ is an approximation to $y_1(x_i)$, the behavior of y_1 , can easily be monitored, and if $u_{1,i}$. increases rapidly from a to b, the shooting technique can be employed in the other direc-tion, that is, solving instead the initial-value problems

$$y'' = p(x)y' + q(x)y + r(x),\ \ a \le x \le b,\ \ y(b) = \beta, \qquad y'(b) = 0,$$

And $y'' = p(x)y' + q(x)y + ,\ \ a \le x \le b,\ \ y(b) = 0\ ,\ \ y'(b) = 1,$

If this reverse shooting technique still gives cancellation of significant digits, and if increased precision does not yield greater accuracy, other techniques must be used, such as those presented later in this chapter. In general, however, if $u_{1,i}$ and $v_{1,i}$ are $O(h^n)$

approximations to $y_1(x_i)$, and $y_2(x_i)$, respectively, for each $i = 0, 1, \ldots, N$, then $w_{1,i}$ will be an $O(h^n)$ approximation to $y(x_i)$, In particular,

$$\left| w_{1,i} - y(x_i) \right| \le Kh'' \left| 1 + \frac{v_{1,i}}{v_{1,N}} \right|,$$

for some constant K.

Example 3.11: Let $u(x) = \sin x$, Let us compute $u'(1) = cos\ 1 = 0.5403023$ ...

by using the finite difference quotient, and so

$$\cos 1 = \frac{\sin(1+h) - \sin 1}{h}$$

(Table 3.3) The result for different values of h is listed.

h	1	0.1	0.01	0.001	0.0001
Approximation	0.067826	0.497364	0.536086	0.539881	0.540260
Error	−0.472476	−0.042939	−0.004216	−0.000421	−0.000042

We observe that reducing the step size by a factor of to $\frac{1}{10}$ reduces the size of the error by approximately the same factor. Thus, to obtain 10 decimal digits of accuracy, we anticipate

needing a step size of about $h = 10^{-11}$. The fact that the error is more of less proportional

to the step size tells us that we are using a first order numerical approximation.

To approximate higher order derivatives, we need to evaluate the function at more than two points. In general, an approximation to the n^{th} order derivative $u^{(n)}(x)$ requires at least $n + 1$ distinct sample points. For example, let us try to approximate $u''(x)$by using the particular sample points $x,$

$x + h \; and \; x - h$. Which combination of the function values

$u(x - h), u\,(x), u(x + h)$ can be used to approximate the derivative $u''(x)$?The answer to such a question can be found by consideration of the relevant Taylor expansions

$$u(x + h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u'''(x)\frac{h^3}{6} + O(h^4)$$

$$u(x - h) = u(x) - u'(x)h - u''(x)\frac{h^2}{2} - u'''(x)\frac{h^3}{6} - O(h^4)$$

where the error terms are proportional to h^4. Adding the two formulae together gives

$$u(x + h) + u(x - h) = 2u(x) + u''(x)h^2 + O(h^4)$$

Rearranging terms, we conclude that

$$u''(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + O(h^2)$$

The result is the simplest finite difference approximation to the second derivative of a function. The error is of order h^2, and depends upon the magnitude of the fourth order derivative of u near x.

Example 3.12: -

Let $u(r) \; = \; e^{x^2},$ with $u"(x) \; = \; (4x^2 \; + \; 2)\; e^{x^2}$.Let us approximate

$u''(x) = 6e = 10.30969097\cdots$ by using the finite difference quotient

$6e \approx \frac{e^{(1+h)^2} - 2e + e^{(1-h)^2}}{h^2}$ The results are listed in the following table.

(Table 3.4) The result $6e$ by using the finite difference quotient.

H	1	0.1	0.01	0.001	0.0001
Approximate	50.1615863	16.4828982	16.3114126	16.3097081	16.3096911
Error	33.8518954	0.17320726	0.00172168	0.00001722	0.00000018

Each reduction in step size by a factor of $\frac{1}{10}$ to reduces the size of the error by a factor of $\frac{1}{100}$ to and a gain of two new decimal digits of accuracy, which is a reflection of the fact that the finite difference formula is of second order, with error proportional to h^2

However, this prediction is not entirely borne out in practice. If we take

$h = 00001$ then

the formula produces the approximation 16.3097002570, with an error of 0.0000092863

which is less accurate that the approximation with $h = 0001$. The problem is that

round-off errors have now begun to affect the computation, and underscores a significant difficulty with numerical differentiation formulae. Such finite difference formulae involve dividing very small quantities, and this can lead to high numerical errors due to round-off. As a result, while they typically produce reasonably good approximations to the derivatives for moderately small step sizes, to achieve high accuracy, one must employ high precision arithmetic. And our expectations about the error for a very small step size were not, in fact justified as the reader may have discovered.

We can improve the order of accuracy of finite difference approximations to derivatives by employing more sample points to form an appropriate linear combination of the function values. For instance, if the first order approximation to the first derivative based on the two points x and a $x + h$ is not sufficiently accurate, one can try combining the finction values at three points $x, \quad x + h$

and $x - h$. To find the appropriate combination of $u(x - h), u(x), u(x + h)$, we return to the Taylor expansions. To solve for $u'(x)$, we subtract the two formulae, and so

$$u(x + h) - u(x - h) = 2u'(x)h + u'''(x)\frac{h^3}{3} + O(h^4).$$

Rearranging the terms, we are led to the well-known centered difference formula

$$u'(x) = \frac{u(x + h) - u(x - h)}{2h} + O(h^2),$$

which is a second order approximation to the first derivative. Geometrically, the centered difference quotient represents the slope of the secant line through the two points $(x - h, u(x - h))$ and

$(x + h, u(x + 1))$ on the graph of u contered symmetrically about

the point x illustrates the geometry behind the two approximations; the advantages in accuracy in the centered difference version are graphically evident. Higher Order approximations can be found by evaluating u at additional points, including, say, $x + 2h, x - 2h$, and so on.

Example 3.13: (Definition 2.10). consider $\frac{dy}{dx} = 3x + 2y, \quad y(0) = 1$. Find (0.1)?

Solution: since $y_1 = y_0 + ak_1 + bk_2$, first we find $k_1$and k_2:

$k_1 = hf(x_0, y_0) =$ (0.1)[3*0+2*1] = 0.2

$k_2 = hf(x_0 + \alpha h, y_0 + \beta k_1)$= $(0.1)f(0 + \frac{3}{2}(0.1), 1 + \frac{3}{2}(0.2)$

= (0.1) f (0.15,1.3) = (0.1) [3*0.15+2*1.3] = 0.235

$$\therefore y(0.1) = y_1 = 1 + \frac{2}{3}(0.2) + \frac{1}{3}(0.305) = 1.235$$

By same way find all $y(0.2)$, $y(0.3)$,$y(0.4)$ and $y(0.5)$.

Again, to find $y(0.2)$?here $h = 0.1$ and $x_1 = 0.1$, $y_1 = 1.235$;

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$f(x,y) = 3x + 2y$; since $y_2 = y_0 + \frac{2}{3}k_1 + \frac{1}{3}k_2$, first we find new k_1 and k_2:

$k_1 = hf(x_1, y_1) = (0.1)f(0.1,1.235) = (0.1)[3 * 0.1 + 2 * 1.235] = 0.277$

$$k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = (0.1)f\left(0.1 + \frac{3}{2}(0.1), 1.235 + \frac{3}{2}(0.277)\right)$$

$$= (0.1)f(0.25,1.6505) = (0.1)[3 * 0.25 + 2 * 1.6505] = 0.4051$$

$$\therefore y(0.2) = y_2 = 1.235 + \frac{2}{3}(0.277) + \frac{1}{3}(0.4051) = 1.5547$$

Example 3.14: Use the Runge-Kutta method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3e^{-2x}, \qquad y(0) = 1, \qquad \dots (1)$$

at $x = 0.1,0.2$

Solution: Again, we rewrite Equation (1) as

$y' = -2y + x^3e^{-2x}, \quad y(0) = 1,$

which is of the form Equation$y' = \mathrm{f(x,y)}$, with

$f(x,y) = -2y + x^3e^{-2x}, \quad x_0 = 0$, and $\quad y_0 = 1.$

The Runge-Kutta method yields

$k_{10} = f(x_0, y_0) = f(0,1) = -2$

$k_{20} = f(x_0 + \frac{h}{2}, y_0 + h\frac{k_{10}}{2}) = f(0.05,1 + (0.05)(-2))$

$= f\ (0.05,0.9) = -2(0.9) + (0.05)^3e^{-0.1} = -1.799886895$

$$k_{30} = f(x_0 + \frac{h}{2}, y_0 + h\frac{k_{20}}{2}) = f(0.05,1 + (0.05)(-1.799886895))$$

$$= f(0.05,0.910005655) = -2(.910005655) + (0.05)^3 e^{-0.1}$$

$$= -1.819898206$$

k_{40}=$f(x_0 + \frac{h}{2}, y_0 + hk_{30}) = f(0.1,1 + (0.1)(-1.819898206)) =$

$f(0.1,.818010179) = -2(0.818010179) + (0.1)^3 e^{-0.2} = -1.635201628$

y_1=$y_1 + \frac{h}{6}(k_{10} + 2k_{20} + 2k_{30} + k_{40}) = 1 + \frac{0.1}{6}(-2 + 2(-1.799886895)) +$

$2(-1.819898206) - 1.635201628)) = 0.818753803,$

$$k_{11} = f(x_1, y_1) = f(0.1,0.818753803) = -2(0.818753803)) + (0.1)^3 e^{-0.2}$$

$$= -1.636688875$$

$$k_{21} = f\left(x_1 + \frac{h}{2}, y_1 + h\frac{k_{11}}{2}\right)$$

$$= f(0.15,0.818753803 + (0.05)(-1.636688875))$$

$$= f(0.15,0.736919359) = -2(0.736919359) + (0.15)^3 e^{-0.3}$$

$$= -1.471338457$$

$$k_{31} = f(x_1 + \frac{h}{2}, y_1 + h\frac{k_{21}}{2}) = f(0.15,0.818753803 + (0.05)(-1.471338457))$$

$$= f(0.15,0.745186880) = -2(0.745186880) + (0.15)^3 e^{-0.3}$$

$$= -1.487873498$$

$$k_{41} = f\left(x_1 + \frac{h}{2}, y_1 + h\frac{k_{31}}{2}\right)$$

$$= f(0.2,0.818753803 + (0.1)(-1.487873498))$$

$$= f(.2,.669966453) = -2(.669966453) + (0.2)^3 e^{-0.4}$$

$$= -1.334570346$$

$y_2 = y_1 + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) = 0.818753803 + \frac{0.1}{6}(-1.636688875 + 2(-1.471338457) + 2(-1.487873498) - (1.334570346) = 0.670592417$

The Runge-Kutta method is sufficiently accurate for most applications.

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پوختە

لەم ڕاپۆرتەدا ئێمە لێکۆڵینەوەمان لە شێوازە ژمارەییەکان کرد بۆ چارەسەرکردنی کێشەی بەهای سنوور.
سەرەتا هەندێک شێوازمان بۆ چارەسەرکردنی ڕاگەیاند وەک

(Runge-kutta, Finite difference, Linear shooting, Euler, Taylor)

لە کۆتاییدا هەندێک نموونە خراونەتەڕ

وو بۆ وەسفکردنی ئەم شێوازانە

ملخص

درسنا في هذا التقرير الطرق العددية لحل مشكلة القيمة الحدية. أولا ، أعلنا عن بعض الطرق لحلها مثل

(Runge-kutta, Finite difference, Linear shooting, Euler, Taylor)

أخيرا ، تم تقديم بعض الأمثلة لتوضيح هذه الأساليب