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Techniques for Nonlinear Differential Equations

Final Year Research Project

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Certification of the Supervisor

I certify that this research was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of science in Mathematics.

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Abstract

This research investigates numerical methods for solving systems of nonlinear equations, emphasizing three key approaches. Firstly, we explore Newton's method, leveraging the Jacobian matrix for multivariable nonlinear equations. Secondly, Broyden's method, a Quasi-Newton technique considered a generalization of the Secant Method, is examined. Lastly, the Finite Difference method is investigated for solving nonlinear boundary value problems in ordinary differential equations. The study includes in-depth discussions on each method's principles, convergence considerations, and practical applications. Examples illustrate the application of Newton's method and the Finite Difference method in real-world scenarios. The research provides insights into the strengths, limitations, and potential applications of these numerical methods, contributing to the understanding of their effectiveness in solving nonlinear mathematical problems.

Keywords: non-linear , Broyden's method, Newton's method, Finite Difference method.

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1.Introduction [Alligood, K.T A,2016]

Nonlinear differential equations whose solutions cannot be found explicitly arise in essentially every branch of modern science, engineering and mathematics. One of the most useful methods available for studying a nonlinear system of ordinary differential equations is to compare it with a single first-order equation derived naturally from an estimate on a system. However, the bounds provided by the comparison method are sometimes difficult or impossible to calculate explicitly. In fact, in many applications explicit bounds are more useful while studying the behavior of solutions of such systems. Another basic tool, which is typical among investigations on this subject, is the use of nonlinear integral inequalities which provide explicit bounds on the unknown functions. Over the last 40 years several new nonlinear integral inequalities have been developed in order to study the behavior of solutions of such systems. the non linear integral inequalities which are very effective in studying the behavior of solutions of systems of differential, integral and integra-differential equations. Applications of some of the inequalities are also presented, and some miscellaneous inequalities which can be used in certain applications are given. Nonlinear equations have been studied for centuries, but the formal study of nonlinear equations began in the 17th century with the work of mathematicians such as Galileo, Kepler, and Newton. Galileo was one of the first mathematicians to recognize that the motion of an object in free fall could not be described by a linear equation. He observed that the acceleration of the object was not constant, but rather increased as the object fell faster. On linear use in Economics: Non-linear equations are used to model the relationship between supply and demand in economics., Biology: Non-linear equations are used to model the growth and spread of populations, diseases, and other biological systems. Physics: Non-linear equations are used to describe the behavior of physical systems, such as the motion of particles, the flow of fluids, and the behavior of waves. Engineering: Non-linear equations are used in the design and analysis of engineering systems, such as structures, control systems, and communication networks. Chemistry: Non-linear equations are used to model chemical reactions and the behavior of substances in various states. Ecology: Non-linear equations are used to model the interactions between species and the environment, and to predict the effects of changes in ecosystems Kepler studied the motion of planets around the sun and found that their orbits were not circular, as had been previously thought, but rather elliptical. This led to the development of a nonlinear equation known as Kepler's equation, which describes the position of a planet as it orbits the sun. The goal of this Research is to

examine three different numerical methods that are used to solve systems of nonlinear equations in several variables. The first method we will look at is Newton's method. This will be followed by Broyden's method, which is sometimes called a Quasi-Newton method; it is derived from Newton's method. Lastly, we will study the Finite Difference method that is used to solve boundary value problems of nonlinear ordinary differential equations. For each method, a breakdown of each numerical procedure will be provided. In addition, there will be some discussion of the convergence of the numerical methods, as well as the advantages and disadvantages of each method.

1.2Preliminaries

In this section, we present the definitions and terms that will be used throughout the project will be presented.

1.2.1 **A system of nonlinear equations**

Definition 1.2.1. [Alligood, K.T A,2017] function $f: \mathbb{R}^n \to \mathbb{R}$ is defined as being nonlinear when it does not satisfy the superposition principle that is

$$f(x_1 + x_2 + \cdots) \neq f(x_1) + f(x_2) + \cdots$$

Now that we know what the term nonlinear refers to we can define a system of nonlinear equations.

Definition 1.2.2 [Alligood, K.,2017] system of nonlinear equations is a set of equations as the following:

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0, \end{aligned}$$

where $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and each f_i is a nonlinear real function, $i = 1,2, \dots, n$.
Example 1.2.3. [Birkhoff, G.,1962] Here is an example of a nonlinear system from Burden and Fairs in :

$$\begin{aligned} \mathbf{3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0} \\ \mathbf{x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0} \\ \mathbf{e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0} \end{aligned}$$

In this article we will use the term root or solution frequently to describe the final result of solving the systems.

Definition 1.2.4. [Birkhoff, G.,2019] A solution of a system of equations $f_1, f_2, \dots, f_n$ in n variables is a point $(a_1, \dots, a_n) \in \mathbb{R}^n$ such that $f_1(a_1, \dots, a_n) = \cdots = f_n(a_1, \dots, a_n) = 0$.

Because systems of nonlinear equations can not be solved as nicely as linear systems, we use procedures called iterative methods.

Definition1.2.5. [Bronstein, M.,2019] An iterative method is a procedure that is repeated over and over again, to find the root of an equation or find the solution of a system of equations.

Definition 1.2.6. [Bronstein, M.,2019] Let **F** be a real function from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$. If $\mathbf{F}(\mathbf{p}) = \mathbf{p}$, for some $p \in D$, then **p** is said to be a fixed point of **F**.

1.2.2 **Convergence** [Burden, R.L.,2014]

One of the things we will discuss is the convergence of each of the numerical methods.

Definition 1.2.7. [Burden, R.L.,2014] We say that a sequence converges if it has a limit.

Definition 1.2.8. [Burden, R.L.,2014] Let p_n be a sequence that converges to p, where $p_n \neq p$. If constants $\lambda, \alpha > 0$ exist such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

Then it is said that p_n converges to p of order α with a constant λ.
There are three different orders of convergences.

Definition 1.2.9. [Burden, R.L.,2014] A sequence p_n is said to be linearly convergent if p_n converges to p with order $\alpha = 1$, for a constant $\lambda < 1$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

Definition1. 2.10. [Cantwell, B.J.,2008] A sequence p_n is said to be quadratically convergent if p_n converges to p with order $\alpha = 2$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

Definition1.2.11. [Cantwell, B.J.,2008] A sequence p_n is said to be superlinearly convergent if

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|}$$

Remark 1.2.12. [Cantwell, B.J.,2008] The value of α measures how fast a sequence converges. Thus the higher the value of α is, the more rapid the convergence of the sequence is. In the case of numerical methods, the sequence of approximate solutions is converging to the root. If the convergence of an iterative method is more rapid, then a solution may be reached in less interations in comparison to another method with a slower convergence.

1.2.3 Jacobian Matrix [Chenciner, A.,2017]

The Jacobian matrix, is a key component of numerical methods in the next section.

Definition1.2.13. [Chenciner, A.,2017] The Jacobian matrix is a matrix of first order partial derivatives

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}.$$

Example 1.2.14. [Chenciner, A.,2017] If we take the system from Example 2.3 we are able to obtain the following Jacobian Matrix:

$$J(\mathbf{x}) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}$$

1.2.4 Hessian Matrix[Diacu, F.,2000]

The Hessian matrix, will be discussed in a future proof.

Definition 1.2.15. [Diacu, F.,2000] The Hessian matrix is a matrix of second order partial derivatives $\mathbf{H} = \left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}\right]_{ij}$ such that

$$H(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f_2}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f_2}{\partial x_2^2} & \cdots & \frac{\partial^2 f_2}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f_n}{\partial x_n \, \partial x_1} & \frac{\partial^2 f_n}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 f_n}{\partial x_n^2} \end{bmatrix}$$

1.2.5 Norms of Vectors [Diacu, F.,2000]

Let $\mathbf{x} \in \mathbb{R}^n$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Definition 1.2.16 [Diacu, F.,2000] A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ that has the following properties:

(1) $\| \mathbf{x} \| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(2) $\| \mathbf{x} \| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(3) $\| \alpha\mathbf{x} \| = |\alpha| \, \| \mathbf{x} \|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(4) $\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

There are two types of vector norms we will discuss, the l_2 and l_∞ norms.

Definition 1.2.17 [Dirac, P.A.M.,2001] The l_2 norm for the vector $\mathbf{x}$ is called the Euclidean norm because it represents the length of the vector denoted by

$$\| \mathbf{x} \| = \| \mathbf{x} \|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Definition 1.2.18 [Dirac, P.A.M.,2001] The l_∞ norm represents the absolute value of the largest component in the vector $\mathbf{x}$. It is denoted by

$$\| \mathbf{x} \|_\infty = \max_{1 \leq i \leq n} |x_i|$$

The following is an example demonstrating the vector norms.

Example1.2.19. [Dirac, P.A.M.,2001] The vector

$$\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

has vector norms

$$\| \mathbf{x} \|_2 = \sqrt{(-2)^2 + (-1)^2 + (1)^2 + (3)^2} = \sqrt{15}$$
$$\| \mathbf{x} \|_\infty = \max(|-2|, |-1|, |1|, |3|) = 3$$

In the next couple of sections, we will examine three different numerical methods that will apply the terms we discussed in this section. These methods include: Newton's method, Broyden's method, and the Finite Difference method.

Chapter 2

Newton's Method [Goldstein, H.,2014]

Newton's method is one of the most popular numerical methods, and is even referred by Burden and Fairs [3] as the most powerful method that is used to solve for the equation $f(x) = 0$. This method originates from the Taylor's series expansion of the function $f(x)$ about the point x_1 :

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \cdots \tag{3.1}$$

where f, and its first and second order derivatives, f' and f'' are calculated at x_1. If we take the first two terms of the Taylor's series expansion we have:

$$f(x) \approx f(x_1) + (x - x_1)f'(x_1). \tag{3.2}$$

We then set (3.2) to zero (i.e $(x) = 0$) to find the root of the equation which gives us:

$$f(x_1) + (x - x_1)f'(x_1) = 0. \tag{3.3}$$

Rearranging the (3.3) we obtain the next approximation to the root, giving us:

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{3.4}$$

Thus generalizing (3.4) we obtain Newton's iterative method:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}, i \in \mathbb{N} \tag{3.5}$$

where $x_i \to \bar{x}$ (as $i \to \infty$), and $\bar{x}$ is the approximation to a root of the function $f(x)$.

Remark 3.1. [Goldstein, H.,2014] As the iterations begin to have the same repeated values i.e. as $x_i = x_{i+1} = \bar{x}$ this is an indication that $f(x)$ converges to $\bar{x}$. Thus x_i is the root of the function $f(x)$.

Proof of Remark 3.1[Goldstein, H., 2014]

Since $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ and if $x_i = x_{i+1}$, then

$$x_i = x_i - \frac{f(x_i)}{f'(x_i)}$$

This implies that

$$\frac{f(x_i)}{f'(x_i)} = 0$$

and thus $f(x_i) = 0$.
Another indicator that x_i is the root of the function is if it satisfies that $|f(x_i)| < \epsilon$, where $\epsilon > 0$ is a given tolerance.

However, (3.5) can only be used to solve nonlinear equations involving only a single variable. This means we have to take (3.5) and alter it, in order to use it to solve a set of nonlinear algebraic equations involving multiple variables

We know from Linear Algebra that we can take systems of equations and express those systems in the form of matrices and vectors. With this in mind and using Definition 2.2, we can express the nonlinear system as a matrix with a corresponding vector. Thus, the following equation is derived:

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J\left(\mathbf{x}^{(k-1)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$$

where $k = 1,2, \dots, n$ represents the iteration, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{F}$ is a vector function, and $J(\mathbf{x})^{-1}$ is the inverse of the Jacobian matrix. This equation represents the procedure of Newton's method for solving nonlinear algebraic systems. However, instead of solving the equation $f(x) = 0$, we are now solving the system $\mathbf{F}(\mathbf{x}) = 0$. We will now go through the equation and define each component.

(1) Let $\mathbf{F}$ be a function which maps $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$\mathbf{F}(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

where $f_i: \mathbb{R}^n \to \mathbb{R}$.

(2) Let $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x}$ represents the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_i \in \mathbb{R}$ and $i = 1,2, \dots, n$.

(3) From Definition 2.13 we know that $J(\mathbf{x})$ is the Jacobian matrix. Thus $J(\mathbf{x})^{-1}$ is

$$J(\mathbf{x})^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}^{-1}$$

Now we describe the steps of Newton's method:

Step 1:

Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right)$ be a given initial vector.

Step 2:
Calculate $J(\mathbf{x}^{(0)})$ and $\mathbf{F}(\mathbf{x}^{(0)})$.

Step 3:
We now have to calculate the vector $\mathbf{y}^{(0)}$, where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

In order to find $\mathbf{y}^{(0)}$, we solve the linear system $J(\mathbf{x}^{(0)})\mathbf{y}^{(0)} = -\mathbf{F}(\mathbf{x}^{(0)})$, using Gaussian Elimination.

Remark 3.2. [Goldstein, H., 2014] Rearranging the system in Step 3, we get that $\mathbf{y}^{(0)} = -J(\mathbf{x}^{(0)})^{-1}\mathbf{F}(\mathbf{x}^{(0)})$. The significance of this is that, since $\mathbf{y}^{(0)} = -J(\mathbf{x}^{(0)})^{-1}\mathbf{F}(\mathbf{x}^{(0)})$, we can replace $-J(\mathbf{x}^{(0)})^{-1}\mathbf{F}(\mathbf{x}^{(0)})$ in our iterative formula with $\mathbf{y}^{(0)}$. This result will yield that

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)}) = \mathbf{x}^{(k-1)} - \mathbf{y}^{(k-1)}$$

Step 4:
Once $\mathbf{y}^{(0)}$ is found, we can now proceed to finish the first iteration by solving for $\mathbf{x}^{(1)}$. Thus using the result from Step 3, we have that

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{y}^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} + \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_n^{(0)} \end{bmatrix}$$

Step 5:
Once we have calculated $\mathbf{x}^{(1)}$, we repeat the process again, until $\mathbf{x}^{(k)}$ converges to $\bar{\mathbf{x}}$. This indicates we have reached the solution to $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, where $\bar{\mathbf{x}}$ is the solution to the system.

Remark 3.3. When a set of vectors converges, the norm $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\| = 0$. This means that

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\| = \sqrt{\left(x_1^{(k)} - x_1^{(k-1)}\right)^2 + \cdots + \left(x_n^{(k)} - x_n^{(k-1)}\right)^2} = 0$$

3.2 Convergence of Newton's Method[Hairer, E.,2013]

Newton's method converges quadratically, (refer to definition 2.10). When carrying out this method the system converges quite rapdily once the approximation is close to the actual solution of the nonlinear system. This is seen as a advantage because Newton's method may require less iterations, compared to another method with a lower rate of convergence, to reach the solution. However, when the system does not converge, this is an indicator that an error in the computations has occured, or a solution may not exist.

In the following proof, we will prove that Newton's method does indeed converge quadratically.

Proof of Newton's Method Quadratic Convergence[Hairer, E.,2013]

In order for Newton's method to converge quadratically, the initial vector $\mathbf{x}^{(0)}$ must be sufficiently close to a the solution of the system $\mathbf{F} = \mathbf{0}$, which is denoted by $\bar{\mathbf{x}}$. As well, the Jacobian matrix at must not be singular, that is, $J(\mathbf{x})^{-1}$ must exist. The goal of this proof is to show that

$$\frac{\|\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}\|}{\|\mathbf{x}^{(k)} - \bar{\mathbf{x}}\|^2} = \lambda$$

where λ denotes a positive constant.
We have that

$$\|\mathbf{e}^{(k+1)}\| = \|\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}\| = \left\|\mathbf{x}^{(k)} - J\left(\mathbf{x}^{(k)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k)}\right) - \bar{\mathbf{x}}\right\|.$$

If we set $\|\mathbf{e}^{(k)}\| = \|\mathbf{x}^{(k)} - \bar{\mathbf{x}}\|$ then have we that

$$\left\|\mathbf{e}^{(k)} - J\left(\mathbf{x}^{(k)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k)}\right)\right\|. \tag{3.6}$$

Next, we want to define the second-order Taylor series as

$$\mathbf{F}\left(\mathbf{x}^{(k)}\right) \approx \mathbf{F}(\bar{\mathbf{x}}) + \mathbf{J}\mathbf{e}^{(k)} + \frac{1}{2}\left(\mathbf{e}^{(k)}\right)^T \mathbf{H}\left(\mathbf{e}^{(k)}\right)$$

where $\mathbf{J} = J\left(\mathbf{x}^{(k)}\right)$ and $\mathbf{H}$ is the Hessian tensor, which is similiar to the Hessian matrix, i.e. $\mathbf{H} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$, when $\mathbf{F} = f$. We then have to multiply each side of the Taylor's series by $\mathbf{J}^{-1}$, which yields

$$\begin{aligned}\mathbf{J}^{-1}\left(\mathbf{F}\left(\mathbf{x}^{(k)}\right)\right) &\approx \mathbf{J}^{-1}\left[\mathbf{F}(\bar{\mathbf{x}}) + \mathbf{J}\mathbf{e}^{(k)} + \frac{1}{2}\left(\mathbf{e}^{(k)}\right)^T \mathbf{H}\left(\mathbf{e}^{(k)}\right)\right] \\ &= \mathbf{e}^{(k)} + \frac{\mathbf{J}^{-1}}{2}\left(\mathbf{e}^{(k)}\right)^T \mathbf{H}\left(\mathbf{e}^{(k)}\right)\end{aligned}$$

Using (3.6) and (3.7) we obtain our last result such that,

$$\begin{aligned}\|\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}\| &= \|\mathbf{e}^{(k+1)}\| \\ &= \left\|\frac{\mathbf{J}^{-1}}{2}\left(\mathbf{e}^{(k)}\right)^T \mathbf{H}\left(\mathbf{e}^{(k)}\right)\right\| \\ &\leq \frac{\|\mathbf{J}^{-1}\| \, \|\mathbf{H}\|}{2}\|\mathbf{e}^{(k)}\|^2.\end{aligned}$$

Thus is shows that Newton's method converges quadratically.

3.3 Advantages and Disadvantages of Newton's Method[Hairer, E.,2003]

One of the advantages of Newton's method is that its not too complicated in form and it can be used to solve a variety of problems. The major disadvantage associated with Newton's method, is that $J(\mathbf{x})$, as well as its inversion has, to be calculated for each iteration. Calculating both the Jacobian matrix and its inverse can be quite time consuming depending on the size of your system is. Another problem that we may be challenged with when using Newton's method is that it may fail to converge. If Newton's method fails to converge this will result in an oscillation between points.

3.4 A Numerical Example of Newton's Method[Hairer, E.,2003]

The following example is a numerical application of Newton's method from.

Example 3.4. [Hairer, E.,2003]Solve the following nonlinear system

$$\begin{aligned}3x_1 - \cos(x_2 x_3) - \frac{1}{2} &= 0, \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0, \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} &= 0,\end{aligned}$$

when the initial approximation is

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix}$$

Solution: (3.7)

Step 1: We have our initial vector

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix}$$

Step 2: Define $\mathbf{F}(\mathbf{x})$ and $(\mathbf{x})$:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} \end{bmatrix}$$

$$J(\mathbf{x}) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}$$

Now that we have defined $\mathbf{F}(\mathbf{x})$ and $J(\mathbf{x})$, we now want to calculate $\mathbf{F}(\mathbf{x}^{(0)})$ and $J(\mathbf{x}^{(0)})$, where $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^\top$:

$$\mathbf{F}(\mathbf{x}^{(0)}) = \begin{bmatrix} 0.3 - \cos(-0.01) - \frac{1}{2} \\ 0.01 - 3.24 + \sin(-0.1) + 1.06 \\ e^{(-0.01)} - 2 + \frac{10\pi - 3}{3} \end{bmatrix} = \begin{bmatrix} -1.19995 \\ -2.269833417 \\ 8.462025346 \end{bmatrix}$$

and

$$J(\mathbf{x}^{(0)}) = \begin{bmatrix} 3 & (-0.1)\sin(-0.01) & 0.1\sin(-0.01) \\ 0.2 & -32.4 & \cos(-0.1) \\ -0.1e^{-0.01} & -0.1e^{-0.01} & 20 \end{bmatrix} = \begin{bmatrix} 3 & 0.000999983 & -0.000999983 \\ 0.2 & -32.4 & 0.995004165 \\ -0.099004984 & -0.099004983 & 20 \end{bmatrix}$$

Step 3: Solve the system $J(\mathbf{x}^{(0)})\mathbf{y}^{(0)} = -\mathbf{F}(\mathbf{x}^{(0)})$, using Gaussian Elimination:

$$\begin{bmatrix} 3 & 0.000999983 & -0.000999983 \\ 0.2 & -32.4 & 0.995004165 \\ -0.099004984 & -0.099004983 & 20 \end{bmatrix} \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ y_3^{(0)} \end{bmatrix} = - \begin{bmatrix} -1.19995 \\ -2.269833417 \\ 8.462025346 \end{bmatrix}$$

After solving the linear system above it yields the result

$$\mathbf{y}^{(0)} = \begin{bmatrix} 0.40003702 \\ -0.08053314 \\ -0.42152047 \end{bmatrix}$$

Step 4: Using the result in Step 3, compute $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{y}^{(0)}$:

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix} + \begin{bmatrix} 0.40003702 \\ -0.08053314 \\ -0.42152047 \end{bmatrix} = \begin{bmatrix} 0.50003702 \\ 0.01946686 \\ -0.52152047 \end{bmatrix}$$

We can use the results of $\mathbf{x}^{(1)}$ to find our next iteration $\mathbf{x}^{(2)}$ by using the same procedure.

Step 5: If we continue to repeat the process, we will get the following results:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|$
0	0.10000000	0.10000000	-0.10000000	-
1	0.50003702	0.01946686	-0.52152047	0.422
2	0.50004593	0.00158859	-0.52355711	0.0179
3	0.50000034	0.00001244	-0.52359845	0.00158
4	0.50000000	0.00000000	-0.52359877	0.0000124
5	0.50000000	0.00000000	-0.52359877	0

From Remark 3.3 we know that when a set of vectors converges the norm

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\| = 0.$$

Thus by our table above, the norm is equal to zero at the fifth iteration. This indicates that our system $\mathbf{F}(\mathbf{x})$ has converged to the solution, which will be denoted by $\bar{\mathbf{x}}$.
Therefore, from our table of our results we know that

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.50000000 \\ 0.00000000 \\ -0.52359877 \end{bmatrix}$$

is an approximation solution of $\mathbf{F}(\mathbf{x}) = 0$.

There are methods that are in the same family of Newton's method, identified as Quasi-Newton methods. A specific Quasi-Newton method, known as Broyden's method, will be examined in the next section.

Chapter 3

Broyden's Method [Hale, J.K.,2011]

In the last chapter, we examined the numerical method known as Newton's method. We established that one of the major disadvantages of this method was that that $J(\mathbf{x})$ and its inverse must be computed at each iteration. We, therefore want to avoid this problem. There are methods known as Quasi-Newton methods, in which Burden and Faires in [3] describe as methods that use an approximation matrix that is updated at each iteration in place of the Jacobian matrix. This implies that the form of the iterative procedure for Broyden's method is almost identical to that used in Newton's method. The only exception being that an approximation matrix A_i is implemented instead of $J(\mathbf{x})$. With that said the following equation is derived:

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - A_i^{-1}\mathbf{F}\left(\mathbf{x}^{(i)}\right).$$

This is defined as Broyden's iterative procedure. .
In [3], A_i is defined as

$$A_i = A_{i-1} + \frac{\mathbf{y}_i - A_{i-1}\mathbf{s}_i}{\|\mathbf{s}_i\|_2^2}\mathbf{s}_i^t$$

$\mathbf{y}_i = \mathbf{F}\left(\mathbf{x}^{(i)}\right) - \mathbf{F}\left(\mathbf{x}^{(i-1)}\right)$ and $\mathbf{s}_i = \mathbf{x}^{(i)} - \mathbf{x}^{(i-1)}$. However, in Broyden's method it involves that computation A_i^{-1}, not A_i, which brings us to the next theorem.

Theorem 3.1. [Hale, J.K.,2011] (Sherman-Morrison Forumula) If A is a nonsingular matrix and $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then $A + \boldsymbol{x}\boldsymbol{y}^t$ is nonsingular provided that $\boldsymbol{y}^t A^{-1}\boldsymbol{x} \neq -1$ and

$$(A + \boldsymbol{x}\boldsymbol{y}^t)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{x}y^t A^{-1}}{1 + \boldsymbol{y}^t A^{-1}\boldsymbol{x}}.$$

The Sherman-Morrison Formula from, is a matrix inversion formula. It allows A_i^{-1} to be computed directly using A_{i-1}^{-1}, rather than computing A_i and then its inverse at each iteration. Now by using Theorem 4.1 and letting $A = A_{i-1}, \mathbf{x} = \frac{\mathbf{y}_i - A_{i-1}\mathbf{s}_i}{\|\mathbf{s}_i\|_2^2}$, and $\mathbf{y} = \mathbf{s}_i$, as well as using A_i as defined above we have that

$$A_i^{-1} = \left(A_{i-1} + \frac{\mathbf{y}_i - A_{i-1}\mathbf{s}_i}{\|\mathbf{s}_i\|_2^2}\mathbf{s}_i^t\right)^{-1}$$

$$= A_{i-1}^{-1} - \frac{A_{i-1}^{-1}\left(A_{i-1} + \frac{\mathbf{y}_i - A_{i-1}\mathbf{s}_i}{\|\mathbf{s}_i\|_2^2}\mathbf{s}_i^t\right)A_{i-1}^{-1}}{1 + \mathbf{s}_i^t A_{i-1}^{-1}\left(\frac{\mathbf{y}_i - A_{i-1}\mathbf{s}_i}{\|\mathbf{s}_i\|_2^2}\right)}$$

$$= A_{i-1}^{-1} - \frac{(A_{i-1}^{-1}y_i - \mathbf{s}_i)\mathbf{s}_i^t A_{i-1}^{-1}}{\|\mathbf{s}_i\|_2^2 + \mathbf{s}_i^t A_{i-1}^{-1}\mathbf{y}_i - \|\mathbf{s}_i\|_2^2}$$

This leaves us with

$$A_i^{-1} = A_{i-1}^{-1} + \frac{(\mathbf{s}_i - A_{i-1}^{-1}\mathbf{y}_i)\mathbf{s}_i^t A_{i-1}^{-1}}{\mathbf{s}_i^t A_{i-1}^{-1}\mathbf{y}_i}.$$

We compute the inverse of the approximation matrix at each iteration with this equation.

We now desribe the steps of Broyden's method:

Step 1:

Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right)$ be the initial vector given.

Step 2:

Calculate $\mathbf{F}\left(\mathbf{x}^{(0)}\right)$.

Step 3:

In this step we compute A_0^{-1}. Because we do not have enough information to compute A_0 directly, Broyden's method permits us to let $A_0 = J\left(\mathbf{x}^{(0)}\right)$, which implies that $A_0^{-1} = J\left(\mathbf{x}^{(0)}\right)^{-1}$.

Step 4:

Calculate $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - A_0^{-1}\mathbf{F}\left(\mathbf{x}^{(0)}\right)$.

Step 5:

Calculate $\mathbf{F}(\mathbf{x}^{(1)})$.

Step 6:

Take $\mathbf{F}(\mathbf{x}^{(0)})$ and $\mathbf{F}(\mathbf{x}^{(1)})$ and calculate $\mathbf{y}_1 = \mathbf{F}(\mathbf{x}^{(1)}) - \mathbf{F}(\mathbf{x}^{(0)})$. Next, take the first two iterations of $\mathbf{x}^{(i)}$ and calculate $\mathbf{s}_1 = \mathbf{x}^{(1)} - \mathbf{x}^{(0)}$.

Step 7:

Calculate $\mathbf{s}_1^t A_0^{-1}\mathbf{y}_1$.

Step 8:

Compute $A_1^{-1} = A_0^{-1} + \left(\frac{1}{\mathbf{s}_1^t A_0^{-1}\mathbf{y}_1}\right)[(\mathbf{s}_1 - A_0^{-1}\mathbf{y}_1)\mathbf{s}_1^t A_0^{-1}]$

Step 9:

Take A_1^{-1} that we found in Step 8, and calculate $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} - A_1^{-1}\mathbf{F}(\mathbf{x}^{(1)})$.

Step 10:

Repeat the process until we converge to $\bar{\mathbf{x}}$, i.e. when $\mathbf{x}^{(i)} = \mathbf{x}^{(i+1)} = \bar{\mathbf{x}}$. This will indicate that we have reached the solution of the system (refer to Remark 3.3).

3.2 Convergence of Broyden's Method[Hale, J.K.,2011]

Unlike Newton's method, Broyden's method as well as all of the Quasi-Newton methods converge super linearly. This means that

$$\lim_{i\to\infty} \frac{\|\mathbf{x}^{(i+1)} - \mathbf{p}\|}{\|\mathbf{x}^{(i)} - \mathbf{p}\|} = 0$$

where $\mathbf{p}$ is the solution to $\mathbf{F}(\mathbf{x}) = 0$, and $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(i+1)}$ are successive approximations to $\mathbf{p}$. This can be proved in a similar manner that proved the convergence of Newton's method.

3.3 Advantages and Disadvantages Of Broyden's Method [Hirsch, M.W.,2020]

The main advantage of Broyden's method is the reduction of computations. More specifically, the way the inverse of the approximation matrix, A_i^{-1} can be computed directly from the previous iteration, A_{i-1}^{-1} reduces the number of computations needed for this method in comparison to Newton's Method. One thing that is seen as a disadvantage of this Quasi-Newton method is that it does not converge quadratically. This may mean that more iterations may be needed to reach the solution, when compared to the number of iterations Newton's method requires. Another disadvantage of Broyden's method is that as described in [3] by Burden and Faires, is that it is not self-correcting. This means that in contrast to Newton's method, it does not correct itself for round off errors with consecutive interations. This may cause only a slight inaccuracy in the iterations compared to Newton's, but the final iteration will be the same.

Now that we have taken a look at numerical methods for solving multivariable nonlinear equations, in the next section we will focus on a numerical method that is used to nonlinear boundary value problems for ordinary differential equations.

3.4 A Numerical Example of Broyden's Method [Hydon, P.E.,2020]

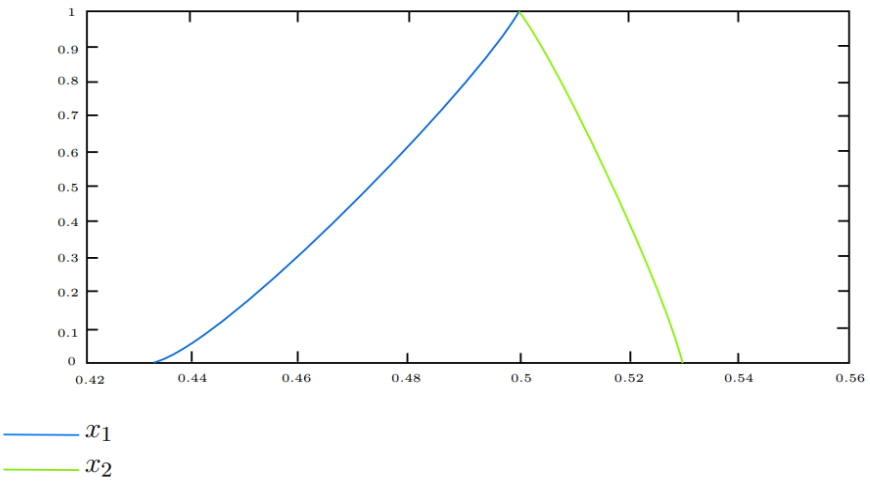

Figure1: Solution Broyden's Method for Example

3.4.1Example[Hydon, P.E.,2004]

$$(3,4,5)x^2 + (1,2,3)x = (1,2,3).$$

Assume that x is positive, without any loss of generality, then the parametric form of this equation is as follows [4,6] :

$$(3+\alpha)\underline{x}^2(\alpha) + (1+\alpha)\underline{x}(\alpha) = (1+\alpha),$$
$$(5-\alpha)\bar{x}^2(\alpha) + (3-\alpha)\bar{x}(\alpha) = (3-\alpha)$$

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Since $A_0(\alpha) = J(\underline{x}_0, \bar{x}_0; \alpha)$

$$A_0(\alpha)^{-1} = J(\underline{x}_0, \bar{x}_0; \alpha)^{-1}.$$

Let $x_i(\alpha) = (\underline{x}_i, \bar{x}_i; \alpha)$, then

$$x_1(\alpha) = x_0(\alpha) - A_0(\alpha)^{-1}F_0(\alpha).$$

In the next step, $x_1(\alpha)$ is used to compute $x_2(\alpha)$. In Broyden's method, the Jacobian matrix is denoted by matrix A_i^{-1}, and it can be computed directly from A_{i-1}^{-1} by using this formula

$$A_1(\alpha)^{-1} = A_0(\alpha)^{-1} + \frac{[s_1(\alpha) - A_0(\alpha)^{-1}y_1(\alpha)]s_1(\alpha)^t A_0(\alpha)^{-1}}{s_1(\alpha)^t A_0(\alpha)^{-1}y_1(\alpha)},$$

where $s_1(\alpha) = x_1(\alpha) - x_0(\alpha)$ and $y_1(\alpha) = F(x_0(\alpha)) - F(x_0(\alpha))$ for $i = 1,2,\dots,n$. Repeat Steps from 4 to 8 until it satisfies the tolerance $\varepsilon \leq 10^{-5}$. After three iterations, the solution was obtained with maximum error 10^{-5}. Details of the solution from $0 \leq \alpha \leq 1$ are given in Figure 1 .

Figure 1 shows that Broyden's method can converge very rapidly to a solution once initial value is obtained which is sufficiently close to true solution. The solution is obtained after two iterations with the maximum error less than 10^{-5}.
Now the second example is proposed.

2.Finite-Difference Method [Ince, E.L.,2004]

In this section, we will examine a numerical method that is used to approximate the solution of a boundary-value problem. We will focus on a two-point boundary-value problem with a second order differential equation which takes the form

$$y'' = f(x, y, y'), a \le x \le b,$$

$$y(a) = \alpha, y(b) = \beta$$

where f is a function, a and b are the end points, and $y(a) = \alpha$ and $y(b) = \beta$ are the boundary conditions.

Example 2.1. [Ince, E.L.,2004] The following example is of a two-point boundary value solve $y''' + y^2y'' - y' = 0, y(0) = y'(0) = 0, y''(1) = 1$. I let $u = dy/dx$ so the new problem is $u'' + y^2u' - u = 0, u(0) = 0, u'(1) = 1, y = \int u$. To try and solve a similar linear problem $u'' + u' - u = 0$. Following code sets up the two matrices so that we can solve $A = b$

Solution:

First, you should split

$$y''' + y^2y'' - y' = 0$$

into a system of three first order ordinary differential equations by letting

$$y' = u$$

and

$$u' = v$$

such that you get

$$v' + y^2v - u = 0$$

After taking another look, I think I might now see what you are trying to do with Simpson's rule. Starting with the definition of Simpson's rule

$$\int_x^{x+2\Delta x} f(x)dx = \frac{\Delta x}{3}(f(x) + 4f(x + \Delta x) + f(x + 2\Delta x))$$

if you consider a grid of n points evenly spaced by Δx where $x_i = i\Delta x$ for $i = 0,1,\dots,n-1$, then for your ODE system you have

have

$$y_{i+2} - y_i = \frac{\Delta x}{3}(u_i + 4u_{i+1} + u_{i+2})$$
$$u_{i+2} - u_i = \frac{\Delta x}{3}(v_i + 4v_{i+1} + v_{i+2})$$
$$v_{i+2} - v_i = \frac{\Delta x}{3}(u_i - y_i^2 v_i + 4(u_{i+1} - y_{i+1}^2 v_{i+1}) + u_{i+2} - y_{i+2}^2 v_{i+2})$$

This is not a linear system of equations in y, u, and v. You can still proceed, but it is difficult. Alternatively, you can integrate these equation in any number of other ways. I recommend Matlabs ODE solvers such as Ode45. Maybe I still don't understand what you are referring to with Simpson's method. If you have simple initial conditions such as $y(0) = u(0) = v(0) = 1$ it would be straight forward to use Ode45. However, it appears you have boundary conditions $y(0) = u(0) = v(1) = 1$ which means you will need to use something like a shooting method (root finding) to figure out which inital conditions $y(0) = u(0) = v(0) = \alpha$ lead you to $v(1) = 1$.

Theorem 5.2. [Iserles, A., 2005] Suppose the function f in the boundary-value problem

$$y'' = f(x, y, y'), a \le x \le b, y(a) = \alpha, y(b) = \beta$$

is continuous on the set

$$D = \{(x, y, y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous in D. If
(1) $f_y(x, y, y') > 0$ for all $(x, y, y') \in D$, and
(2) a constant M exists with $|f_{y'}(x, y, y')| \le M$ for all $(x, y, y') \in D$,

then the boundary-value problem has a unique solution.

The numerical method we will be looking at is the Finite-Difference method. This method can be used to solve both linear and nonlinear ordinary differential equations. We will just survey the nonlinear Finite-Difference method.

A nonlinear boundary-value problem takes on the form of

$$y'' = f(x, y, y'), a \le x \le b, y(a) = \alpha, y(b) = \beta$$

In order for the Finite-Difference method to be carried out we have to assume f satisfies the following conditions as described in [4]:

(1) f and the partial derivatives f_y and $f_{y'}$ are all continuous on

$$D = \left((x, y, y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\right)$$

(2) $f_y(x, y, y') \ge \delta$ on D,for some $\delta > 0$.

(3) Constants k and L exist, with

$$k = \max_{(x,y,y')\in D} \left|f_y(x, y, y')\right|, \text{ and } L = \max_{(x,y,y')\in D} \left|f_{y'}(x, y, y')\right|$$

With f satisfying these conditions, Theorem 5.2 implies that a unique solution exists. When solving a linear boundary-value problem using the Finite-Difference, the second order boundary-value equation

$$y'' = p(x)y' + q(x)y + r(x)$$

is expanded using y in a third Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, where a formula called the centered-difference formula for both $y''(x_i)$ and $y'(x_i)$ is derived. Burden and Faires in define the centered-difference formula for $y''(x_i)$ and $y'(x_i)$ as follows

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i)$$

for some ξ_i in (x_{i-1}, x_{i+1}), and

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6}y'''(\eta_i)$$

for some η_i in (x_{i-1}, x_{i+1}).

Now we can begin to form the procedure for the Finite-Difference method.

Step 1:
We first want to divide the interval $[a, b]$ into $(N + 1)$ equal subintervals which gives us

$$h = \frac{(b-a)}{(N+1)}$$

with end points at $x_i = a + ih$ for $i = 0,1,2, \dots, N + 1$.
Step 2:
Next we will take

$$y''(x_i) = f\big(x_i, y(x_i), y'(x_i)\big)$$

and substitute equations (5.1) and (5.2) into it. This will give us:

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = f\left(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i)\right) + \frac{h^2}{12}y^{(4)}(\xi_i)$$

for some ξ_i and η_i in the interval (x_{i-1}, x_{i+1}).

Step 3:
The Finite-Difference method results by using (5.3), and the boundary conditions to define:

$$w_0 = \alpha, w_{N+1} = \beta$$

and

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0$$

for each $i = 1,2, \dots, N$.
Step 4:
Once we define the boundary conditions in Step 3, an $N \times N$ nonlinear system, $F(\mathbf{w})$, is produced from the Finite Difference method defined in as:

$$\begin{aligned}
2w_1 - w_2 + h^2 f\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) - \alpha &= 0 \\
-w_1 + 2w_2 - w_3 + h^2 f\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right) &= 0 \\
&\vdots \\
-w_{N-2} + 2w_{N-1} - w_N + h^2 f\left(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}\right) &= 0 \\
-w_{N-1} + 2w_N + h^2 f\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) - \beta &= 0
\end{aligned}$$

Step 5:
We can take $\mathbf{F}(\mathbf{w})$, and implement Newton's method to approximate the solution to this system. We can do this by taking an initial approximation $\mathbf{w}^{(0)} = \left(w_1^{(0)}, w_2^{(0)}, \dots, w_N^{(0)}\right)^t$, $\mathbf{F}\left(\mathbf{w}^{(0)}\right)$ and defining the Jacobian matrix as follows:
$J(w_1, w_2, \dots, w_N)_{ij} = -1 + \frac{h}{2}f_{y'}\left(x_i, w_i \frac{w_{i+1} - w_{i-1}}{2h}\right)$, for $i = j - 1$ and $j = 2, \dots, N$

$J(w_1, w_2, \dots, w_N)_{ij} = 2 + h^2 f_y\left(x_i, w_i \frac{w_{i+1}-w_{i-1}}{2h}\right)$, for $i = j$ and $j = 1, \dots, N$

$J(w_1, w_2, \dots, w_N)_{ij} = -1 - \frac{h}{2} f_{y'}\left(x_i, w_i \frac{w_{i+1}-w_{i-1}}{2h}\right)$, for $i = j + 1$ and $j = 1, \dots, N-1$

where $w_0 = \alpha$ and $w_{N+1} = \beta$.

Remark 2.3. [Landau, L.D., 2019] We can find the initial approximation $\mathbf{w}^{(0)}$ by using the following equation equation

$$\mathbf{w}^{(0)} = \alpha + \frac{\beta - \alpha}{b - a}(x_i - a)$$

where $x_i = a + ih$ for $i = 1,2, \dots, N$

In the Finite-Difference method, $J(w_1, w_2, \dots, w_N)$ is tridiagonal with ij th entry. This means that there are non-zero entries on the main diagonal, non-zero entries on the diagonal directly below the main diagonal, and there are non-zero entries on the diagonal directly above the main diagonal.

Crout LU Factorization

Since $J(\mathbf{w})$ is tridiagonal, it takes on the form:

$$J(\mathbf{w}) = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & \dots & \dots & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & \dots & \dots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & a_{i-1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{i,j-1} & a_{ij} \end{bmatrix}.$$

Crout's LU Factorization factors the matrix above into two triangular matrices L and U. These two matrices can be found in the form:

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & l_{32} & l_{33} & 0 & \dots & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_{i,j-1} & l_{ij} \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & u_{12} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & u_{23} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & u_{34} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & u_{i-1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Once we have expressed our original matrix $J(\mathbf{w})$ in terms of L and U, we need to compute the entries of each of these matrices. This procedure involves:

(1) Computing the first column of L, where $l_{i1} = a_{i1}$

(2) Computing the first row of U, where $u_{1j} = \frac{a_{1j}}{l_{11}}$

(3) Alternately computing the columns of L and the rows of U, where

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}, \text{ for } j \leq i, i = 1,2, \dots, N$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}}{l_{ii}}, \text{ for } i \leq j, j = 2,3, \dots, N$$

Once the entries of the LU matrices are determined, we want to solve the system $J(w_1, \dots, w_N)(v_1, \dots, v_n)^t = -\mathbf{F}(w_1, w_2, \dots, w_N)$.

We solve this system using the following procedure:

(1) Set up and solve the system $L\mathbf{z} = \mathbf{F}\left(\mathbf{w}^{(k)}\right)$, where $\mathbf{z} \in \mathbb{R}^n$.

(2) Set up and solve the system $U\mathbf{v} = \mathbf{z}$. (Remember $\mathbf{v} = (v_1, \dots, v_n)^t$)

Once we are able to obtain $\mathbf{v}$, we can proceed with computing $\mathbf{w}_i^{(k)} = \mathbf{w}_i^{(k-1)} + \mathbf{v}_i$, and thus repeating Newton's method for the next iteration.As a result once we can obtain the initial approximation $\mathbf{w}^{(0)}$ and form a $N \times N$ system, we can follow the iterative process for Newton's method described in Chapter 3, with the addition of Crout's LU factorization in place of the Gaussian Elimination, to solve the boundary-value probem, i.e. the values of $y(x_i)$, where $x_i = a + ih$ and $i = 0,1,2, \dots, N + 1$. This implies that the procedure for the Finite-Difference method consists of converting the boundary-value problem into a nonlinear algebraic system. Once a nonlinear algebraic system is formulated, we can use Newton's method to solve this system.

Conclusion

Rom this research, it is safe to say that numerical methods are a vital strand of mathematics. They are a powerful tool in not only solving nonlinear algebraic equations with one variable, but also systems of nonlinear algebraic equations. Even equations or systems of equations that may look simplistic in form, may in fact need the use of numerical methods in order to be solved. Numerical methods are also influential in solving for boundary value problems of nonlinear ordinary differential equations. Solving for boundary value problems of linear ordinary differential equations can be difficult enough. Thus, it would be nearly impossible to solve boundary value problems of nonlinear ordinary differential equations without implementing numerical methods. In this paper, we only examined three numerical methods, however, there are several other ones that we have yet to take a closer look at. The main results of this paper can be highlighted in to two different areas: Convergence and the role of Newton's method. With regards to convergence, we can summarize that a numerical method with a higher rate of convergence may reach the solution of a system in less iterations in comparison to another method with a slower convergence. For example, Newton's method converges quadratically and Broyden's method only converges superliner ally. The implication of this would be that given the exact same nonlinear system of equations denoted by F, Newton's method would arrive at the solution of F=0 in less iterations compared to Broyden's method. The second key result from this paper, is the significance of Newton's method in numerical methods. In the case of both Broyden's method and the Finite-Difference method, Newton's method is incorporated into each of their algorithms. Broyden's method had an almost identical algorithm as Newton's method, with the exception of the use of approximation matrix. The Finite-Difference method implemented Newton's method once the boundary value problem was converted into a nonlinear algebraic system. Not only was Newton's method a part of these methods, but also other various numerical methods that I had come across. This demonstrates the diversity that Newton's method possesses; it can be applied to many problems. This mean we can make a conjecture that Newton's method is a notable process in the area of numerical methods.

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پوختە

ئەم توێژینەوەیە لێکۆڵینەوە لە ڕێگا ژمارەییەکان دەکات بۆ چارەسەرکردنی سیستەمەکانی هاوکێشە ناهێڵەکان، جەخت دەکاتەوە لەسەر سێ ڕێگای سەرەکی. یەکەم، ئێمە میتۆدی نیوتن دەدۆزینەوە، بەکارهێنانی ماتریکسی جاکۆبی بۆ هاوکێشە نا هێڵییە فرەگۆڕەکان. دووەم، میتۆدی برۆیدن، تەکنیکێکی کواسی نیوتن کە بە گشتگیرکردنی میتۆدی سێکانت دادەنرێت، لێکۆڵینەوەی لەسەر دەکرێت. لە کۆتاییدا، میتۆدی جیاوازی کۆتایی لێکۆڵینەوەی لەسەر دەکرێت بۆ چارەسەرکردنی کێشەکانی بەهای سنووری نا هێڵی لە هاوکێشە جیاوازەکانی ئاساییدا. لێکۆڵینەوەکە گفتوگۆی قوولی لەخۆ دەگرێت لەسەر بنەماکانی هەر ڕێگایەک، ڕەچاوکردنی نزیکبوونەوە، و بەکارهێنانی کرداری. نموونەکان بەکارهێنانی میتۆدی نیوتن و شێوازی جیاوازی کۆتایی لە سیناریۆکانی جیهانی ڕاستەقینە نیشان دەدەن. توێژینەوەکە تێڕوانینێک دەربارەی خاڵە بەهێزەکان و سنووردارکردنەکان و بەکارهێنانە شیاوەکانی ئەم ڕێگا ژمارەییانە پێشکەش دەکات کە بەشدارن لە تێگەیشتن لە کاریگەرییان لە چارەسەرکردنی کێشە بیرکارییە ناهێڵەکان