Bank of Questions

Three stages

Subject: Numerical Analysis

EXERCISES 1

1. Compute the absolute and relative errors when approximating p by the value p^*.

   a) $p = \pi$ by $p^* = 3.14$

   b) $p = e^1$ by $p^* = 1699/625$

2. If x=2.536, find the absolute error and relative error when

   (i) x is rounded to two decimal digits. (ii) x is chopped to two decimal digits.

3. Suppose that 1.414 is used as an approximation to . Find the absolute error and relative errors.

4. Find the absolute, relative and percentage errors, if x is rounded off to three decimals digits given x=0.005998.

5. If 0.333 is approximate value of ,

   find the absolute, relative and percentage errors.

6. Find the absolute, relative and percentage errors, if x is rounded off to three decimal digits given x=0.005998.

7. Starting with the intervals [0.5, 0.6], use 3 steps of the Bisection method to find a smaller interval containing a root of $f(x)=\exp(x)-2\cos(x)$. What will be the length of the final interval after 10 steps of the method?

8. Rearranging the equation $x^3=0.5$, gives the iterative formula $x_{n+1}=g(x_n)$, where $g(x) = (2x^2)^{-1}$.

(a) Starting with $x_0 = 1$, compute the x_n up to n=6, and describe what is happening.

(b) Explain the behavior of (a) by evaluating $g'(x)$ at $x=2^{-1/3}$.

(c) Repeat (a) and (b) for the two iterative schemes $x_{n+1} = \frac{x_n}{2} + \frac{1}{4x_n^2}$ and $x_{n+1} = \frac{2x_n}{3} + \frac{1}{6x_n^2}$. In particular: which of them gives faster convergence, and why? [Work to four decimal places.]

9. Apply the Regula-Falsi method for 3 steps starting with $x_0 = 0$, $x_1 = 1$ to find approximations to the root of $f(x) = \frac{1}{x+1} - \frac{3}{4}$.

10. Use the method of false position to find the smallest root of the equation $f(x) = 2\sin(x) + x - 2 = 0$, stopping when $|f(x_n)| < 5 * 10^{-5}$.

11. Compare the results obtained when you use

(a) the bisection method,

(b) the method of false position, and

(c) the secant method with the starting values of 0.7 and 0.9 to solve the equation $3\sin(x) = x + \frac{1}{x}$.

12. If we are asked to find the root of $f(x) = x^2 - 1$ in [0,2], give a function $g(x)$ for which the root is a fixed point.

13. State the criteria for the function $g(x)$ to have a fixed point in the interval [a, b]. Does the function $g(x)=\sqrt{x}$ have a fixed point in the interval [0.5,2]?

14. State the criteria for a function $g(x)$ used in a fixed point iteration method to converge on an interval [a, b]. Does the function $g(x) = 0.5x + \frac{1}{x}$ meet these criteria on the interval [1, 2]?

15. Solve $4\cos(x) = 2^{x}$ with an accuracy of 10^{-1}, by using Newton-Raphson method.

16. Find the positive root of $\cos(x) - x\,e^{x} = 0$ for two steps, use false position method.

17. Use Secant method to write the iterative formula for finding $\sqrt{8}$.

18. For what values of x_0 and x_1 can the secant method be used to solve the following equation?

$$f(x) = \frac{4x - 7}{x - 2} = 0.$$

19. Write three rearrangements of $4x^{3} - e^{x} = 0$ which are converges, use fixed point method.

20. Use Aitken method to find the approximate root of $f(x) = x - e^{-x}$, $x_0 = 1$ for only two steps.

21. Show that $f(x) = 1 - xe^{1-x}$ has a double root at x=1.

22. The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots which is greater than -1. Calculate these roots correct to three decimal places, use Bisection method.

23. Find an approximate root of $x\log_{10}(x) - 1.2 = 0$, use False position method.

24. Can Newton-Raphson method be used to solve $f(x)$=0 if $f(x) = \sqrt{x-3}$ and the starting value is x_0=4? Explain your answer.

25. Find an approximate root of $x^{3} - x - 1 = 0$ in [1, 2] with an 10^{-5} of accuracy, first by Newton-Raphson method and then by secant method.

26. How should the constants k be chosen to ensure the fastest possible convergence with the iteration formula $x_{n+1} = \frac{k\,x_n + x_n^{-2} + 1}{k + 1}$.

27. Use the fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2} for $g(x) = \pi + 0.5\sin(x)$ on the interval $x \in [0, \pi]$.

28. Find the third approximation to the $(5)^{1/3}$ using the bisection method.

29. Find the order of convergence for $x_{n+1} = \frac{1}{2}x_n(1 + \frac{a}{x_n^2})$ at $\sqrt{a}$.

30. Use Newton-Raphson method to solve the non-linear equation $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) - 8^{-x} = 0$, where $x_0 = 0.5$, for only one step.

31. Which of the following iteration forms

1. $x_{n+1} = x_n(3 - 3cx_n + c^2x_n^2)$ 2. $x_{n+1} = x_n(2 - cx_n)$

should be used for finding an approximate value of $\frac{1}{c}$? Explain your answer.

32. Find *a*, *b* and *c* so that the order of the iteration form $x_{n+1} = ax_n^{-2} + cd^2x_n^{-5}$ for $\sqrt[3]{d}$ becomes as high as possible.

33. Consider the iteration form $x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \; n = 0,1,2,\ldots$ where $g(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$. Show that this is quadratically convergent, under suitable hypotheses.

34. Suppose a zero of $g(x)=x$ is approximated by bisection. How many iterations of the bisection method on [-1, 4] are sufficient to compute a solution to $g(x)=x$ to error less than 10^{-6} ?

35. Find the conditions on α to ensure that the iteration $x_{n+1} = x_n - \alpha f(x_n)$, n=0, 1, … will converge linearly to the zero of f if started near the zero.

36. Starting with the interval [1.1, 1.2], use 3 steps of the bisection method to find a smaller interval containing a root of $f(x)=e^x-2-x$. Give the interval after the 3 steps, the estimate for the root, and the maximum error.

37. Newton-Raphson method converge to r of order two if $f(r)=0$ and $f'(r)\neq 0$. What special properties must a function f have if Newton-Raphson method applied to f converges cubically to a zero of f?

38. Show that the sequence $p_n=\frac{1}{n}$ converges linearly to $p=0$, and determine the number of the terms require

Exercises 3

39. For x=(4,-5,1,-1,3), calculate $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$.

40. For $A=\begin{bmatrix} 1 & 7 & -4 \\ 4 & -4 & 9 \\ 12 & -1 & 3 \end{bmatrix}$, find $\|A\|_1, \|A\|_2$ and $\|A\|_\infty$.

41. Find LU factorization for $A=\begin{pmatrix} 6 & -5 & 2 \\ 3 & 2 & -1 \\ 12 & 1 & 1 \end{pmatrix}$.

42. Use Jacobi and Gauss-Seidel methods to solve the following linear system with an accuracy of 1×10^{-1}:

$$x-y+10z=-7$$
$$20x+3y-2z=51$$
$$2x+8y+4z=25.$$

43. Use Gauss -Seidel method to solve the following linear systems

a. $$\begin{aligned} x_1-x_2+10x_3&=-7 \\ 20x_1+3x_2-2x_3&=51 \\ 2x_1+4x_3+8x_2&=25, \end{aligned}$$ (stop iteration after three steps).

b. $\begin{aligned} 3x - 6y + 2z &= 23 \\ -4x + y - z &= -8 \\ x - 3y + 7z &= 17, \end{aligned}$ where $x^{(0)} = \begin{bmatrix} 0.9 \\ -3.1 \\ 0.9 \end{bmatrix}$, (stop iteration after three steps).

44. Show that (a) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$, and that (b) $\|x\|_\infty \leq \|x\|_1 \leq n\,\|x\|_\infty$