Outline

Outline

Review for ODE

- Consider $ay'' + by' + cy = 0$ with $a \neq 0$.
- Characteristic equation: $ar^2 + br + c = 0$.
- Quadratic formula: $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Real distinct roots $r_1 \neq r_2$:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- Complex roots $r_{1,2} = \lambda \pm i\mu$:

$$y(t) = c_1 e^{\lambda t} cos(\mu t) + c_2 e^{\lambda t} sin(\mu t).$$

- Real repeated roots $r_1 = r_2 = r$:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Review for ODE

- Separation of variables:
- Example: Solve $y' = \frac{y}{t}$.
- $\frac{dy}{dt} = \frac{y}{t} \Rightarrow \frac{1}{y}dy = \frac{1}{t}dt \Rightarrow \int \frac{1}{y}dy = \int \frac{1}{t}dt$.
- Then $ln\,|y| = ln\,|t| + c_1 \Rightarrow e^{ln|y|} = e^{ln|t|+c_1}$.
- Hence $|y| = e^{c_1}\,|t| \Rightarrow y = \pm e^{c_1}t$.
- Finally, $y(t) = ct$.

Wave equation

- Consider the wave equation with homogeneous Dirichlet boundary condition on a bounded domain:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases}$$

- Basic idea: Separation of variables.
- Assume $u(x,t) = X(x)T(t)$.
- Then $u_{tt} = X(x)T''(t)$ and $u_{xx} = X''(x)T(t)$.
- Hence $X(x)T''(t) = c^2 X''(x)T(t)$.
- $\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$.

Wave equation

- $\frac{T''(t)}{c^2T(t)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial x} = 0.$
- $\frac{X''(x)}{X(x)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial t} = 0.$
- Then λ is a constant.
- Also, $\frac{T''(t)}{c^2T(t)} = -\lambda \Rightarrow T''(t) + \lambda c^2 T(t) = 0.$
- And $\frac{X''(x)}{X(x)} = -\lambda \Rightarrow X''(x) + \lambda X(x) = 0.$
- $u(x,t) = X(x)T(t)$ and $u(0,t) = u(l,t) = 0$ implies that for any t,
$$X(0)T(t) = X(l)T(t) = 0.$$
- Hence $X(0) = X(l) = 0.$

Wave equation

- We first consider $X''(x) + \lambda X(x) = 0$ with $X(0) = X(l) = 0$.
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $X''(x) + \beta^2 X(x) = 0$.
- Characteristic equation: $r^2 + \beta^2 = 0$.
- $r_{1,2} = \pm \beta i$.
- So $X(x) = Ccos(\beta x) + Dsin(\beta x)$.
- $X(0) = 0 \Rightarrow C = 0$.
- Then $X(x) = Dsin(\beta x)$
- $X(l) = 0 \Rightarrow Dsin(\beta l) = 0$.
- If $D = 0$, then $X(x) \equiv 0$. Trivial case!

Wave equation

- If $D \neq 0$, then $sin(\beta l) = 0 \Rightarrow \beta l = n\pi \; (n = 1, 2, 3, \cdots)$.
- Hence $\beta_n = \frac{n\pi}{l} \; (n = 1, 2, 3, \cdots)$.
- Recall that $\lambda = \beta^2$ and $X(x) = Dsin(\beta x)$.
- Eigenvalues:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2 \; (n = 1, 2, 3, \cdots).$$

- Eigenfunctions:

$$X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right) \; (n = 1, 2, 3, \cdots)$$

by taking $D = 1$.

Wave equation

- Substep 2: $\lambda = 0$.
- Then $X''(x) = 0 \Rightarrow X(x) = C + Dx$.
- $0 = X(0) = C \Rightarrow X(x) = Dx$.
- $0 = X(l) = Dl \Rightarrow D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Wave equation

- Substep 3: $\lambda < 0$.
- Define $w = \sqrt{-\lambda}$, then $X''(x) - w^2 X(x) = 0$.
- Characteristic equation: $r^2 - w^2 = 0$.
- $r_1 = w$ and $r_2 = -w$.
- So $X(x) = Ce^{wx} + De^{-wx}$.
- $X(0) = X(l) = 0 \Rightarrow C + D = 0, \; Ce^{wl} + De^{-wl} = 0$.
- Then $C = 0$ and $D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Wave equation

- Now we consider $T''(t) + \lambda c^2 T(t) = 0$.
- Since $u(x,t) = X(x)T(t)$, we only need to discuss the cases which give non-trivial $X(x)$.
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $T''(t) + \beta^2 c^2 T(t) = 0$.
- Characteristic equation: $r^2 + \beta^2 c^2 = 0$.
- $r_{1,2} = \pm \beta c i$.
- So $T(t) = Acos(\beta ct) + Bsin(\beta ct)$.
- Then for $n = 1, 2, 3, \cdots$:

$$T_n(t) = A_n cos(\beta_n ct) + B_n sin(\beta_n ct) = A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)$$

Wave equation

- Each of $X_n(x)T_n(t)$ $(n = 1, 2, 3, \cdots)$ is a solution.
- Recall eigenfunctions:

$$X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right) \quad (n = 1, 2, 3, \cdots)$$

.

- Finally, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)\right] sin\left(\frac{n\pi}{l}x\right).$$

Wave equation

- What if we also have the initial conditions: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$?
- $\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{l}x\right).$
- $u_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \frac{n\pi c}{l} sin\left(\frac{n\pi ct}{l}\right) + B_n \frac{n\pi c}{l} cos\left(\frac{n\pi ct}{l}\right)\right] sin\left(\frac{n\pi}{l}x\right).$
- $\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} sin\left(\frac{n\pi}{l}x\right).$
- How do we work out A_n and B_n? Chapter 5!

Diffusion/heat equation

- Consider the diffusion/heat equation with homogeneous Dirichlet boundary condition on a bounded domain:

$$\begin{cases} u_t = ku_{xx}, & \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases}$$

- Basic idea: Separation of variables.
- Assume $u(x,t) = X(x)T(t)$.
- Then $u_t = X(x)T'(t)$ and $u_{xx} = X(x)''T(t)$.
- Hence $X(x)T'(t) = kX(x)''T(t)$.
- $\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$.

Diffusion/heat equation

- $\frac{T'(t)}{kT(t)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial x} = 0.$
- $\frac{X''(x)}{X(x)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial t} = 0.$
- Then λ is a constant.
- Also, $\frac{T'(t)}{kT(t)} = -\lambda \Rightarrow T'(t) + \lambda k T(t) = 0.$
- And $\frac{X''(t)}{X(t)} = -\lambda \Rightarrow X''(x) + \lambda X(x) = 0.$
- $u(x,t) = X(x)T(t)$ and $u(0,t) = u(l,t) = 0$ implies that for any t,
$$X(0)T(t) = X(l)T(t) = 0.$$
- Hence $X(0) = X(l) = 0.$

Diffusion/heat equation

- We first consider $X''(x) + \lambda X(x) = 0$ with $X(0) = X(l) = 0$.
  This is exactly the same as above!
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $X''(x) + \beta^2 X(x) = 0$.
- Characteristic equation: $r^2 + \beta^2 = 0$.
- $r_{1,2} = \pm\beta i$.
- So $X(x) = Ccos(\beta x) + Dsin(\beta x)$.
- $X(0) = 0 \Rightarrow C = 0$.
- Then $X(x) = Dsin(\beta x)$
- $X(l) = 0 \Rightarrow Dsin(\beta l) = 0$.

Diffusion/heat equation

- If $D = 0$, then $X(x) \equiv 0$. Trivial case!
- If $D \neq 0$, then $sin(\beta l) = 0 \Rightarrow \beta l = n\pi$ $(n = 1, 2, 3, \cdots)$.
- Hence $\beta_n = \frac{n\pi}{l}$ $(n = 1, 2, 3, \cdots)$.
- Recall that $\lambda = \beta^2$ and $X(x) = Dsin(\beta x)$.
- Eigenvalues:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2 \ (n = 1, 2, 3, \cdots).$$

- Eigenfunctions:

$$X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right) \ (n = 1, 2, 3, \cdots)$$

by taking $D = 1$.

Diffusion/heat equation

- Substep 2: $\lambda = 0$.
- Then $X''(x) = 0 \Rightarrow X(x) = C + Dx$.
- $0 = X(0) = C \Rightarrow X(x) = Dx$.
- $0 = X(l) = Dl \Rightarrow D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Diffusion/heat equation

- Substep 3: $\lambda < 0$.
- Define $w = \sqrt{-\lambda}$, then $X''(x) - w^2 X(x) = 0$.
- Characteristic equation: $r^2 - w^2 = 0$.
- $r_1 = w$ and $r_2 = -w$.
- So $X(x) = Ce^{wx} + De^{-wx}$.
- $X(0) = X(l) = 0 \Rightarrow C + D = 0,\ \ Ce^{wl} + De^{-wl} = 0$.
- Then $C = 0$ and $D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Diffusion/heat equation

- Now we consider $T'(t) + \lambda k T(t) = 0 \Rightarrow T'(t) = -\lambda k T(t)$.
- Since $u(x,t) = X(x)T(t)$, we only need to discuss the cases which give non-trivial $X(x)$.
- Substep 1: $\lambda > 0$.
- $\frac{dT}{dt} = -\lambda k T \Rightarrow \frac{1}{T} dT = -\lambda k dt \Rightarrow \int \frac{1}{T} dT = \int -\lambda k dt$.
- $ln|T| = -\lambda kt + c_1$.
- $|T| = e^{-\lambda kt + c_1} = e^{c_1} e^{-\lambda kt}$.
- So $T(t) = \pm e^{c_1} e^{-\lambda kt} = A e^{-\lambda kt}$.
- Then for $n = 1, 2, 3, \cdots$:

$$T_n(t) = A_n e^{-\lambda_n kt} = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}.$$

Diffusion/heat equation

- Each of $X_n(x)T_n(t)$ $(n = 1, 2, 3, \cdots)$ is a solution.
- Recall eigenfunctions:

$$X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right) \quad (n = 1, 2, 3, \cdots)$$

.

- Finally, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} sin\left(\frac{n\pi}{l}x\right).$$

Diffusion/heat equation

- What if we also have the initial conditions: $u(x, 0) = \phi(x)$?
- $\phi(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{l}x\right)$.
- How do we work out A_n? Chapter 5!

Outline

Wave equation

- Consider the wave equation with homogeneous Neumann boundary condition on a bounded domain:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{for } 0 < x < l, \\ u_x(0,t) = u_x(l,t) = 0. \end{cases}$$

- Basic idea: Separation of variables.
- Assume $u(x,t) = X(x)T(t)$.
- Then $u_{tt} = X(x)T''(t)$ and $u_{xx} = X(x)''T(t)$.
- Hence $X(x)T''(t) = c^2 X(x)''T(t)$.
- $\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$.

Wave equation

- $\frac{T''(t)}{c^2T(t)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial x} = 0.$
- $\frac{X''(x)}{X(x)} = -\lambda \Rightarrow \frac{\partial\lambda}{\partial t} = 0.$
- Then λ is a constant.
- Also, $\frac{T''(t)}{c^2T(t)} = -\lambda \Rightarrow T''(t) + \lambda c^2 T(t) = 0.$
- And $\frac{X''(t)}{X(t)} = -\lambda \Rightarrow X''(x) + \lambda X(x) = 0.$
- $u(x,t) = X(x)T(t)$ and $u_x(0,t) = u_x(l,t) = 0$ implies that for any t,
$$X'(0)T(t) = X'(l)T(t) = 0.$$
- Hence $X'(0) = X'(l) = 0.$

Wave equation

- We first consider $X''(x) + \lambda X(x) = 0$ with $X'(0) = X'(l) = 0$.
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $X''(x) + \beta^2 X(x) = 0$.
- Characteristic equation: $r^2 + \beta^2 = 0$.
- $r_{1,2} = \pm \beta i$.
- $X(x) = Ccos(\beta x) + Dsin(\beta x)$.
- $X'(x) = -C\beta sin(\beta x) + D\beta cos(\beta x)$.
- $X'(0) = 0 \Rightarrow D\beta = 0 \Rightarrow D = 0$.
- $X(x) = Ccos(\beta x)$ and $X'(x) = -C\beta sin(\beta x)$.
- $X'(l) = 0 \Rightarrow -C\beta sin(\beta l) = 0$.

Wave equation

- If $C = 0$, then $X(x) \equiv 0$. Trivial case!
- If $C \neq 0$, then $sin(\beta l) = 0 \Rightarrow \beta l = n\pi \ (n = 1, 2, 3, \cdots)$.
- Hence $\beta_n = \frac{n\pi}{l} \ (n = 1, 2, 3, \cdots)$.
- Recall that $\lambda = \beta^2$ and $X(x) = Ccos(\beta x)$.
- Eigenvalues:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2 \ (n = 1, 2, 3, \cdots).$$

- Eigenfunctions:

$$X_n(x) = cos(\beta_n x) = cos\left(\frac{n\pi}{l} x\right) \ (n = 1, 2, 3, \cdots)$$

by taking $C = 1$.

Wave equation

- Substep 2: $\lambda = 0$.
- Then $X''(x) = 0 \Rightarrow X(x) = C + Dx$.
- $X'(x) = D$.
- $0 = X'(0) = D, \; 0 = X'(l) = D \Rightarrow X(x) = C$.
- Eigenvalue:

$$\lambda_0 = 0.$$

- Eigenfunction:

$$X_0(x) = \frac{1}{2}$$

by taking $C = \frac{1}{2}$.

Wave equation

- Substep 3: $\lambda < 0$.
- Define $w = \sqrt{-\lambda}$, then $X''(x) - w^2 X(x) = 0$.
- Characteristic equation: $r^2 - w^2 = 0$.
- $r_1 = w$ and $r_2 = -w$.
- So $X(x) = Ce^{wx} + De^{-wx} \Rightarrow X'(x) = Cwe^{wx} - Dwe^{-wx}$.
- $X'(0) = X'(l) = 0 \Rightarrow Cw - Dw = 0,\ Cle^{wl} - Dle^{-wl} = 0$.
- $C - D = 0,\ Ce^{wl} - De^{-wl} = 0$.
- Then $C = 0$ and $D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Wave equation

- Now we consider $T''(t) + \lambda c^2 T(t) = 0$.
- Since $u(x,t) = X(x)T(t)$, we only need to discuss the cases which give non-trivial $X(x)$.
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $T''(t) + \beta^2 c^2 T(t) = 0$.
- Characteristic equation: $r^2 + \beta^2 c^2 = 0$.
- $r_{1,2} = \pm \beta c i$.
- So $T(t) = Acos(\beta ct) + Bsin(\beta ct)$.

Wave equation

- Then for $n = 1, 2, 3, \cdots$:

$$T_n(t) = A_n cos(\beta_n ct) + B_n sin(\beta_n ct) = A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)$$

- Substep 2: $\lambda = 0$.

- Then

$$T''(t) + \lambda c^2 T(t) = 0 \Rightarrow T''(t) = 0 \Rightarrow T_0(t) = A_0 + B_0 t.$$

- Each of $X_n(x)T_n(t)$ $(n = 0, 1, 2, 3, \cdots)$ is a solution.

Wave equation

- Recall eigenfunctions:

$$X_0(x) = \frac{1}{2},\ X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right)\ \ (n = 1, 2, 3, \cdots)$$

.

- Finally, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}[A_n cos\left(\frac{n\pi ct}{l}\right) \\ &\quad + B_n sin\left(\frac{n\pi ct}{l}\right)]cos\left(\frac{n\pi}{l}x\right). \end{aligned}$$

Wave equation

- What if we also have the initial conditions: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$?
- $\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=0}^{\infty} A_n cos\left(\frac{n\pi}{l}x\right).$
- $u_t(x,t) =$
$\frac{1}{2}B_0 + \sum_{n=1}^{\infty}\left[-A_n\frac{n\pi c}{l}sin\left(\frac{n\pi ct}{l}\right) + B_n\frac{n\pi c}{l}cos\left(\frac{n\pi ct}{l}\right)\right] cos\left(\frac{n\pi}{l}x\right).$
- $\psi(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n\frac{n\pi c}{l}cos\left(\frac{n\pi}{l}x\right).$
- How do we work out A_n and B_n? Chapter 5!

Diffusion/heat equation

- Consider the diffusion/heat equation with homogeneous Neumann boundary condition on a bounded domain:

$$\begin{cases} u_t = ku_{xx}, & \text{for } 0 < x < l, \\ u_x(0,t) = u_x(l,t) = 0. \end{cases}$$

- Basic idea: Separation of variables.
- Assume $u(x,t) = X(x)T(t)$.
- Then $u_t = X(x)T'(t)$ and $u_{xx} = X(x)''T(t)$.
- Hence $X(x)T'(t) = kX(x)''T(t)$.
- $\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$.

Diffusion/heat equation

- $\frac{T'(t)}{kT(t)} = -\lambda \Rightarrow \frac{\partial \lambda}{\partial x} = 0.$
- $\frac{X''(x)}{X(x)} = -\lambda \Rightarrow \frac{\partial \lambda}{\partial t} = 0.$
- Then λ is a constant.
- Also, $\frac{T'(t)}{kT(t)} = -\lambda \Rightarrow T'(t) + \lambda k T(t) = 0.$
- And $\frac{X''(t)}{X(t)} = -\lambda \Rightarrow X''(x) + \lambda X(x) = 0.$
- $u(x,t) = X(x)T(t)$ and $u_x(0,t) = u_x(l,t) = 0$ implies that for any t,
$$X'(0)T(t) = X'(l)T(t) = 0.$$
- Hence $X'(0) = X'(l) = 0.$

Diffusion/heat equation

- We first consider $X''(x) + \lambda X(x) = 0$ with $X'(0) = X'(l) = 0$. This is exactly the same as above!
- Substep 1: $\lambda > 0$.
- Define $\beta = \sqrt{\lambda}$, then $X''(x) + \beta^2 X(x) = 0$.
- Characteristic equation: $r^2 + \beta^2 = 0$.
- $r_{1,2} = \pm \beta i$.
- $X(x) = Ccos(\beta x) + Dsin(\beta x)$.
- $X'(x) = -C\beta sin(\beta x) + D\beta cos(\beta x)$.
- $X'(0) = 0 \Rightarrow D\beta = 0 \Rightarrow D = 0$.
- $X(x) = Ccos(\beta x)$ and $X'(x) = -C\beta sin(\beta x)$.

Diffusion/heat equation

- $X'(l) = 0 \Rightarrow -C\beta sin(\beta l) = 0$.
- If $C = 0$, then $X(x) \equiv 0$. Trivial case!
- If $C \neq 0$, then $sin(\beta l) = 0 \Rightarrow \beta l = n\pi \ (n = 1, 2, 3, \cdots)$.
- Hence $\beta_n = \frac{n\pi}{l} \ (n = 1, 2, 3, \cdots)$.
- Recall that $\lambda = \beta^2$ and $X(x) = Ccos(\beta x)$.
- Eigenvalues:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2 \ (n = 1, 2, 3, \cdots).$$

- Eigenfunctions:

$$X_n(x) = cos(\beta_n x) = cos\left(\frac{n\pi}{l}x\right) \ (n = 1, 2, 3, \cdots)$$

by taking $C = 1$.

Diffusion/heat equation

- Substep 2: $\lambda = 0$.
- Then $X''(x) = 0 \Rightarrow X(x) = C + Dx$.
- $X'(x) = D$.
- $0 = X'(0) = D, \ 0 = X'(l) = D \Rightarrow X(x) = C$.
- Eigenvalue:

$$\lambda_0 = 0.$$

- Eigenfunction:

$$X_0(x) = \frac{1}{2}$$

by taking $C = \frac{1}{2}$.

Diffusion/heat equation

- Substep 3: $\lambda < 0$.
- Define $w = \sqrt{-\lambda}$, then $X''(x) - w^2 X(x) = 0$.
- Characteristic equation: $r^2 - w^2 = 0$.
- $r_1 = w$ and $r_2 = -w$.
- So $X(x) = Ce^{wx} + De^{-wx} \Rightarrow X'(x) = Cwe^{wx} - Dwe^{-wx}$.
- $X'(0) = X'(l) = 0 \Rightarrow Cw - Dw = 0,\ Cle^{wl} - Dle^{-wl} = 0$.
- $C - D = 0,\ Ce^{wl} - De^{-wl} = 0$.
- Then $C = 0$ and $D = 0$.
- Hence $X(x) \equiv 0$. Trivial case!

Diffusion/heat equation

- Now we consider $T'(t) = -\lambda k T(t)$.
- Since $u(x,t) = X(x)T(t)$, we only need to discuss the cases which give non-trivial $X(x)$.
- Substep 1: $\lambda > 0$.
- $\frac{dT}{dt} = -\lambda k T \Rightarrow \frac{1}{T} dT = -\lambda k dt \Rightarrow \int \frac{1}{T} dT = \int -\lambda k dt$.
- $ln|T| = -\lambda kt + c_1$.
- $|T| = e^{-\lambda kt + c_1} = e^{c_1} e^{-\lambda kt}$.
- So $T(t) = \pm e^{c_1} e^{-\lambda kt} = Ae^{-\lambda kt}$.
- Then for $n = 1, 2, 3, \cdots$:

$$T_n(t) = A_n e^{-\lambda_n kt} = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}.$$

Diffusion/heat equation

- Substep 2: $\lambda = 0$.
- $T'(t) = -\lambda k T(t) \Rightarrow T'(t) = 0 \Rightarrow T_0(t) = A_0$.
- Each of $X_n(x)T_n(t)$ $(n = 0, 1, 2, 3, \cdots)$ is a solution.
- Recall eigenfunctions:

$$X_0(x) = \frac{1}{2},\ X_n(x) = sin(\beta_n x) = sin\left(\frac{n\pi}{l}x\right)\ (n = 1, 2, 3, \cdots)$$

.

- Finally, the solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} cos\left(\frac{n\pi}{l}x\right).$$

Diffusion/heat equation

- What if we also have the initial conditions: $u(x,0) = \phi(x)$?
- $\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi}{l}x\right).$
- How do we work out A_n? Chapter 5!

Other types of problems (HW)

- Mixed boundary conditions $u(0, t) = u_x(l, t) = 0$ or $u_x(0, t) = u(l, t) = 0$ for the wave equation $u_{tt} = c^2 u_{xx}$ or the diffusion/heat equation $u_t = k u_{xx}$.
- Schrödinger equation: $u_t = i u_{xx}$ with different types of boundary conditions.